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THE AMERICAN MATHEMATICAL MONTHLY.

MONTEREY, VIRGINIA, DEC. 26, 1893.

DEAR SIR:—

As there is no regular Mathematical Journal devoted to the solutions of problems and the discussion of mathematical subjects published in the United States, we have taken upon ourselves the great responsibility of publishing The American Mathematical Monthly, the first number of which will appear in January, 1894.

In this journal, we shall endeavor to publish all problems sent to us by contributors, whether they be the easy problems in Arithmetic or the difficult problems in the Calculus, Mechanics, or Probability, and we earnestly invite every lover of mathematics to send us problems and solutions, and their contributions will receive the proper recognition. We, therefore, earnestly solicit your aid in this undertaking in the way of subscriptions and contributions. The journal is to be issued promptly each month, each issue containing 20 pages or more, 10x7 1-2 inches. The paper is to be of good quality and the typography will receive close attention.

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Yours very truly,

B. F. Finkel, A. M.,
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MATHEMATICS AND LOGIC

A brief survey serving as preface to a review of
The Philosophy of Bertrand Russell

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1. Reduction of mathematics to set theory: the logical apparatus. The reduction of mathematics to set-theory was the achievement of the epoch of Dedekind, Frege and Cantor, roughly between 1870 and 1895. As to the basic notion of set (to which that of function is essentially equivalent) there are two conflicting views: a set is considered either a collection of things (Cantor), or synonymous with a property (attribute, predicate) of things. In the latter case "x is a member of the set γ," in formula $x \epsilon \gamma$, means nothing but that x has the property γ. The property of being red or being odd is certainly prior to the set of all red bodies or of all odd numbers. On the other hand, if with regard to a bag of potatoes or a curve drawn by pencil on paper, the property of a potato to be in the bag or of a point to lie on the curve is introduced, then the set (or a more concrete structure representing the set) is prior to the property. Whatever the epistemological significance of this distinction, it leaves the mathematician cool, since for any property γ we may speak of the set γ of all elements which have the property, and with respect to a given set γ we may speak of the property to be a member of γ. When he adopts the term *set* in preference to *property* the mathematician indicates his intention to consider co-extensive properties as identical, two properties α and β being co-extensive if every element that has the property α has also the property β and *vice versa* (set = "Begriffsumfang," Frege). Thus he will identify red and round in spite of their different "meanings" if every red body in the world happens to be round and *vice versa*.

The property π of "being prime" is represented by the propositional function $P(x)$, read "x is prime," with an argument x the range of significance of which is circumscribed by the concept "number." (The natural numbers 1, 2, 3, $\cdots$ shall simply be called numbers; other numbers will be specified as rational, real, *etc.*) Indeed, understanding of the (false) proposition "6 is prime" requires that one understand what it means for *any* number x to be prime. Hence the proposition $P(6)$ arises from the propositional function $P(x)$ by the substitution $x=6$. Besides properties we must consider binary, ternary, $\cdots$, relations, represented by propositional functions of 2, 3, $\cdots$ arguments.

Although the mathematician need care little whether the language of properties or sets is used, he cannot afford to ignore another distinction sometimes confused with it: the distinction between what is considered as *given* and what he *constructs* from the given by the iterated combination of certain explicitly described constructive processes. For instance, in the axiomatic setup of elementary geometry one considers as given three categories of objects—points, lines (=straight lines) and planes, and a few relations between these objects (as "point *lies on* plane"). More complicated relations must be "defined," *i.e.*, logically constructed from these primitive relations. In that intuitive theory of natu-

ral numbers (arithmetic) which is truly basic for all mathematics, even the objects are not given but constructed from the first number 1 by iterating one process, addition of 1; while all arithmetical relations are logically constructed from the one basic relation thus established: $y=x+1$, "x is followed by y." On the other hand, in a phenomenology of nature one would have to deal not only with categories of objects, as "bodies" or "events," but also with whole categories of properties which are prior to all construction, *e.g.* with the continuum of color qualities.

Logical construction of propositional functions from other propositional functions consists in the combined iterated application of a few elementary operations. Among them are the primitive logical operators $\sim$ (not), $\cap$ (and), $\cup$ (or), and the two quantifiers $(\exists x)$, "there exists an x such that," and $(\forall x)$, "for all x." For instance, from two propositional functions $S(x)$, $T(x)$ we form

$$\sim S(x), \qquad S(x) \cap T(x), \qquad S(x) \cup T(x), \qquad (\exists x)S(x), \qquad (\forall x)S(x).$$

The quantifiers carry an argument x as index and "kill" that argument in the propositional function following the quantifier, just as the substitution of an individual number, say $x=6$, does. The arithmetic operations $+$ and $\times$ are primarily applied to numbers and from there carry over to functions, while the process of integration with respect to a variable x by its very nature refers to a function $f(x)$. Just so, the logical operations $\sim$, $\cap$, $\cup$ deal primarily with propositions, while $(\exists x)$, $(\forall x)$ refer to propositional functions involving a variable x. The operator $\cup$ is primitive in the sense that the truth value (true or false) of $a \cup b$ depends only on the truth values of a and b. The same holds for $\sim$ and $\cap$. It is convenient to add the primitive operator of implication for which I use Hilbert's symbol $\rightarrow$: $a \rightarrow b$ is false if a is true and b false, but true for the three other combinations: a true, b true; a false, b true; a false, b false. Propositions without argument result when all variables have been eliminated by substitution of explicitly given individuals or by quantifiers. The construction of arithmetic propositions and propositional functions constitutes their "meanings."

In introducing properties of numbers we presuppose that we know what we mean by "*any* number"; we shall say that the *category* of the elements to which the argument in the propositional function under consideration refers must be given. We are assuming that this category is a closed realm of things existing in themselves, or, as we shall briefly say, is *existential*, when we ask with respect to a given property γ of its elements whether there *exists* an element of the property γ, with the expectation that, whatever the property γ, the question has a definite meaning and that there either exists such an element or every one of its elements has the opposite property $\sim\gamma$. In number theory or in elementary geometry we assume the numbers or the points, lines, planes, to constitute existential categories in this sense. However, we envisage only single individual properties and relations, never anything like the category of "all possible properties of numbers." This situation changes radically with the set-theoretic approach.

There we are forced to consider properties of numbers x as *objects* ξ of a new type to which the numbers stand in the relation $x \epsilon \xi$. The proposition "6 is prime" is now looked upon as arising from the binary relation $x \epsilon \xi$ by substituting 6 for x, and the property π of being prime for the argument ξ. The "copula" ϵ corresponds to the word "is" in the spoken sentence "6 is prime." Any propositional function $P(x)$, like "x is prime," gives rise to a corresponding property $\pi = [x]P(x)$ (the property of being prime) such that $P(x)$ is equivalent to $x \epsilon \pi$. The operator $[x]$ which effects transition from the propositional function to the corresponding property or set kills the argument x. For the sake of uniformity of notation we shall write henceforth $\epsilon(x; \xi)$ instead of $x \epsilon \xi$. In the same way a binary propositional function $P(x, y)$ defines a relation $\pi = [x, y]P(x, y)$ and $\epsilon(xy; \pi)$ expresses the same as $P(x, y)$, namely that x and y are in this relation π.

2. Two examples. Let our further reflections be guided by two typical examples taken from Dedekind's set-theoretic analysis of the two decisive steps in the building up of mathematics: his analysis of the sequence of numbers (in "*Was sind und was sollen die Zahlen*" 1887) and of the continuum of real numbers (in "*Stetigkeit und Irrationalzahlen*" 1872). Our critical attention will be kept more alert by using Frege's terminology of properties rather than that of sets.

I. A property α of numbers is said to be hereditary if for any number x that has the property α, the follower $x+1$ also has it. Dedekind defines: A number b is less than a if there *exists* a hereditary property that a has but b has not.

Here it is not only supposed that we know what we mean by *any* property, but we refer to the totality of all possible properties. In applying the quantifiers to properties of numbers as well as to numbers, it is absolutely imperative to look upon properties as secondary objects related to our primary objects, the numbers, by the copula relation ϵ. Heredity is even a property of properties. To be consistent we have to imagine objects of type 1 (the numbers), of type 2 (the properties of numbers), type 3, $\cdots$, and the fundamental relation $\epsilon(x_n; x_{n+1})$ connects a variable x_n of type n with one x_{n+1} of type $n+1$. Let $\mathrm{I}(\xi)$ denote the proposition that ξ is hereditary. The definition of this propositional function whose Greek argument refers to the category "properties (or sets) of numbers" is as follows:

$$\mathrm{I}(\xi) = (\forall x)\{\epsilon(x; \xi) \rightarrow \epsilon(x + 1; \xi)\},$$

and Dedekind's definition of $x < y$:

$$(x < y) = (\exists \xi)\{\mathrm{I}(\xi) \cap \epsilon(y; \xi) \cap \sim \epsilon(x; \xi)\}.$$

II. Dedekind, as Eudoxos had done more than 2000 years before him, characterizes a non-negative real number α by the set of all positive rational numbers (fractions) $x > \alpha$. But for him any arbitrarily constructed (non-empty) set α of fractions (satisfying a certain condition, namely that along with any fraction b it contains every fraction $> b$) creates a corresponding real number α. The real number α is but a *façon de parler* for this set α. A set α consists of all fractions x satisfying a certain propositional function $A(x)$; $\alpha = [x]A(x)$. Let $\mathrm{I}(\xi)$ be a

propositional function whose argument ξ refers to properties of fractions. The (greatest) lower bound $\gamma = [x]C(x)$ of a set of non-negative real numbers ξ can then be obtained as the *join* of all sets ξ for which $\mathrm{I}(\xi)$ holds:

$$C(x) = (\mathfrak{A}\xi)\{\mathrm{I}(\xi) \cap (x \in \xi)\}. \tag{1}$$

In this way Dedekind *proves* that any set of non-negative real numbers has a lower bound. Again the quantifier $(\mathfrak{A}\xi)$ is applied to "all possible properties of fractions."

3. Levels or no levels? The constructive and the axiomatic standpoints. But now let us pause to think what we have been doing. Properties of fractions are constructed by combined iterated application of a number of elementary logical operations $O_1, O_2, \cdots, O_h$. Let us call any property α thus obtained a *constructible* property, or a property of level 1. We can then interpret $(\mathfrak{A}\xi)$ in the definition (1) as "There exists a *constructible* property ξ," and this application of the quantifier is legitimate provided we admit its applicability to natural numbers. Indeed, the different manners in which one can form finite sequences with iteration out of h symbols $O_1, \cdots, O_h$ is not essentially more complicated than the possible finite sequences of one symbol 1. But the property $\gamma = [x]C(x)$ defined by (1) is certainly not identical in its meaning with any of the properties of level 1 because it is defined *in terms of the totality of all properties of level* 1. It is therefore a property of higher level 2. Nevertheless it may be *coextensive* with a property of level 1, and this is exactly what Russell's "axiom of reducibility" claims. But if the properties are constructed there is no room for an axiom here; it is a question which ought to be decided on the ground of the construction; and in our case that is a hopeless business. On the other hand, the edifice of our classical analysis collapses if we have to admit different levels of real numbers such that a real number is of level $l+1$ when it is defined in terms of the totality of real numbers of level l. If we wish to save Dedekind's proof we must abandon the constructive standpoint and assume that there is *given*, independently of all construction, an existential category "properties" or "second-type objects" (among which the constructible ones form but a small part) such that the following axiom holds, replacing the definition (1): For a given third-type object $\mathfrak{i}$ there exists a second-type object γ such that

$$x \in \gamma \cdot \equiv \cdot (\mathfrak{A}\xi)\{(x \in \xi) \cap (\xi \in \mathfrak{i})\}.$$

Here the arguments x and ξ refer to objects of first and second types, and $\equiv$ means "coextensive." That is a bold, an almost fantastic axiom; there is little justification for it in the real world in which we live, and none at all in the evidence on which our mind bases its constructions. With the assumption that properties constitute an existential category of given objects we return from Dedekind, who wanted to *construct* the real numbers out of the rational ones, to Eudoxos, for whom they were *given* by the points on a line; and instead of proving the existence of the lower bound on the ground of a definition of real numbers, we accept it as an axiom.

In reflecting on the source of the antinomies which had shown up at the fringe of Cantor's general set-theory, Russell realized the necessity of distinguishing between the several levels [1]. No doubt, by this fundamental insight, which he expressed somewhat loosely by his vicious circle principle: "No totality can contain members defined in terms of itself," he cured the disease but, as shown by the Dedekind example, also imperiled the very life of the patient. Classical analysis, the mathematics of real variables as we know it and as it is applied in geometry and physics, has simply no use for a continuum of real numbers of different levels. With his axiom of reducibility Russell therefore abandoned the road of logical analysis and turned from the constructive to the existential-axiomatic standpoint,—a complete volte-face.* After thus abolishing the several levels of properties, he still has the hierarchy of types: primary objects, their properties, properties of their properties, *etc.* And he finds that this alone will stop the known antinomies. But in the resulting system mathematics is no longer founded on logic, but on a sort of logician's paradise, a universe endowed with an "ultimate furniture" of rather complex structure and governed by quite a number of sweeping axioms of closure. The motives are clear, but belief in this transcendental world taxes the strength of our faith hardly less than the doctrines of the early Fathers of the Church or of the scholastic philosophers of the Middle Ages.

4. The Russell universe. Let us describe this structure in a little more detail. A few primary categories of elements are given; they are the ranges of significance for the lowest types of arguments. In a relation, each of the n ($=1$ or 2 or 3 or $\cdots$) arguments $x_1, \cdots, x_n$ refers to a certain "type" $k_1, \cdots, k_n$; the relation itself is of a type $k^* = \{k_1, \cdots, k_n\}$, determined by $k_1, \cdots, k_n$, that stands higher than any of the constituents $k_1, \cdots, k_n$. Draw a diagram representing k^* by a dot and $k_1, \cdots, k_n$ by a row of dots below k^* joined with it by straight lines, as you would depict a man k^* and his descendants in a geneological tree. Descending from k^* to its n constituents, and from them to their constituents, *etc.*, one obtains a "topological tree" in which each *end point* is associated with one of the primary categories: this diagram describes the type k^*. The fundamental relation is $\epsilon(x_1 \cdots x_n; x^*)$ where $x_1, \cdots, x_n$ refer to given types $k_1, \cdots, k_n$ and x^* to $k^* = \{k_1, \cdots, k_n\}$. Existential categories of elements are supposed to be given, one for each possible type (including the lowest, the primary types). We mentioned at the beginning with what data axiomatic elementary geometry operates, or a phenomenology of nature may have to operate; our present "Russell universe" U is seen to be incomparably richer.

* I know very well that this is at odds with Russell's own interpretation; he in the course of time became more and more inclined to visualize sets as "logical fictions." "Though," as Gödel adds, "perhaps the word fiction need not necessarily mean that these things do not exist, but only that we have no direct perceptions of them." In the second edition of Volume I of the Principia Mathematica, an attempt is made to prove independently of the axiom of reducibility that at least all levels of *natural numbers* can be reduced to the five lowest. But as Gödel observes, the proof is far from being conclusive. [2]

We give a few of the more obvious axioms on which its theory is to be erected. The universal normal form for propositional functions involving n variables $x_1, \cdots, x_n$ of given types $k_1, \cdots, k_n$ is $\epsilon(x_1 \cdots x_n; a^*)$. Indeed, such a relation is itself an element a^* of type $k^* = \{k_1, \cdots, k_n\}$. The relation of identity $(x=y)$ between elements of type k is itself an element of type $\{k, k\}$; call it $I=I_k$. Similarly, let $E=E_{k_1 \cdots k_n}$ be the ϵ relation with its $n+1$ arguments of types $k_1, \cdots, k_n$, $k^* = \{k_1, \cdots, k_n\}$. The existence of these special elements must be stipulated explicitly:

Axiom 1. There is an element $I=I_k$ of type $\{k, k\}$ such that $\epsilon(xy; I)$ holds if and only if the elements x, y of type k are identical. There is an element $E=E_{k_1 \cdots k_n}$ of type $K = \{k_1, \cdots, k_n, k^*\}$ such that $\epsilon(x_1 \cdots x_n x^*; E)$ is coextensive with $\epsilon(x_1 \cdots x_n; x^*)$, the variables $x_1, \cdots, x_n$ and x^* ranging over their respective categories $k_1, \cdots, k_n$ and $k^* = \{k_1, \cdots, k_n\}$.

The composite property "red or round" is no longer constructed from the descriptive properties "red," "round," but belongs with them to the category of properties given prior to all construction. Its existence must be guaranteed by one of the simpler axioms of closure.

Axiom 2. Given an element a^* and an element b^* of type $k^* = \{k_1, \cdots, k_n\}$ there exists an element c^* of the same type such that $\epsilon(x_1 \cdots x_n; c^*)$ is coextensive with

$$\epsilon(x_1 \cdots x_n; a^*) \cup \epsilon(x_1 \cdots x_n; b^*)$$

(each x_i varying over its category k_i).

Substitution of a definite element b for a variable is taken care of by the

Axiom 3. Given an element a^* of type $k^* = \{k_1, \cdots, k_n\}$ and an element b_n of type k_n, there exists an element c of type $k = \{k_1, \cdots, k_{n-1}\}$ such that $\epsilon(x_1 \cdots x_{n-1}; c)$ is coextensive with $\epsilon(x_1 \cdots x_{n-1} b_n; a^*)$ (if $x_1, \cdots, x_{n-1}$ vary over the categories $k_1, \cdots, k_{n-1}$).

Elimination of a variable x_n by the corresponding quantifier $(\exists x_n)$ changes a relation a^* of type k^* into a relation a of type k:

Axiom 4. Given an element a^* of type $k^* = \{k_1, \cdots, k_n\}$, there exists an element a of type $k = \{k_1, \cdots, k_{n-1}\}$ such that $\epsilon(x_1 \cdots x_{n-1}; a)$ is coextensive with $(\exists x_n)\epsilon(x_1 \cdots x_{n-1} x_n; a^*)$.

These are only a few typical axioms that indicate the direction. The reader must not look for them in the *Principia Mathematica*, which are conceived in a different style. But our system U embodies the same ideas in a form that seems to me both natural in itself and advantageous for a comparison with other systems W, Z presently to be discussed.

Our axioms serve as a basis for deduction in the same way as, for instance, the axioms of geometry; deduction takes place by that sort of logic on which one is used to rely in geometry or analysis, including the free use of "there exists" and "all" with reference to any fixed type in the hierarchy of types and the elements of the corresponding categories. While these categories, as well as the basic relation ϵ, are considered as undefined, the logical terms like "not" $\sim$,

"if then" $\rightarrow$, "there is" $(\exists x)$, *etc.*, have to be understood in their meaning and do not form part of the axiomatic system: the formalism of symbolic logic is merely used for the sake of conciseness.

If the *Principia Mathematica* set out to base mathematics on pure logic the result, as we now see, is quite different: an axiomatic world system has taken the place of logic. Its very structure, the hierarchy of types, cannot be described without resort to the intuitive concept of iteration. To develop, in Dedekind-Frege fashion, a theory of natural numbers from this system is therefore an enterprise of doubtful value.

5. A constructive compromise. Realizing the highly transcendental character of the axiomatic universe from which this system deduces mathematics, one wonders whether it is not possible to stick, in spite of everything, to the constructive standpoint, which seems so much more natural to the mathematician. We accept the hierarchy of types; but we assume only one category of primary objects, the numbers; and one basic binary relation between numbers, namely "x is followed by y." All other relations of the various types are explicitly constructed, the quantifiers $(\exists x)$ and $(\forall x)$ being applied only to numbers and not to arguments of higher type. No axioms are postulated. What we can get in this way constitutes the ground level, or level 1. One could build over it a second level containing relations which are constructed by applying the quantifiers to the totality of relations of this or that type constructible on the first level, and proceed in the same manner from the second to a third level, *etc.* One would obtain a "ramified hierarchy" of types and levels. But in this way, as we have said before, nothing resembling our classical Calculus will result. The temptation to pass beyond the first level of construction must be resisted; instead, one should try to make the range of constructible relations as wide as possible *by enlarging the stock of basic operations*. It is *a priori* clear that *iteration* in some form must find a place among these irreducible principles of construction—contrary to the Dedekind-Frege program.

We begin again at the beginning. Let $R(x, \xi)$ be a binary propositional function the two arguments x, ξ of which are of types k, κ respectively; for instance the relation "x is less than ξ" between numbers. We can then speak of the property of a number "to be less than ξ" (or of the set of all numbers less than ξ). This is obviously a property depending on ξ. In general terms we can form $[x]R(x, \xi) = r^*(\xi)$, which is an element of $\{k\}$ depending on ξ such that $R(x, \xi)$ is coextensive with $\epsilon(x; r^*(\xi))$. If in addition to $R(x, \xi)$ we have a propositional function $S(x^*)$ whose argument is of type $k^* = \{k\}$, we can form $T(\xi) = S(r^*(\xi))$, or more explicitly,

$$T(\xi) = S([x]R(x, \xi)).$$

This is the *process of substitution* generating $T(\xi)$ from $R(x, \xi)$ and $S(x^*)$.

Take now the particular situation that $\kappa = \{k\}$. Then argument and value of the function $r^*(\xi) = [x]R(x, \xi)$ are of the same type κ, and whenever that happens *iteration* becomes possible. Thus we define by complete induction a

relation $T(n, \xi)$ in which the argument n refers to the primary category of numbers, as follows:

$$T(1, \xi) = S(\xi);$$
$$T(n + 1, \xi) = T(n, r^*(\xi)) \qquad (n = 1, 2, \cdots).$$

Adding the operation of substitution and iteration, as illustrated by these examples, to the other elementary logic operations, but without applying the quantifiers to anything else than numbers, the writer was able (in *Das Kontinuum*, 1918) to build up in a purely constructive way, and without axioms, a fair part of classical analysis, including for instance Cauchy's criterion of convergence for infinite sequences of real numbers.* In this system iteration plays the role which in set-theory was played by the uninhibited application of quantifiers. Our construction honestly draws the consequences of Russell's logical insight into the tower of levels, which Dedekind had ignored inadvertently and Russell himself, afraid of its consequences, razed to the ground by his axiom of reducibility. Considering their common origin the axiomatic system U as outlined in section 4 and this constructive approach are remarkably different. But even here we have adhered to the belief that "there is" and "all" make sense when applied to natural numbers: in addition to logic we rely on this existential creed and the idea of iteration.

6. Brouwer's intuitionistic mathematics. Essentially more radical and a further step toward pure constructivism is Brouwer's intuitionistic mathematics [3]. Brouwer made it clear, as I think beyond any doubt, that there is no evidence supporting the belief in the existential character of the totality of all natural numbers, and hence the principle of excluded middle in the form "Either there is a number of the given property γ, or all numbers have the property $\sim\gamma$" is without foundation. The first part of the sentence is an *abstract* from some statement of fact in the form: The number thus and thus constructed has the property γ. The second part is one of *hypothetic* generality, asserting something only if $\cdots$; *viz.*, *if* you are actually given a number, you may be sure that it has the property $\sim\gamma$. The sequence of numbers which grows beyond any stage already reached by passing to the next number, is a manifold of possibilities open towards infinity; it remains forever in the status of creation, but is not a closed realm of things existing in themselves. That we blindly converted one into the other is the true source of our difficulties, including the antinomies—a source of more fundamental nature than Russell's vicious circle principle indicated. Brouwer opened our eyes and made us see how far classical mathematics, nourished by a belief in the "absolute" that transcends all human possibilities of realization, goes beyond such statements as can claim real meaning and truth founded on evidence. According to his view and reading of history, classical logic was abstracted from the mathematics of finite sets and

* But of course the theorem of the lower bound of an arbitrary set of non-negative real numbers could not be upheld.

their subsets. (The word finite is here to be taken in the precise sense that the members of such a set are explicitly exhibited one by one.) Forgetful of this limited origin, one afterwards mistook that logic for something above and prior to all mathematics, and finally applied it, without justification, to the mathematics of infinite sets. This is the Fall and original sin of set-theory, for which it is justly punished by the antinomies. Not that such contradictions showed up is surprising, but that they showed up at such a late stage of the game!

Thanks to the notion of "Wahlfolge," that is a sequence *in statu nascendi* in which one number after the other is *freely chosen* rather than determined by law, Brouwer's treatment of real variables is in the closest harmony with the intuitive nature of the continuum; this is one of the most attractive features of his theory. But on the whole, Brouwer's mathematics is less simple and much more limited in power than our familiar "existential" mathematics. It is for this reason that the vast majority of mathematicians hesitate to go along with his radical reform.

7. The Zermelo brand of axiomatics; sets and classes. From this excursion to the left wing of the "constructionists" we return to the universe U with its hierarchy of types. Once one has committed oneself to the existential or axiomatic viewpoint, can one not go forth in the same direction and even erase all differences of types, taking only such precautions as are absolutely necessary to avoid the known contradictions? This is what Zermelo did in his *Untersuchungen über die Grundlagen der Mengenlehre*, 1908 [4]. His axioms deal with only one (existential) category of objects called elements or sets, and one basic relation $x \in y$, "x is member of y." But he is forced to give up the principle that any well-defined property γ determines an element c such that $x \in c$ whenever the element x has the property γ and *vice versa*. Properties are used by him merely to cut out subsets *from a given set*. Hence his axiom of selection: "Given a well-defined property γ and an element a, there is an element a' such that $x \in a'$ if and only if x is member of a and has at the same time the property γ." The notion of a well-defined property which enters into it is somewhat vague. But we know that we can make it precise by constructing properties by iterated combined application of some elementary constructive processes. Instead of saying that x has the property γ, let us say that x is a member of the class γ, $x \in \gamma$. We thus distinguish between elements or sets on the one hand, classes on the other, and formulate the axioms in terms of two undefined categories of objects, elements and classes. Since we postulate that two elements a, b are identical in case $x \in a$ and $x \in b$ are coextensive, and since each element a is associated with the class α of all elements x satisfying the condition $x \in a$, we are justified in identifying a with that class α. Then *every element is a class* and the axioms deal with one undefined fundamental relation $x \in \xi$, "the element x is member of the class ξ," which has absorbed Zermelo's relation $x \in y$ between elements. The principles for the construction of properties are replaced by corresponding axioms for classes; *e.g.*, given two classes α and β, there exists a class γ

such that the statement $(x \in \alpha) \cup (x \in \beta)$ about an arbitrary element x is co-extensive with $x \in \gamma$.

Since the axiom of selection can only generate smaller sets out of a given set, we need some vehicle that carries us in the opposite direction. Therefore two axioms are added guaranteeing the existence of the set of all subsets of a given set and the join of a given set of sets. It is essential that they be limited to sets = elements and do not apply to classes.

With the introduction of classes, which is due to Fraenkel, von Neumann, Bernays, and others, the axioms assume the same self-sustaining character as, for instance, the axioms of geometry; no longer do such general notions as "any well-defined property" penetrate into the axiomatic system from the outside. A complete table of axioms for this system, Z as we shall call it, is to be found on the first pages of Gödel's monograph, *Consistency of the Continuum Hypothesis*, Annals of Mathematics Studies No. 3, Princeton, 1940 [5]. Even before the turn of the century Cantor himself had moved in the same direction by distinguishing "consistent classes" = sets and inconsistent classes [6]. Not the hierarchy of types, but the non-admission among the decent "sets" of such classes as are too "big," averts here the disaster of the familiar antinomies.

One might object to a system like Z on the ground that it does not rest on a real insight into the causes of the antinomies but patches up Cantor's original conception by a minimum of concessions necessary to avoid the known contradictions. Indeed, we have no assurance of the consistency of Z, except from the empirical fact that so far no contradictions have resulted from it. But we are in no better position toward Russell's universe U. And Z has its great advantages over U: it is of an essentially simpler structure and seems to be the most adequate basis for what is actually done in present-day mathematics. In particular, the "existential" Dedekind-Frege theory of numbers can be derived from it (Zermelo), and Gödel was able to show (*l.c.*) that Zermelo's far-reaching axiom of choice in a very sharp form is consistent with the other axioms of Z

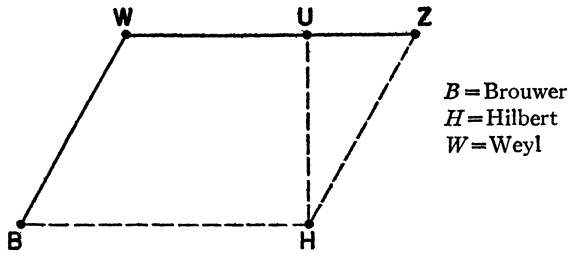
8. Complete formalization and the question of consistency. Pessimistic conclusions. A new turn in the axiomatization of mathematics of paramount importance was inaugurated by Hilbert's "Beweistheorie" (since 1922) [7]. Hilbert sets out *to prove* (not the truth, but) *the consistency of mathematics*. He realizes that to that end mathematics and logic must first be completely formalized: all statements are to be replaced by formulas in which now also the logical operators $\sim$, $\cap$, $(\exists x)$ *etc.*, appear as undefined symbols. Thus formalized logic is absorbed into formalized mathematics.* The formulas have no meaning. A mathematical demonstration is a concrete sequence of formulas in which a formula is derived from the preceding ones according to certain rules comprehensible without recourse to any meaning of the formulas—just as in a game of chess each position is derived from the preceding one by a move obeying certain

* In this regard the ground was well prepared for Hilbert by the Principia Mathematica. For the sake of comparison I mention one other completely formalized system, that of Quine. [8]

rules. *Consistency*, the fact that no such game of deduction may end with the formula $\sim(1=1)$, must be proved by intuitive reasoning about the formulas which rests on evidence rather than on axioms, and respects throughout the limits of evidence as disclosed by Brouwer. But in this thinking about demonstrations, in following a hypothetic sequence of formulas leading up to the end formula $\sim(1=1)$ our mind cannot help using that type of evidence in which the possibility of iteration is founded. In the axiomatization of mathematics Hilbert is forced to proceed with more restraint than Zermelo: if he is too liberal with his axioms he will lose all chance of ever proving their consistency; he is guided by an at least vaguely preconceived plan for such a proof. It is for this reason that he finds it advisable, for instance, to distinguish various levels of variables.

Hilbert's formulas are concrete structures consisting of concrete symbols; the order in which the symbols follow one another in a formula, and also their identity in the same or different formulas, must be recognizable irrespective of little variations in their execution. Handling these symbols, we move on the same level of understanding as guides our daily life in our relationship to such tools as hammer or table or chair. Hilbert sees in it the most important prelogical foundation of mathematics, in fact of all science. But in addition his axioms of mathematics and the intuition of iteration of which the metamathematical non-axiomatic reasoning about mathematics makes use are other extralogical ingredients of his system.

Our brief survey may be summarized in a little diagram in which the constructive tendency increases toward the left, the axiomatic toward the right, and also the relative "depth" of the foundations is indicated. Frege and after him Russell had hoped (1) to develop the theory of natural numbers on a sure basis without recourse to the intuitive idea of indefinite iteration, and (2) to make mathematics a part of logic. We have now seen that none of the systems



discussed gives any hope of accomplishing (2), the subjugation of mathematics by logic. In U and Z elaborate systems of axioms form the basis, in W, B, and to a lesser degree also in U, the intuitive idea of iteration is indispensable. The extra-logical foundations of Hilbert's theory have just been described. The only system which in a sense can claim to reach the goal (1) is Z. But even there the theory of numbers does not rest on logic alone, but on a highly transcendental system of axioms (the belief in whose consistency is supported by empirical facts,

but not by reasons). Poincaré has proved right in his defense of mathematical induction as an indispensable and irreducible tool of mathematical reasoning.

It is likely that all mathematicians ultimately would have accepted Hilbert's approach had he been able to carry it out successfully. The first steps were inspiring and promising. But then Gödel dealt it a terrific blow (1931), from which it has not yet recovered. Gödel enumerated the symbols, formulas, and sequences of formulas in Hilbert's formalism in a certain way, and thus transformed the assertion of consistency into an arithmetic proposition. He could show that this proposition can neither be proved nor disproved within the formalism [9]. This can mean only two things: either the reasoning by which a proof of consistency is given must contain some argument that has no formal counterpart within the system, *i.e.*, we have not succeeded in completely formalizing the procedure of mathematical induction; or hope for a strictly "finitistic" proof of consistency must be given up altogether. When G. Gentzen finally succeeded in proving the consistency of arithmetic [10] he trespassed those limits indeed by claiming as evident a type of reasoning that penetrates into Cantor's "second class of ordinal numbers."

From this history one thing should be clear: we are less certain than ever about the ultimate foundations of (logic and) mathematics. Like everybody and everything in the world today, we have our "crisis." We have had it for nearly fifty years. Outwardly it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable practical influence on my mathematical life: it directed my interests to fields I considered relatively "safe," and has been a constant drain on the enthusiasm and determination with which I pursued my research work. This experience is probably shared by other mathematicians who are not indifferent to what their scientific endeavors mean in the context of man's whole caring and knowing, suffering and creative existence in the world.

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UNIFORM ISOHEDRAL TORI

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1. Introduction. A polyhedron will be said to be *uniform* if the same number of edges radiates from each vertex. It is *isohedral*, in the euclidean sense, if all of its faces are congruent or are mirror images. "Isohedral" is used here in the *local* rather than the *complete* sense, *i.e.* the symmetry groups of the tori considered are not transitive on the faces.

The Euler equation states $F+V-E=2(1-g)$ for a polyhedron of genus g with F faces, V vertices, and E edges. For a uniform isohedral torus (abbreviated u.i.t.) with p-sided faces and e edges per vertex $g=1$, $V=pF/e$, $E=pF/2$, and the Euler equation becomes $F(1-\frac{1}{2}p+p/e)=0$. If $F\neq 0$ the only solutions in positive integers are $p=3$, $e=6$; $p=4=e$; and $p=6$, $e=3$. Thus there may be three classes of u.i.t. with triangular, quadrilateral, or hexagonal faces. In this paper we show the existence of the first two classes by constructing u.i.t. with 1) $p=3$, $e=6$, $F=12n$ $(n\geqq 3)$ or 2) $p=4=e$, $F=8n$ $(n\geqq 4)$. These fragmentary results leave the following questions unanswered. Are there u.i.t. in the hexagonal class? For what values of F can u.i.t. of each class be constructed, and in particular, what is the minimum F in each class? It would also be amusing to know whether there exist knotted u.i.t.

2. Triangular tori. We shall construct triangular u.i.t. by combining suitable octahedral cells. Consider the six points $A(0, 0, h)$, $B(-a, 0, c)$, $C(a, 0, c)$, $D(0, -a, -c)$, $E(0, a, -c)$, and $F(0, 0, -h)$ for a, c, and $h>0$. From the set of segments determined by pairs of these points remove AF, BC, and DE. If $h\neq c$ the remaining segments are edges of an isohedral octahedron. It is easily shown that this is rigid by cutting along edges BD, DC, CE, and EB and hinging edges AB, AC, AD, and AE of one of the halves. For this half DE is a strictly monotone decreasing continuous function of BC. In order to be able to assemble the octahedron we need $BC=DE$. Since both BC and DE can be zero, there is a unique BC such that $BC=DE$. Hence the octahedron is rigid. Since the faces themselves are rigid, excision of CEF and ABD leaves a rigid polyhedron. This implies that *our triangular u.i.t. are rigid.*

Reflect the isohedral octahedron $ABCDEF$ in the plane of CEF to obtain a congruent octahedron $A'B'CD'EF$, where A', B', and D' are the images of A, B, and D respectively. We shall call this pair of apposed octahedra a *wedge.* To prevent interpenetration of octahedra we require them to be convex. Convexity occurs if and only if $h>c$. By symmetry planes BAD, CEF, and $B'A'D'$ intersect in a line. Consider the dihedral angle between these planes which contains our wedge. The dihedral angle between planes BAD and $B'A'D'$ is twice that between planes BAD and CEF. Let the latter angle be α. By taking n congruent wedges with $\alpha=\pi/n$ and apposing face BAD of one wedge to its mirror image $B'A'D'$ in another wedge—in short, by repeated reflections—we

combine the wedges into a torus. Since the octahedra have four edges per vertex and each pair of apposed vertices shares a pair of edges from two octahedra, the torus is uniform with $e=6$. Since each of the $2n$ octahedra shares two faces with other octahedra, the u.i.t. has $12n$ exposed faces.

It remains to determine possible values for n. A simple calculation yields

$$\cos\alpha = (a^2 + 2h^2 - 2c^2)/(a^2 + 2h^2 + 2c^2).$$

Rewrite this as

$$2c^2(1 + \cos\alpha) = (a^2 + 2h^2)(1 - \cos\alpha) \tag{2.1}$$

and recall $0<c<h$ to obtain

$$0 < a^2/h^2 < 2\cos\alpha \csc^2 \tfrac{1}{2}\alpha. \tag{2.2}$$

This implies $\alpha<\frac{1}{2}\pi$. Hence if $\alpha=\pi/n$, $n\geqq 3$. For such an n choose any positive a and h that satisfy (2.2). Then (2.1) determines c. For any $n\geqq 3$ there are two parameters at our disposal. Since their ratio determines the scale or size of our u.i.t. this yields a one parameter family of dissimilar u.i.t. for each $n\geqq 3$.

If one of these u.i.t. has acute angled faces the construction can be modified to permit the insertion of additional faces. First we show that for each $n\geqq 3$ there are u.i.t. with acute-angled faces. Apply the law of cosines to face ACE of our fundamental octahedron to find

$$EA\cdot AC\cos EAC = h^2 - c^2,$$
$$AC\cdot CE\cos ACE = a^2 - 2ch + 2c^2,$$
$$CE\cdot EA\cos CEA = a^2 + 2ch + 2c^2.$$

Since $h>c$, EAC and CEA are acute. ACE will be acute only if $a^2-2ch+2c^2>0$. Use (2.1) to rewrite this as $a^2>2h\cos^2\frac{1}{2}\alpha(c-h\tan^2\frac{1}{2}\alpha)$. Since $h>c$ this will certainly be satisfied if $a^2/h^2>2\cos\alpha$. This lower bound for a^2/h^2 is compatible with the upper bound (2.2) for $\alpha=\pi/n$ for $n\geqq 3$. Hereafter suppose ACE is acute-angled. We now construct an isohedral octahedron different from that previously described. Consider $G(0, 0, z)$, $H(x, 0, 0)$, $I(0, y, 0)$, $J(-x, 0, 0)$, $K(0, -y, 0)$ and $L(0, 0, -z)$ with $xyz\neq 0$. From the set of segments determined by pairs of these points remove GL, HJ, and IK. The remaining segments are the edges of an isohedral octahedron. For suitable x, y, z GHI can be made congruent to any acute angled triangle, in particular to ACE. As before, produce a wedge by reflecting $GHIJKL$ in the plane of JKL. The planes of GHI and their images $G'H'I'$ are parallel. This fact permits the use of this wedge to stretch or spread the u.i.t. previously described.

In the u.i.t. treated heretofore the plane of the common face of any pair of adjacent octahedra will also contain the common face of another different pair of adjacent octahedra. Cut the torus into congruent or enantiomorphic halves by such a plane. If we insert two of our new wedges symmetrically between these halves so that faces GHI are apposed to freshly exposed faces $A'C'E'$ and $G'H'I'$ to ACE we again obtain u.i.t., but with 24 additional faces. By varying

the location of the cut or the number of the new type of wedges symmetrically inserted we can produce u.i.t. with $12n+24k$, $k \geqq 0$, faces congruent or enantiomorphic to ACE. Two u.i.t. with the same n but different k's are dissimilar, so the set of dissimilar u.i.t. with faces congruent to ACE is infinite.

Figure 1 shows a triangular u.i.t. with $n=3$, $k=0$.

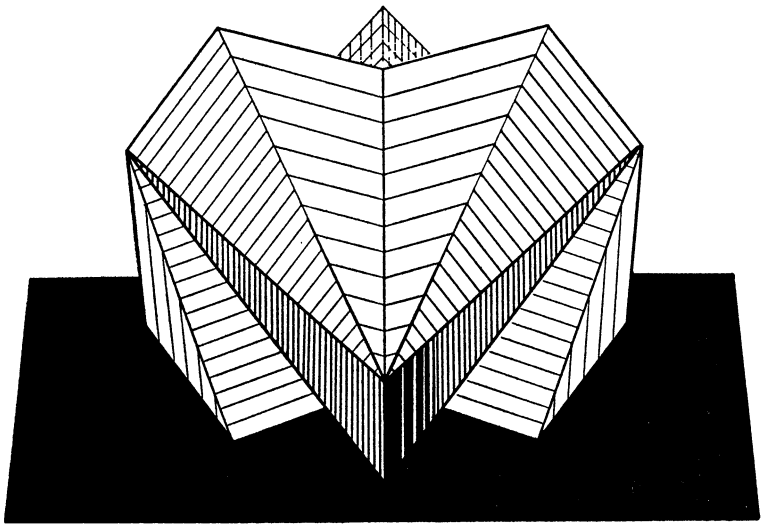


FIG. 1.

3. Quadrilateral tori. We can build $8n$-faced quadrilateral u.i.t. with isohedral hexahedra in a way similar to the first method for constructing triangular u.i.t. with isohedral octahedra. Thus it suffices to construct an isohedral hexahedron, to measure the dihedral angle α between a pair of opposite faces, and to determine the values of n for which we may set $\alpha=\pi/n$.

Consider the points $A(0, 0, h)$, $B(r, 0, c)$, $C(-\frac{1}{2}r, \frac{1}{2}r\sqrt{3}, c)$, $D(-\frac{1}{2}r, -\frac{1}{2}r\sqrt{3}, c)$, $E(r\cos\theta, r\sin\theta, -c)$, $F[r\cos(\theta+\pi/3), r\sin(\theta+\pi/3), -c]$, $G[r\cos(\theta-\pi/3), r\sin(\theta-\pi/3), -c]$, and $H(0, 0, -h)$, where h, r, and $c>0$. It proves helpful to observe that AH is perpendicular to the planes of the congruent equilateral triangles BCD and EFG and passes through their circumcenters. Now we arbitrarily demand that A, B, C, and E be coplanar. This occurs if and only if

$$h = (h - c)[\sin \pi/6 + \sin(\theta + \pi/6)]. \tag{3.1}$$

If (3.1) holds, the following are congruent plane quadrilaterals: $ABCE$, $ACDF$, $ADBG$, $HEFC$, $HFGD$, and $HGEB$. Thus our six points are vertices of an isohedral hexahedron. Let α be the dihedral angle between the planes of the opposite faces $ABCE$ and $HFGD$. By a simple calculation

$$\cos\alpha = [r^2 + 4(h - c)^2 \sin(\theta + \pi/6)]/[r^2 + 4(h - c)^2]. \tag{3.2}$$

Rewrite this as

$$(h - c)^2 = r^2(1 - \cos \alpha)/[\cos \alpha - \sin (\theta + \pi/6)] \tag{3.3}$$

to find $0<\alpha+\theta<\pi/3$ since $(h-c)^2>0$. Since $0<\theta\leqq\pi/3$, if $\alpha=\pi/n$, then $n\geqq 4$. For such an n we may choose any θ in $0<\theta<\pi/3-\pi/n$ and any r. (3.3) determines $h-c$, and (3.1) determines h. Since r determines the size or scale of the hexahedron, and hence of the u.i.t., the arbitrariness of θ yields a one parameter family of dissimilar u.i.t. for each $n\geqq 4$.

We shall show that the polyhedron that remains after excision of the opposite faces $ABDG$ and $CEFH$ of our isohedral hexahedron is rigid if $\theta\neq\pi/3$. This implies that *our quadrilateral u.i.t. are rigid.* Suppose faces $ABCE$ and $ACFD$ are hinged along AC and opened to some dihedral angle ϕ, $0\leqq\phi\leqq\pi$. The perpendiculars from B and D to AC have a common foot, and if d is the distance from B to AC

$$BD^2 = 2d^2(1 - \cos \phi) \equiv P(\phi).$$

Let e and f be the lengths of the perpendiculars from E and F to AC and g the distance between their feet. Then

$$EF^2 = e^2 + f^2 + g^2 - 2ef \cos \phi \equiv Q(\phi).$$

For faces $BEGH$ and $DFGH$ hinged along GH, by symmetry

$$BD^2 = Q(\phi) \quad \text{and} \quad EF^2 = P(\phi)$$

in terms of the dihedral angle ϕ with edge GH. To permit these four faces to be assembled, there must exist ϕ_1 and ϕ_2 such that

$$P(\phi_1) = Q(\phi_2), \qquad P(\phi_2) = Q(\phi_1). \tag{3.4}$$

Suppose $P(\phi)-Q(\phi)\equiv 0$. Then for $\phi=0$, $BD=EF=0$, so $BE=CF=CE$, whence $\theta=\pi/3$. By (3.2) $\alpha=0$, which does not interest us. Hence suppose $P(\phi)-Q(\phi)\not\equiv 0$. Since P, Q, and $P-Q$ are monotone continuous functions $\phi_1=\phi_2$. For by (3.4)

$$P(\phi_1) - Q(\phi_1) = -[P(\phi_2) - Q(\phi_2)]. \tag{3.5}$$

By monotonicity $P(\phi)-Q(\phi)=0$ has at most one solution. If $\phi_1\neq\phi_2$, by (3.5) $P(\phi)-Q(\phi)$ actually changes sign so there is a unique ϕ_0 such that $P(\phi_0)-Q(\phi_0)=0$. If $\phi_1<\phi_0$, (3.5) implies $\phi_0<\phi_2$. By (3.4) and monotonicity of P and Q

$$Q(\phi_2) > Q(\phi_1) = P(\phi_2) > P(\phi_1) = Q(\phi_2),$$

a contradiction. $\phi_1>\phi_0$ yields a similar contradiction. Thus the only solution of (3.4) is $\phi_1=\phi_2=\phi_0$, so the hexahedron minus faces $ABDG$ and $CEFH$ is rigid.

Equations (2.1) or (3.1) and (3.3) will provide dimensions of u.i.t. in case it is desired to make cardboard models. Great care must be exercised in cutting out faces to assure that the model will fit together reasonably well. Though it involves extra labor, it will increase the strength of the model to make it out of complete octahedral or hexahedral units as suggested by the construction.

THREE-LINE LATIN RECTANGLES—II

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1. Introduction. This note gives a new formula for the numbers, ${}_3K_n$, of reduced three-line Latin rectangles of n elements. It may be recalled from a former paper (this MONTHLY, Oct. 1944, pp. 450–452) that a Latin rectangle is an array in which each row is a permutation of n elements and each column has distinct elements, and that a rectangle is reduced when the elements in the first row are in a standard order.

The formula is as follows:

$$ {}_3K_n = \sum_{i=0}^{[\frac{1}{2}n]} \binom{n}{i} D_i D_{n-i} u_{n-2i}, $$

$$ D_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)!, \tag{1} $$

$$ u_n = \sum_{i=0}^{n} (-1)^i \frac{2n}{2n-i} \binom{2n-i}{i} (n-i)!, \qquad u_0 = 1. $$

Here $D_n = \Delta^n 0!$ is sub-factorial n, a "rencontres number," and u_n is a "menages number," defined as in the former paper (except that there $u_0 = 2$). An equally explicit but more complicated formula has recently been given by Dulmage (this MONTHLY, Oct. 1945, p. 458).

For numerical concreteness, a short table of all numbers in the formula follows:

n	0	1	2	3	4	5	6	7	8
D_n	1	0	1	2	9	44	265	1854	14833
u_n	1	−1	0	1	2	13	80	579	4738
${}_3K_n$				2	24	552	21280	1073760	70299264

Since $D_n < n!$, $u_n < n!$ (by definition of the combinatorial problems they solve), an immediate consequence of the formula is

$$ \frac{{}_3K_n - D_n u_n}{n!^2} < \sum_{i=2}^{[\frac{1}{2}n]} \binom{n}{i} i!(n-i)! \frac{(n-2i)!}{n!^2} < \frac{n}{2} \frac{1}{(n-3)^4}, $$

which is sufficient to prove the asymptotic formula

$$ {}_3K_n \sim n!^2 e^{-3} $$

of the former paper.

Further improvements in the inequality above are of course possible but are perhaps of less interest now because of the development of a general asymptotic formula, as yet unpublished, for r-rowed rectangles by Drs. Erdös and Kaplansky, who have kindly sent me a preliminary communication.

2. Proof of Formula (1). By equation (4) of the former paper

$$ {}_3K_n = \sum C_{[a]} u_{[a]}, \tag{2} $$

summation being over all non-unitary partitions of n, each partition denoted by $[a] \equiv [2^{a_2}3^{a_3}, \cdots, n^{a_n}]$, say, with $\sum i a_i = n$, and

$$ C_{[a]} = \frac{n!}{2^{a_2}a_2! \cdots n^{a_n}a_n!}, \tag{3} $$

$$ u_{[a]} = \sum u_{2\pm 2\pm \cdots \pm 2\pm \cdots \pm n\pm \cdots \pm n}. \tag{4} $$

On the right of (4) the subscript contains a_2 2's, a_3 3's, *etc.*, and summation is over the 2^{k-1}, $k = a_2 + a_3 + \cdots + a_n$, possible assignments of $+$ and $-$ signs. Note that $u_{-n} = u_n$. Ex. gr.

$$ \begin{aligned} u_{[532^2]} &= \sum u_{2\pm 2\pm 3\pm 5} \\ &= u_{12} + 2u_8 + u_6 + u_4 + 3u_2. \end{aligned} $$

It is clear that $u_{[a]}$ may be expanded in terms of u_{n-2i}, say

$$ u_{[a]} = \sum_i b_{n,i}^{[a]} u_{n-2i}; \tag{5} $$

and formula (1) will follow if

$$ \sum_{[a]} C_{[a]} b_{n,i}^{[a]} = \binom{n}{i} D_i D_{n-i}, \tag{6} $$

where summation is again over all non-unitary partitions of n (indicated by $[a]$).

For this, the following lemmas are needed.

LEMMA 1: *$b_{n,i}^{[a]}$ is the number of ways of choosing (non-unitary) partitions of i from the parts in $[a]$, repeated parts taken as distinct.*

Proof: The coefficient of u_{n-2i} in the expansion of $u_{[a]}$ is obtained by choosing $-$signs for all parts, or sets of parts, of $[a]$ whose sum is i. Ex. gr. for $i=5$ and $[a]=532^2$, $n=12$, the coefficient of u_2, as given above, is 3, corresponding to parts 5, 32_1, 32_2, writing 2_1 and 2_2 for the repeated parts. These correspond, respectively, to the following choices of sign:

$$ \begin{aligned} &2 + 2 + 3 - 5 \\ &2 - 2 - 3 + 5 \\ &-2 + 2 - 3 + 5 = 2 - 2 + 3 - 5. \end{aligned} $$

LEMMA 2: *The number of ways in which a partition $p_1^{i_1}p_2^{i_2} \cdots$ of i may be chosen from parts $p_1^{j_1}p_2^{j_2} \cdots$ of $[a]$ is*

$$ \binom{j_1}{i_1}\binom{j_2}{i_2} \cdots . $$

Proof: Each repeated part p_k may be chosen in $\binom{j_k}{i_k}$ ways, independently of the others.

Remark: $b_{n,i}^{[a]}$ is the sum of the result of this lemma over all partitions of i contained in $[a]$. *E.g.*, for the example already given,

$$b_{12,4}^{[532^2]} = 1 + \binom{1}{1}\binom{2}{1} = 3.$$

Lemma 3. The contribution of any partition $p_1^{i_1}p_2^{i_2}\cdots$ to the product $b_{m,i}^{[a]}C_{[a]}$ is

$$\binom{n}{i}\frac{i!}{p_1^{i_1}i_1!p_2^{i_2}i_2!\cdots}\,\frac{(n-i)!}{p_1^{j_1-i_1}(j_1-i_1)!p_2^{(j_2-i_2)}(j_2-i_2)!\cdots}.$$

Proof: This follows at once from the identity

$$\binom{j_1}{i_1}\binom{j_2}{i_2}\cdots\frac{p^{i_1}i_1!p^{i_2}i_2!\cdots}{p_1^{j_1}j_1!p_2^{j_2}j_2!\cdots} = \frac{1}{p_1^{j_1-i_1}(j_1-i_1)!p_2^{j_2-i_2}(j_2-i_2)!\cdots}.$$

Equation (6) now follows by summation, noting that

$$\sum_{[a]} C_{[a]} = D_n,$$

which may be regarded as an equation of definition of D_n. Ex. gr. take $i=4$, $n=11$; the partitions of 4 in question are 4 and 2^2, the partitions of 11 may be taken in two classes with coefficients as follows:

Partition:	74	542	4^23	432^2;	72^2	52^3	432^2	32^4
Coefficient:	1	1	2	1	1	3	1	6

Note that 432^2 appears in both classes, since it contains both 4 and 2^2. Then by lemma 3

$$\sum b_{11,4}^{[a]}C_{[a]} = \binom{11}{4}4!\left[\frac{1}{4}+\frac{1}{2^2\cdot 2!}\right]\cdot 7!\left[\frac{1}{7}+\frac{1}{5.2}+\frac{1}{4.3}+\frac{1}{3.2^2.2!}\right]$$
$$= \binom{11}{4}D_4D_7.$$

When $n=2m$, $i=m$, equation (6) fails; the corresponding result is

$$\sum b_{2m,m}^{[a]}C_{[a]} = \binom{n}{m}\frac{1}{2}D_m^2. \tag{7}$$

This exceptional case is avoided in (1) by the choice $u_0=1$.

THE GRAM MATRIX AND HADAMARD THEOREM

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1. Introduction. The recent book of Halmos [2] on vector spaces has as its object the development of the theory by elegant "coordinate-free" methods which lend themselves to generalization to infinite spaces. We offer here in the same manner proofs of some theorems on the Gram matrix. The complex case is considered, the real case may be treated analogously (and more simply).

2. Inner product spaces. Let $V(C)$ be a *vector space* over the complex field C, *i.e.*, a set of vectors $(0, \xi, \eta, \cdots)$ forming an abelian group under addition, and admitting the complex field $C=(0, 1, a, b, \cdots)$ as operator domain. This means that for every ξ, η of V and every a, b of C, we suppose $(\xi+\eta)a=\xi a+\eta a$; $\xi(a+b)=\xi a+\xi b$; $\xi(ab)=(\xi a)b$; $\xi 1=\xi$.

We say $V(C)$ is an *inner product space* in case for every ξ, η of V exists a complex number (ξ, η) called *inner product*, such that $(\xi+\eta, \zeta)=(\xi, \zeta)+(\eta, \zeta)$; $(\xi, \eta a)=(\xi, \eta)a$; $(\xi, \eta)=\overline{(\eta, \xi)}$ (complex conjugate); $(\xi, \xi)\geqq 0$, and $(\xi, \xi)=0$ if and only if $\xi=0$.

In such a space, vectors ξ, η are called *orthogonal* in case $(\xi, \eta)=0$, and a set of vectors $(\delta_1, \cdots, \delta_n)$ are said to be *orthonormal* in case $(\delta_i, \delta_j)=\delta_{ij}$, the "Kronecker delta." An orthonormal set is of course linearly independent in the sense that $\sum\delta_i a_i=0$ implies that all $a_i=0$.

3. Inequalities of Bessel and Schwarz. Suppose $(\delta_1, \cdots, \delta_m)$ is an orthonormal set in $V(C)$, and ξ is arbitrary. Define $x_i=(\delta_i, \xi)$ as the *δ_i-component* of ξ (Fourier coefficient). We recall the following important relation (R) easily verified by direct computation:

$$(R) \qquad 0 \leqq (\xi - \sum \delta_i x_i, \xi - \sum \delta_i x_i) = (\xi, \xi) - \sum \bar{x}_i x_i \leqq (\xi, \xi).$$

From this follows the *Bessel inequality*

$$(B) \qquad \sum |x_i|^2 = \sum \bar{x}_i x_i \leqq (\xi, \xi)$$

where equality holds if and only if $\xi=\sum\delta_i x_i$.

Note also that the latter inequality of (R) becomes equality if and only if all $x_i=(\delta_i, \xi)=0$, that is, if and only if ξ is orthogonal to all δ_i.

It seems not generally recognized that the Schwarz inequality

$$(S) \qquad |(\xi, \eta)|^2 \leqq (\xi, \xi)(\eta, \eta)$$

is a trivial special case of the Bessel inequality. Most recent books establish (S) independently, and with less elegance.

If $\eta=0$, (S) is trivial. If $\eta\neq 0$, define $\delta=\eta/(\eta, \eta)^{1/2}$ which is then an orthonormal set of *one* element. Using (B) we have for $x=(\delta, \xi)$, $|(\delta, \xi)|^2\leqq(\xi, \xi)$. Substituting for δ and multiplying we get (S). Moreover equality holds in (S) exactly when it does correspondingly in (B), that is when $\xi=\delta(\delta, \xi)$ and ξ, η are dependent.

4. The Schmidt Process. In any treatment of inner product spaces, the Schmidt orthonormalization process plays a fundamental rôle. It may be stated in the following form:

If $(\delta_1, \cdots, \delta_m, \xi_{m+1})C$, where the δ_i are orthonormal, is the subspace generated by the independent vectors δ_i, ξ_{m+1}, then $(\delta_1, \cdots, \delta_m, \xi_{m+1})C = (\delta_1, \cdots, \delta_m, \delta_{m+1})C$, where the latter are *all* orthornormal. Explicitly, one defines $\delta^*_{m+1} = \xi_{m+1} - \sum_1^m \delta_i(\delta_i, \xi_{m+1})$ and then normalizes by choosing $\delta_{m+1} = \delta^*_{m+1}/(\delta^*_{m+1}, \delta^*_{m+1})^{1/2}$.

Thus, if $(\xi_1, \cdots, \xi_n)$ is any set of independent vectors, one obtains for the space $W = (\xi_1, \cdots, \xi_n)C$ an orthonormal basis $(\delta_1, \cdots, \delta_n)$ so that $W = (\delta_1, \cdots, \delta_n)C$, by first defining $\delta_1 = \xi_1/(\xi_1, \xi_1)^{1/2}$ and then inductively producing bases $(\delta_1, \xi_2, \cdots, \xi_n)$, $(\delta_1, \delta_2, \xi_3, \cdots, \xi_n)$, $\cdots$, $(\delta_1, \cdots, \delta_n)$ for W by means of the definition $(\delta_1, \cdots, \delta_m, \xi_{m+1}, \xi_{m+2}, \cdots, \xi_n) = (\delta_1, \cdots, \delta_m, \delta_{m+1}, \xi_{m+2}, \cdots, \xi_n)T_{m+1}$, $m = 0, 1, \cdots, n-1$. One verifies readily that for the process indicated,

$$T_{m+1} = \begin{bmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & x_1 & & & \\ & & & \vdots & & & \\ & & & x_m & & & \\ & & & d_{m+1} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \\ 0 & & & & & & \end{bmatrix}$$

where $x_i = (\delta_i, \xi_{m+1})$, and $d_1 = (\xi_1, \xi_1)^{1/2}$, $d_{m+1} = (\delta^*_{m+1}, \delta^*_{m+1})^{1/2} \leqq (\xi_{m+1}, \xi_{m+1})^{1/2}$. The latter inequality is due to (R) of §3, and as indicated there, *is equality if and only if* ξ_{m+1} *is orthogonal to* $\delta_1, \cdots, \delta_m$, *the* $x_i = (\delta_i, \xi_{m+1})$ *being zero in* T_{m+1}.

Successive substitution shows that the final basis $(\delta_1, \cdots, \delta_n)$ is related to the original by $(\xi_1, \cdots, \xi_n) = (\delta_1, \cdots, \delta_n)T$, where $T = T_n T_{n-1} \cdots T_1$. It is then clear* that $|T|^2 = d_1^2 \cdots d_n^2 \leqq (\xi_1, \xi_1) \cdots (\xi_n, \xi_n)$.

We show that the equality holds here if and only if the ξ_i are mutually orthogonal. Indeed, if equality holds, *every* $d_i^2 = (\delta_i^*, \delta_i^*) = (\xi_i, \xi_i)$. But $(\delta_2^*, \delta_2^*) = (\xi_2, \xi_2)$ implies, by the italicized remark, that ξ_2 is orthogonal to $\delta_1 = \xi_1/(\xi_1, \xi_1)^{1/2}$, hence to ξ_1. Moreover, δ_2^* and δ_2 are then defined as scalar multiples of ξ_2. An obvious induction completes the proof in this direction.

Now supposing that all ξ_i are mutually orthogonal, we have ξ_2 orthogonal to $\delta_1 = \xi_1/(\xi_1, \xi_1)^{1/2}$, hence by the italicized remark, $(\delta_2^*, \delta_2^*) = (\xi_2, \xi_2)$. Moreover, δ_2 is then defined as a scalar multiple of ξ_2, and ξ_3 is orthogonal to δ_1 and δ_2. Hence $(\delta_3^*, \delta_3^*) = (\xi_3, \xi_3)$, and so on. Finally $|T|^2 = (\xi_1, \xi_1) \cdots (\xi_n, \xi_n)$.

Thus we have

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* The notation $|T|$ denotes the determinant of the matrix T.

THEOREM 1. *If* $\xi_1, \cdots, \xi_n$ *are independent, then the Schmidt process gives* $(\xi_1, \cdots, \xi_n)C = (\delta_1, \cdots, \delta_n)C$ *where* $(\xi_1, \cdots, \xi_n) = (\delta_1, \cdots, \delta_n)T$, *with* $|T|^2 \leqq (\xi_1, \xi_1) \cdots (\xi_n, \xi_n)$, *equality holding if and only if the* ξ_i *are mutually orthogonal.*

5. The Gram matrix. For vectors $\xi_1, \cdots, \xi_n$ in $V(C)$ we define the Gram matrix $G(\xi) = ((\xi_i, \xi_j))$, and prove

THEOREM 2. *The rank of the Gram matrix* $G(\xi)$ *is equal to the maximum number of linearly independent vectors in the set* $\xi_1, \cdots, \xi_n$. *If the* ξ_i *are independent,* $G(\xi) = \overline{T}^\tau T$, *where* T *is the Schmidt matrix of* §4†. *Thus* $|G(\xi)| = |T|^2 > 0$.

Note first that if ξ_i, η_i, $i = 1, \cdots, n$ are any vectors for which $\xi_i = \sum \eta_j t_{ji}$, we have $(\xi_i, \xi_k) = \sum \bar{t}_{ji}(\eta_j, \eta_l)t_{lk}$ so that for the Gram matrices, $G(\xi) = \overline{T}^\tau G(\eta)T$.

Now the rank of $G(\xi)$ is unchanged under permutations of rows and columns, so that we may suppose the ξ_i numbered so that the first r are independent, while the remaining ξ_i are expressible in terms of them. Applying the Schmidt process to the first r ξ's enables us to write $(\xi_1, \cdots, \xi_r, \xi_{r+1}, \cdots, \xi_n) = (\delta_1, \cdots, \delta_r, \xi_{r+1}, \cdots, \xi_n)T$ where

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & 0 \end{bmatrix}$$

and T_{11} is the Schmidt matrix. Thus $G(\xi) = \overline{T}^\tau G(\delta_1, \cdots, \delta_r, \xi_{r+1}, \cdots, \xi_n)T$ where the latter G has in the upper left corner the rxr identity I_r.

Since the rank of the product of matrices is not greater than that of any factor [3, p. 218], rank $G(\xi) \leqq r$. But the upper left corner of $G(\xi)$ is the nonsingular $\overline{T}_{11}^\tau T_{11}$. Thus rank $G(\xi) \geqq r$. Hence $|G(\xi)| = 0$ unless the ξ are independent. In the latter case, $G(\xi) = \overline{T}^\tau IT = \overline{T}^\tau T$ and $|G(\xi)| = |T|^2 > 0$.

6. The Hadamard theorem. *We now obtain trivially* [1]

THEOREM 3. (*Hadamard-Fischer*) *If* $\xi_1, \cdots, \xi_n$ *are vectors in* $V(C)$ *then* $|G(\xi)| \leqq (\xi_1, \xi_1) \cdots (\xi_n, \xi_n)$, *equality holding if and only if the* ξ *are mutually orthogonal.*

If the ξ are dependent $|G(\xi)| = 0$ and the theorem is trivial. If independent, $G(\xi) = \overline{T}^\tau T$ and the result follows from THEOREM 1.

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† If $T = (t_{ij})$ then T^τ means the matrix (t_{ji}), *i.e.*, the "transposed" matrix with rows and columns interchanged.

DISCUSSIONS AND NOTES

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The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

BINOMIAL IDENTITIES

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The use of probability concepts to establish algebraic identities is not new. The problems of Chapters II and III of Uspensky's *Introduction to Mathematical Probability* furnish several examples of such use. This article will be concerned with certain non-trivial identities for binomial coefficients which were discovered by the use of probability considerations.

The authors were led to this problem when they noted that computed values of Q and R, as defined below, were equal for fixed N, n, and m, where $0 \leqq n \leqq N$, $0 \leqq m \leqq N$. The defining relations for Q and R are

$$Q = \sum_{p} \binom{N-n}{p}\binom{n}{m-p}\binom{N-p}{n}, \tag{1}$$

$$\max(0, m-n) \leqq p \leqq \min(m, N-n),$$

and

$$R = \sum_{q} \binom{n}{q}\binom{N-n}{m-q}\binom{N-q}{N-n}, \tag{2}$$

$$\max(0, m+n-N) \leqq q \leqq \min(n, m).$$

If one actually tabulates the binomial coefficients above, certain similarities in Q and R may be observed. This similarity is brought into focus by making the substitution $q=(m-p)$ in (2). One then obtains

$$R = \sum_{p} \binom{n}{m-p}\binom{N-n}{p}\binom{N-m+p}{N-n}, \tag{3}$$

$$\max(0, m-n) \leqq p \leqq \min(n, N-n),$$

in which the similarity to (1) is more marked. Attempts to find a direct proof of the identity $Q \equiv R$ by manipulations with factorials were unsuccessful. A proof was devised, however, in which a double summation form would reduce to Q when one of the indicated summations had been carried out, and would reduce to R if the other indicated summation had been carried out. The method of this proof was used to establish a more general set of binomial relations.

Let there be three boys, Tom, Dick, and Harry, and let them make independent choices of k, m, and n integers respectively from the set of the first N

positive integers. What is the probability that there is no integer common to the three choices? Or, since the use of probabilities introduces denominator terms, what is the total number of ways in which the three choices can be made so that there is no common integer?

Suppose that Tom's set of k integers and Dick's set of m integers contain p duplications. If Harry's set of n integers does not duplicate any of these p integers, a "favorable" case has been obtained. Hence one way of counting the number of favorable cases is to introduce the parameter p, count the number of favorable cases for each allowable value of p, and perform a summation on these terms. Let Tom select his k integers from the entire collection of N. This can be done in $\binom{N}{k}$ ways. For fixed p, Dick's choice of m integers may be regarded as the sum of two choices: p of Dick's selections are to coincide with selections already made by Tom, and the remaining $(m-p)$ of Dick's selections must be chosen from the $(N-k)$ integers not selected by Tom. Thus Dick has $\binom{k}{p}\binom{N-k}{m-p}$ ways of selecting his m integers so that there are exactly p duplications between his selection and Tom's selection. To insure no integer common to all three selections, Harry need only select his n integers from the $(N-p)$ integers which are not included in both Tom's and Dick's selections. Harry, therefore, has $\binom{N-p}{n}$ ways of making a favorable selection. Altogether there are

$$\binom{N}{k}\binom{k}{p}\binom{N-k}{m-p}\binom{N-p}{n}$$

ways in which Tom and Dick have exactly p duplications none of which is chosen by Harry. All favorable cases are obtained by summing on all allowable values of the parameter p. The following inequalities limit p,

$$\max(0, m+k-N) \leqq p \leqq \min(k, m, N-n).$$

The number of favorable cases, P, is given by

$$P = \sum_p \binom{N}{k}\binom{k}{p}\binom{N-k}{m-p}\binom{N-p}{n} \tag{4}$$

with the range on p given above.

A similar expression for P can be obtained by interchanging Tom's and Dick's roles. One obtains

$$P = \sum_p \binom{N}{m}\binom{m}{p}\binom{N-m}{k-p}\binom{N-p}{n}. \tag{5}$$

Both (4) and (5) reduce to

$$P = \sum_{p} \binom{N-p}{n} \frac{N!}{p!(k-p)!(m-p)!(N-k-m+p)!}. \tag{6}$$

Thus a direct proof of the symmetric nature of P with respect to the variables k and m is evident from (6).

In general, it will not be necessary to specify the range of the summation variables, since the binomial coefficients may be assigned the value zero whenever negative factorials are involved.

There are four other expressions for the number P similar to (4) and (5). These four expressions are obtainable from a count of the number of favorable cases when other groupings of Tom, Dick, and Harry (or other permutations of k, m, and n) are taken. Since P is the same number in all these cases, it follows that the form (5) is symmetric in the three variables k, m, and n. Since a direct proof of the symmetry of form (5) for all three variables does not follow from simple algebraic manipulations of the factorials, form (5) may be termed a non-trivial symmetric form.

To exhibit the full symmetry of form (5) in the three variables k, m, and n, another method of calculating the number P is preferable. Let Tom and Dick choose p integers in common, Dick and Harry choose q integers in common, and Harry and Tom choose r integers in common. Then Tom has $(k-p-r)$ choices left, Dick has $(m-p-q)$, and Harry has $(n-q-r)$. There are $(N-k-m-n+p+q+r)$ integers not chosen at all. The number of ways of choosing these seven categories can be summed over all allowable values for p, q, and r, and one obtains

$$P = \sum_{pqr} \frac{N!}{p!q!r!(k-p-r)!(m-p-q)!(n-q-r)!(N-k-m-n+p+q+r)!} \tag{7}$$

for the number of favorable cases. Bearing in mind that p, q, and r are dummy summation parameters, the symmetry in k, m, and n is evident in the above formulation.

Form (7) can be summed on one of the summation parameters to give other expressions for P. Thus

$$\begin{aligned} &\sum_{r} \frac{1}{r!(k-p-r)!(n-q-r)!(N-k-m-n+p+q+r)!} \\ &= \sum_{r} \binom{k-p}{r} \binom{N-m-k+p}{n-q-r} \frac{1}{(k-p)!(N-m-k+p)!} \\ &= \binom{N-m}{n-q} \frac{1}{(k-p)!(N-m-k+p)}. \end{aligned} \tag{8}$$

The substitution of (8) in (7) yields

$$P = \sum_{pq} \frac{N!(N-m)!}{p!q!(m-p-q)!(n-q)!(k-p)!(N-m-n+q)!(N-m-k+p)!}. \tag{9}$$

Form (9) (or the two forms symmetric thereto) can also be obtained by counting up the number of favorable cases in another way. For example, Tom's set of k integers and Dick's set of m integers may have p integers in common while Dick and Harry may have chosen q in common which do not duplicate any of the set of p integers. These choices may be counted up as before, and the resulting expression for P will reduce to form (9) when factorials are used.

If form (7) be summed out on two of the parameters, using identities similar to (8), a form similar to (4), (5) or (6) is obtained. Thus a direct proof of the symmetrical nature of form (5) can be obtained independently of probability considerations by showing that the various forms may all be derived from (7) by algebraic operations.

To return to the first non-trivial identity, $Q \equiv R$, note that if $k = (N-n)$, and if $\sum_r \sum_q$ be performed on (7), $\binom{N}{n} Q$ is obtained, and that if $\sum_r \sum_p$ be performed on (7), $\binom{N}{n} R$ is obtained. Thus a direct proof of the identity follows by performing two sets of transforming operations on (7).

A THEOREM CONCERNING CIRCLES*

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The note published by W. M. Stewart under the above title (this MONTHLY, 1937, p. 165) is editorially accompanied by the following theorem of F. E. Wood:

If three circles are drawn having as centers the vertices of a triangle and radii ρk, ρl, ρm, respectively, where k, l, m are fixed, then as ρ varies, the locus of the radical center of the three circles is a straight line (or part of a line if only real circles are used).

This proposition is implicitly contained in a study by J. Neuberg (*Mathesis*, 1923, p. 298) in connection with a problem concerning some circles proposed by E. Lemoine (*Assoc. Franc. pour Avanc. des Sciences*, 1888, p. 170). It was also pointed out by A. Dechilly (*Journal de Vuibert*, t. 36, 1912, p. 91) with an application to the case where k, l, m are equal to the opposite sides of the triangle.

On our part, we have extended the problem to the tetrahedron and have obtained the following theorem which led us, in particular, to the determination of a *sphere of Longchamps* for the tetrahedron analogous to the circle of the same name for a triangle (*Mathesis*, 1932, pp. 223–229; *L'Enseignement Mathématique*, Genève, 1937, pp. 81–99).

THEOREM. *In a tetrahedron $T \equiv ABCD$, the locus of the center of the sphere (ω), for which the powers of the vertices A, B, C, D have the forms kl^2, km^2, kn^2, kp^2, (l, m, n, p being given values and k being an arbitrary coefficient), as k varies, is a straight line drawn perpendicularly through the circumcenter of T to the plane having barycentric coordinates l^2, m^2, n^2, p^2.*

* Translated from the French by Howard Eves.

This theorem has a number of particular cases of which some have given rise to interesting remarks contained in our aforementioned articles. (*Cf.* also V. Thébault, *Gazeta Matematica*, Bucarest, t. 41, pp. 172–175.)

Recently we have considered four spheres (ω), (ω'), (ω''), (ω''') such that the powers of the vertices A, B, C, D of T, with respect to each of them, are (kl^2, km^2, kn^2, kp^2), (km^2, kn^2, kp^2, kl^2), (kn^2, kp^2, kl^2, km^2), (kp^2, kl^2, km^2, kn^2). Concerning these spheres we have the following

THEOREM. *The sphere orthogonal to the spheres* (ω), (ω'), (ω''), (ω''') *has for center the centroid* G *of* T *and for the square of its radius*

$$\sigma^2 = [4k(l^2 + m^2 + n^2 + p^2) - S]/16,$$

S being the sum of the squares of the edges of T.

If one interchanges the same two powers of the set kl^2, km^2, kn^2, kp^2 in each of the sequences of powers for (ω), (ω'), (ω''), (ω''') one obtains four new spheres (ω_1), (ω_2), (ω_3), (ω_4), which we call spheres *semi-associated* with (ω). Concerning these spheres we have the

THEOREM. *Four spheres semi-associated with* (ω) *have for radical center the centroid* G *of* T. *The radical planes of* (ω) *with respect to its semi-associated spheres pass through the medians* AG, BG, CG, DG *of* T. *A sphere* (ω) *and all its semi-associated spheres are orthogonal to the same sphere* (G, σ).

Finally, the values $(-kl^2, -km^2, -kn^2, -kp^2)$ are the powers of the vertices A, B, C, D with respect to a sphere (ω', ρ') whose center ω' is the symmetric with respect to the circumcenter O of T of the center ω of the sphere (ω, ρ) corresponding to the powers (kl^2, km^2, kn^2, kp^2), and, if R is the circumradius of T, we have the relation

$$\rho'^2 + \rho^2 = 2(R^2 + d^2),$$

which becomes $\rho'^2 = 2(R^2 + d^2)$ when the sphere (ω, ρ) reduces to its center ω, d being the distance $O\omega$.

If one changes the sign of one of the powers of A, B, C, D to obtain successively $(-kl^2, km^2, kn^2, kp^2)$, $(kl^2, -km^2, kn^2, kp^2)$, $(kl^2, km^2, -kn^2, kp^2)$, $(kl^2, km^2, kn^2, -kp^2)$, these new powers correspond to some spheres (Ω_1), (Ω_2), (Ω_3), (Ω_4). We may state the following theorem concerning these spheres.

THEOREM. *The radical planes of the sphere* (ω, ρ) *with respect to each of the spheres* (Ω_1), (Ω_2), (Ω_3), (Ω_4) *are the planes of the faces of* T, *and the radical centers of quadruples of spheres obtained by taking the sphere* (ω, ρ) *with triples of the spheres* (Ω_1), (Ω_2), (Ω_3), (Ω_4) *are the vertices of* T. *The radical center of the spheres* (Ω_1), (Ω_2), (Ω_3), (Ω_4) *has barycentric coordinates* $1/l^2$, $1/m^2$, $1/n^2$, $1/p^2$, *and coincides with the pole with respect to* T *of the radical plane of the spheres* (O, R) *and* (ω, ρ).

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1944–45

Pi Mu Epsilon, Northwestern University

Six regular meetings of the Illinois Beta Chapter were held throughout the year. The topics presented were as follows:

The dynamics of the solar system, by Dr. Oliver Lee

Pi by chance, by Dr. Elliot Buell

Alexandria, the home of mathematics, by Dr. H. T. Davis

Mathematics and electrical engineering courses, by Dr. Robert Beam

Soap films and the problem of Plateau, by Dr. W. T. Reid

Some applications of elementary calculus to economic theory, by Dr. William Jaffe

The geometry of numbers, by Dr. A. T. Lonseth.

A party was held in the fall at the home of the social chairman. The annual initiation banquet was held on May 19, with Dr. Richard Wolfe as the toastmaster. The banquet address was

Mathematical research during the war, by Dr. E. J. Moulton.

Officers for the year were: President, Isabel Barrett; Vice-President, John Pederson; Secretary, Betty McDonald; Treasurer, Paul Axt; Social Chairman, Priscilla Williams; Scholarship Chairman, Julia Bissell; Faculty Adviser, Dr. Elliot Buell; Permanent Secretary, Dr. E. H. C. Hildebrandt.

Mathematics—Physics Club, College of Saint Teresa

The club met monthly and devoted its meetings to the history of mathematics and physics. There were reports on

Mathematicians and physicists: Galileo, Johannson, Richards

Women in science: Herschel, Germain, Agnesi, Noether

Catholic scientists: Galvani, Volta, Ampère, Roentgen

American mathematicians: D. E. Smith, Cajori, Karpinski.

Two plays were given, entitled

Archimedes, and

The cubic equation.

The members were addressed by a guest speaker on

Science and industry.

At the close of the year a picnic was given in the nature of a Mathematical Convention of Historical Characters.

Officers were: President, Josephine Schmelzle; Vice-President, Kathleen Raleigh; Secretary, Sara Horn; Treasurer, Joyce Hassett.

Mathematical Society, University of Adelaide, South Australia

On June 22, 1944, a Mathematical Society was founded with the object of promoting interest in mathematics among undergraduates, graduates, and teachers of mathematics at schools in the metropolitan area. Meetings are held once a fortnight and papers are read dealing either with mathematical topics that have been discussed or mentioned in advanced courses given during the year at the University, or with original mathematical work actually done by the speaker.

Membership is informal and open to all persons who have some fundamental knowledge of mathematics in general and an interest in the development of some branch of Pure or Applied Mathematics. One object of the Society is to enable those who are interested in mathematics to keep in touch with modern developments in the subject, and it is hoped that teachers in secondary schools and students taking post-graduate courses will for that reason become active members.

Following the preliminary meeting on June 22, the following topics were presented during the winter term from June to October 1944.

The theory of cyclic groups, by G. E. Wall

Vector algebra and matrix calculus, by A. T. James

Some fundamental notions of mathematical statistics, by B. Worthley

On approximative methods for the solution of analytic equations, by D. R. Blaskett

The isoperimetric problem—an application of Fourier series, by H. Schwerdtfeger

Steffensen's formula on divided differences, by G. J. Aitchison

Arithmetical functions, by Mrs. M. Sved

Derivation of the equation of the vibrating string from physical principles, by W. O. Gibberd.

Mathematics Club, University of Dayton

Biweekly meetings were held during the year at which members of the club presented the following papers:

Problems in logic, by Frank Levin

Topology, by Theodore Brown

Probability, by Catherine Ens

Mathematics and the ancients, by Kenneth Trimbach

The hatchet planimeter, by George Igel

A mathematical interpretation of a passage from the fourth Georgic of Vergil, by Allan Braun

Transfinite numbers, by Frank Levin

Roman methods of calculating, by Alice Blaeser

Diophantine analysis, by John Thalheimer

Application of graphs to the solution of rate and mixture problems, by Margaret Magin

The Doolittle methods of solving linear equations, by Helen Lechner.

Faculty talks presented in the first and second semesters were:

The summability of a divergent series, by Dr. K. C. Schraut

Strategy in mathematical thinking, by Professor Henry Blumberg of Ohio State University.

During the first semester a new form of social entertainment was introduced into the Club, namely a Monte Casino night. The purpose of this form of entertainment was to illustrate the laws of probability. Toward the close of the second semester a banquet was held at Wishing Well Inn.

At the formal initiation ceremony eighteen new members were issued certificates of membership into the Club. The Vice-President's charge to the new members was entitled

The Mathematics Club—1944, by Allan Braun.

The Dean of Science Award was conferred upon Theodore Brown for having delivered before the Club the most interesting paper of the first semester and upon Allan Braun for the second semester. The book chosen for the Awards was *Isaac Newton* by L. T. More. The Mathematics Club Alumni Award for Excellence in Advanced Mathematics was conferred at the June Commencement upon George Igel of the senior class and upon Alice Blaeser of the junior class.

During the year, the Mathematics Club was encouraged to sponsor mathematics clubs in several high schools in the vicinity of Dayton. The project proved to be so popular that an organizing committee composed of representatives from these high schools and the University of Dayton decided to establish an organization which has come to be known as The National Mathematics Honor Society of Secondary Schools. The Mathematics Club of Purcell High School in Cincinnati and Chaminade High School in Dayton were admitted as the first Chapters of the Society. In May the University of Dayton Mathematics Club sponsored a colloquium at which representatives of the High School Chapters presented papers. Awards for the best papers were presented at a banquet which followed.

The officers of the Club during the year were: President, George Igel; Vice-President and Secretary, Allan Braun; Treasurer, Kenneth Trimbach; Faculty Adviser, Professor K. C. Schraut.

Pi Mu Epsilon, University of Nebraska

This year six meetings were held including business meetings and general meetings. The following talks were presented:

Trisecting the angle, by Miss Mary Kathryn Cooper and Mr. Oscar Ellis

Interpolation, by Mr. Kotaro Murai, Mr. Anton Kashas, and Miss Natalie Burn.

The officers for 1945–46 are: Director, Margaret Stewart; Secretary, Mary Lou Weaver; Treasurer, Mary Ann Mattoon. The Faculty Adviser is Professor F. S. Harper.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Spherical Trigonometry after the Cesàro Method. by J. D. H. Donnay. New York, Interscience Publishers, Inc., 1945. 11+83 pages. $1.75.

Among the sundry ills of the teaching of elementary mathematics, there are two which in the reviewer's opinion are serious, widespread, and chronic. One is the occurrence of rote methods including, of course, emphasis on the mere acquisition of manipulative techniques. The other is essentially a frame of mind—a rigid and reactionary orthodoxy that insists on the strict segregation of mathematical concepts into compartments in accordance with well established custom. The first of these need not concern us here, since it is not prevalent at the college level. The second however operates to make freshman courses less *interesting*, *comprehensive*, and *fundamental* than the subject matter warrants. Perhaps there is a tendency to assume of our first year students too much computational persistence and not enough conceptual maturity. Certainly, it is not true that all of the concepts, processes, and methods of higher mathematics are beyond the understanding of lower division students. Various topics from vector analysis, the ausdehnungslehre, and the domain of invariance are admirably suited to the exposition of the theory of plane and spherical triangles and two and three dimensional geometry in general. The advantages are simplicity, conciseness, the introduction of the concepts themselves with the resulting expansion of the student's horizon, the development of mathematical maturity, and finally the enthusiasm engendered by the use of powerful methods. A fresh instance of the elementary application of advanced ideas (here the idea is conformal mapping by stereographic projection) is furnished by the Cesáro method for spherical trigonometry—the subject matter of the book under review.

This work is dedicated to the memory of Giuseppe Cesàro (1849–1939), crystallographer and mineralogist, professor at the University of Liége. It was originally planned as a joint writing of the author and Cesàro. We learn from the preface that the material has been used for some time in Belgium and recently at the Johns Hopkins University and Laval University. It seems to the reviewer that the method presented is far superior to the usual procedure.

The general plan is to reduce spherical trigonometry to its plane counterpart by projecting the given triangle onto a plane. Specifically, one of the three vertices A is singled out, call it the north pole, and the triangle is projected onto the equatorial plane from the south pole as center of projection. The two sides which join at the north pole go into straight line segments and the third side goes into the arc of a circle. Angles are preserved. The chord line is drawn and the resulting plane triangle is taken as the mate of the original. The angles

of the plane triangle are then A, $B-E$, $C-E$—if E denotes one-half of the spherical excess. Surely, this is the simplest way to introduce the excess. The sides of the plane triangle are $\sin a/2$, $\sin b/2 \cos c/2$, $\sin c/2 \cos b/2$. By completing the lune determined by sides a and c of the given triangle a second plane triangle, called the derived triangle, is formed. Its angles and sides are $180-A$, E, $A-E$, $\cos a/2$, $\sin b/2 \sin c/2$, $\cos b/2 \cos c/2$. The same procedure may be applied to the polar triangles if desired.

Turning to the details, the following propositions are proved: Circles are projected as circles or straight lines; The angle between two circles is projected in true magnitude. The methods are synthetic and the tools are well known theorems of plane geometry. A knowledge of solid geometry is not required. Some of the demonstrations are believed to be original. All are in elegant form.

After the various parts of the plane triangles have been evaluated in terms of the elements of the original spherical triangle, one could proceed to solve numerical problems in spherical trigonometry by a direct treatment of the corresponding plane triangles. The author prefers to develop the standard formulas of spherical trigonometry and does this in chapters III and IV by applying ordinary trigonometry to the plane triangles and simplifying the results. Thus he first obtains Napier's proportions by applying the law of tangents to the various triangles. From the law of sines he gets Delambre's formulas. Euler's formula for $\cos E$ is obtained by applying the law of cosines to the derived triangle. The derived triangle yields also Lhuilier's and Cagnoli's formulas and a relationship expressing $\cot E$ in terms of two sides and the included angle.

Chapter IV presents the following material: (1) the equation, $\cos a = \cos b \cos c + \sin b \sin c \cos A$; (2) expressions for the half angles in terms of the sides; (3) the well known relationship between two sides and the angles opposite; (4) the formula $\cos A = -\cos B \cos C + \sin B \sin C \cos a$; (5) the expressions of the half-sides in terms of the angles; and (6) some relations connecting four consecutive parts.

Chapter V is devoted to right triangles. Chapter VI is taken up with numerical computations. The final chapter presents a miscellany of theorems which may be established by means of the formulas of spherical trigonometry. The book contains an adequate index, two appendices (spherical areas and formulae of plane trigonometry) and fifty-four problems most of which have answers.

In conclusion, the reviewer feels that the author has made a substantial contribution in presenting the Cesàro method. The general idea of solving a difficult problem by setting up a correspondence with a simpler problem is exciting in itself, and certainly such notions as conformal mapping, stereographic projection, projection and section are well worth acquiring. Thus if the Cesàro route is selected instead of the usual procedure, the student will not only have a more interesting trip through the subject but he will gain more in mathematical maturity—and mathematical maturity makes up perhaps a major portion of the profit derived from studying mathematics.

H. V. CRAIG

A Course of Mathematical Analysis for Higher Technical Institutes. Second Edition. By A. F. Bermant. Moscow and Leningrad, OGIZ, GOSTEKHIZDAT, 1944. Vol. I, 320 pages+120 Figs. $2.00. Vol. II, 336 pages+90 Figs. $3.60.

During the past few years, higher mathematics has been employed to an unprecedented extent in the course of technical investigations, especially in such domains as aerodynamics, elasticity, plasticity, electronics, electrical network analysis, *etc.*; with the further development of research in these fields, it is not unreasonable to expect a parallel increase in the role which mathematics will play in these investigations. Moreover, the recent development of automatic calculating devices, such as punched-card machines, improved differential analyzers and electrical network computers, offers new and unanticipated possibilities for mathematics to supplement experiment in the construction and elaboration of physical theories. These developments in engineering methods demand corresponding modifications in the curriculum and point of view of the engineering school.

Two stringent requirements must be satisfied by a modern mathematical text-book designed for use in a technical school. On the one hand, the book must make a strong appeal to the student's intuition; the student must always be made aware of the relation between abstract mathematical theory and the geometrical, mechanical and physical problems in which it can be applied; on the other, the student must be equipped to study the more advanced parts of higher mathematics, so that he may be able to acquire the knowledge and techniques he may need in his work, and that he may come to appreciate and understand the abstract approach characteristic of mathematical reasoning today. Consequently, the course must help in developing a mental flexibility and resourcefulness which will allow these engineers who devote themselves to theoretical research to formulate practical problems in abstract mathematical terms and at the end to translate the results of analysis into the language of the practicing engineer.

These problems have been brilliantly solved by A. F. Bermant in his "Course of Mathematical Analysis." Professor Bermant is a well known Russian mathematician, managing editor of the *Matematicheski Sbornik* (*Recueil mathematique*), with many years' experience teaching in technical schools. It is quite evident that this experience has strongly influenced the choice of materials, while the intuitive presentation, expository skill and lucidity of style combine to make this book the basis for an excellent course in mathematics for engineering students. Because the technical student regards mathematics primarily as a tool, and not an end in itself, the author avoids formalism whenever he can, appealing constantly to the student's common sense. No opportunity to give a geometrical or mechanical interpretation of mathematical concepts is overlooked, and an illustrative diagram is usually added. Considerable effort is made to present analysis as coherent and rational. For example, each chapter has a brief introduction which indicates what topics are to be considered, where they fit in the framework

of analysis, how they are related to the student's previous knowledge, and why they are important to the engineer. It was obviously impossible to give rigorous proofs of all assertions in a book of this kind, but all statements and definitions satisfy modern standards of accuracy, and what cannot be proved is at least made intuitively clear, the student being referred to more elaborate treatises for the proofs. It is regrettable that the barrier of language renders this book unavailable for use by American students; it would be desirable if it were to be translated. Some idea of the subjects treated may be obtained from the following abbreviated Table of Contents.

VOLUME I: INTRODUCTION. Chap. I. THE FUNCTION CONCEPT; Chap. II. LIMITS; Chap. III. DERIVATIVES AND DIFFERENTIALS. DIFFERENTIAL CALCULUS; Chap. IV. THE BEHAVIOR OF FUNCTIONS. CURVE TRACING; Chap. V. DEFINITE INTEGRAL; Chap. VI. INDEFINITE INTEGRAL. INTEGRAL CALCULUS; Chap. VII. APPLICATIONS OF THE INTEGRAL; VOLUME II: FUNCTIONS OF SEVERAL VARIABLES: Chap. VIII. DERIVATIVES AND DIFFERENTIALS; Chap. IX. DIFFERENTIAL CALCULUS AND SIMPLE APPLICATIONS; Chap. X. MULTIPLE INTEGRALS AND ITERATED INTEGRALS; Chap. XI. LINE INTEGRALS AND SURFACE INTEGRALS; Chap. XII. DIFFERENTIAL EQUATIONS; Chap XIII. SERIES.

An indication of the author's approach may be obtained from the following detailed contents of two representative chapters, those on Differential Equations and Infinite Series. Chap. XII. DIFFERENTIAL EQUATIONS 1. Equations of the first order (Generalities; Equations whose variables can be separated; Total differential equations; Integrating factor.) 2. Equations of the first order (continuation) (Direction fields; Approximate solutions of first order equations; Singular solutions; Clairaut's equation; Orthogonal and isogonal trajectories). 3. Equations of second and higher order (Generalities; Equations of special types; Linear homogeneous equations; Linear non-homogeneous equations). 4. Linear equations with constant coefficients (Homogeneous equations; Non-homogeneous equations). 5. Vibrations (Mechanical vibrations; Free and forced vibrations) Resonance; Oscillations in electric circuits. 6. Concluding remarks. Linear equations which can be reduced to equations with constant coefficient—Systems of equations. Chap. XIII. SERIES 1. Numerical series (The nature of series; Convergence; Series with positive terms; Sufficient conditions for convergence; Leibniz's criterion; Absolute convergence). 2. Series of functions (Definitions. Uniform convergence; Differentiation and integration of series of functions). 3. Power series (Interval of convergence; Abel's theorem, General properties; Taylor series (Application to the solution of differential equations) Definition of the elementary functions; Euler's formulas). 4. Trigonometric series (Preliminary remarks; Fourier series; Expansion of periodic functions; Points of discontinuity of the first kind; Dirichlet's theorem; Remarks on the expansion of functions in Fourier series; Examples; Practical harmonic analysis; Equation of the vibrating string).

STEFAN BERGMAN

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 701. *Proposed by N. S. Mendelsohn, Queen's University*

Suppose the distance between the front and rear wheels of a car is b, and the distance between wheels from side to side is a. The right front wheel is turned through a counterclockwise angle θ from its neutral position. Through what angle must the left front wheel be turned so that in the car's subsequent motion all four wheels will roll without slipping? Describe the motion.

E 702. *Proposed by J. Brenner, Palmer Laboratory, Princeton, N. J.*

The smallest consecutive cubes whose difference is a square are 8^3 and 7^3. What is the next pair?

E 703. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In which scales of notation can a four-digit number $abbc$ be the square of a two digit number mn, if $c=a+1$ and $n=m+1$? (*Cf.* E 652 [1945, 42])

E 704. *Proposed by D. H. B owne, Buffalo, N. Y.*

Sum the series $1+1/3-1/5-1/7+1/9+\cdots$.

E 705. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

A point in the interior of a simple closed curve, which is nowhere concave, will be called a *power point* if the product of the segments of a variable chord through the point is constant. The existence of how many distinct power points insures that the curve is a circle?

SOLUTIONS

The Generalized Water-fetching Puzzle

E 451 [1941, 65]. *Proposed by W. E. Buker, Pittsburgh Public Schools*

There are three containers, having capacities of a, b, c quarts, where $a>b>c$ (positive integers). With the largest container full and the others empty it is desired to divide the liquid into two equal portions, using these containers and no others. For what values of a, b, c is a solution possible?

Solution by J. C. Currie, Northeast Junior College of L. S. U., Monroe, Louisi-

ana. A set of three integers for which the prescribed division is possible will be referred to as *a solution.*

This discussion will be restricted by the following conditions:

(1) $a<b+c-2$. For discussion of solutions not thus restricted see D. H. Browne's partial solution [1942, 125–27].

(2) b and c have no common factor.

(3) a is not a multiple of c (to eliminate a trivial case).

The discussion is based on the conclusions of the editorial note to Browne's solution; the object is to seek solutions not given by the ENSC (Editorial Note Sufficient Condition). We define

$$\sigma_0 = 0, \qquad \sigma_1 = c,$$

$$\sigma_{i+1} = \sigma_i + c \text{ or } \sigma_i - b \text{ according as } \sigma_i < b \text{ or } \sigma_i \geqq b.$$

The σ's form a cycle of $b+c$ distinct integers, a permutation of $0, 1, \cdots, b+c-1$. A set of integers (a, b, c) is a solution if and only if it is possible to pass from either end of the cycle to $\frac{1}{2}a$ without passing, en route, any number greater than a.

If there exists an integer j such that $\sigma_i \leqq 2\sigma_j < b+c-2$ either for every $i<j$ or for every $i>j$ then $(2\sigma_j, b, c)$ is a solution. Of course we reject the j's for which $2\sigma_j$ does not meet the requirement $a>b$, and condition (3).

We shall refer to this as the σ-test.

Let $b=mc+r$, $0<r<c$. Let $c=qr+t_1$, $0<t_1<r$, where condition (2) implies that r and t_1 have no common factor. Let $t_0=0$, $t_{i+1}=t_i+t_1$ or t_i+t_1-r according as $t_i+t_1<r$ or $t_i+t_1 \geqq r$. The t's thus form a cycle of r distinct integers, a permutation of $0, 1, \cdots, r-1$. The σ-test will be replaced by an equivalent test on the t-cycle, by means of which we may construct all solutions for given r and t_1.

The structure of the σ-cycle is revealed to best advantage in a row and column array; for example, for $b=41$, $c=12$ (with $m=3$, $r=5$, since $41=3\cdot 12+5$):

0	12	24	36	48
	7	19	31	43
2	14	26	38	50
	9	21	33	45
4	16	28	40	52
	11	23	35	47
	6	18	30	42
1	13	25	37	49
	8	20	32	44
3	15	27	39	51
	10	22	34	46
	5	17	29	41

Note that the array has c rows, $m+2$ columns; the horizontal lines divide the array into r sections such that within each section successive rows differ by r. The first row contains only multiples of c; r is the last (and least) number in the second column and b is the last (and least) number in the last column. Finally, the rt's are in the first column, the c next larger numbers are in the second column, *etc.*, the c the largest numbers being in the last column.

THEOREM 1. *If* (a, b, c) *is a solution, with* $b>3c$, *then* $(a-2c, b-2c, c)$ *also is a solution.*

The proof depends on the fact that the σ-array for $b-2c$ and c may be obtained from the σ-array for b and c simply by deleting the last two columns. Since there is a difference of c between elements of successive columns of the array, if the greatest number passed on the way to σ_j in the original array does not exceed $2\sigma_j$, then the greatest number passed on the way to σ_j-c in the reduced array does not exceed $2\sigma_j-2c$ (σ_j in the first row of multiples of c having been excluded). The fact that σ_j-c is actually in the reduced array follows from the fact that $\sigma_j=\frac{1}{2}a$ cannot be in the last column of the original array (because of condition (1)) so that σ_j-c cannot be in either of the two deleted columns.

By induction we obtain

THEOREM 2. *If* (a, b, c) *is a solution, then* $(a-2kc, b-2kc, c)$ *also is a solution for every positive integer* k *for which* $b-2kc>c$.

The largest k for which this is possible gives what we may call a "primitive" solution (a, b, c), in which $b=c+r$ or $b=2c+r$, and from which non-primitive solutions may be constructed by adding $2kc$ to a and to b. Note that no non-primitive solutions satisfy the ENSC, except the trivial cases in which a is a multiple of c. For example, the solution (18, 17, 4) given by William Scott [1944, 592] is non-primitive, its primitive being (10, 9, 4).

Case A. Here $b=c+r$, in which case the array has three columns. Inspection of the array for this case reveals that $\frac{1}{2}a$ can only be one of the elements $c+t_j$ (for each j, the first number in the second column of the $(j+1)$th section), and that the elements $2c+t_i$ (for each i, the first number in the third column of the $(i+1)$th section) are the crucial blocking elements; *i.e.*, passage by these implies passage by other elements. Thus the inequalities of the σ-test may be written in the sharper form $2c+t_i\leqq 2c+2t_j<b+c-2$, implying the following test for case A:

Given r and t_1, if there exists an integer $j<r$ such that $t_i\leqq 2t_j<r-2$, either for every $i<j$ or for every $i\geqq j$, then $(2c+2t_j, c+r, c)=(2qr+2t_j+2t_1, (q+1)r+t_1, qr+t_1)$ is a solution for $q=1, 2, \cdots$.

Most of the resulting solutions are of the ENSC type, but for $r\leqq 20$ there are 14 j's which give solutions not of this type. The first one is $r=13$, $t_1=4$, $j=11$, giving solutions $(26q+18, 13q+17, 13q+4)$ for $q=1, 2, \cdots$.

Case B. Here $b=2c+r$, in which case the array has four columns. Inspection of the array for this case reveals that $\sigma_j=\frac{1}{2}a$ can only be in the third column of the array since elements in first and second columns cannot satisfy the left

inequality of the σ-test, and elements of the fourth column cannot satisfy the right inequality. Of the third column elements, those in the rth or last section, $c+nr$, give solutions $(2c+2nr,\ 2c+r,\ c)$ for every positive integer $n \leqq \frac{1}{2}(q+1)$. All these solutions satisfy the ENSC and, with the ones mentioned under case A, exhaust the solutions of the ENSC type.

Since, as stated above, $\frac{1}{2}a$ must be in the third column of the array, for sections other than the rth, $\frac{1}{2}a$ must be of the form $c+nr+t_j$, $1 \leqq n \leqq q+1$. Then $a=2c+2nr+2t_j$. We note that the r largest numbers in the last column, $3c+t_i$, are crucial blocking elements, so that the inequalities of the σ-test may be recast: $3c+t_i \leqq 2c+2nr+2t_j < 3c+r-2$. Recalling that $c=qr+t_1$, this implies $t_i \leqq (2n-q)r+2t_j-t_1 < r-2$, which permits only $n=\frac{1}{2}q$ if q is even and $n=\frac{1}{2}(q+1)$ if q is odd. This leads to two tests for case B.

I. When q is even.

Given r and t_1, if there exists an integer $j<r$ such that $t_i \leqq 2t_j-t_1 < r-2$ either for every $i<j$ or for every $i \geqq j$, then $(3qr+2t_j+2t_1,\ (2q+1)r+2t_1,\ qr+t_1)$ is a solution for $q=2, 4, 6, \cdots$.

II. When q is odd.

Given r and t_1, if there exists an integer $j<r$ such that $t_i \leqq r+2t_j-t_1 < r-2$ either for every $i<j$ or for every $i \geqq j$, then $((3q+1)r+2t_j+2t_1,\ (2q+1)r+2t_1,\ qr+t_1)$ is a solution for $q=1, 3, 5, \cdots$.

There are, for $r \leqq 20$, 164 examples under *B-I* and 110 under *B-II*. The three tests, under *A*, *B-I*, *B-II*, may be conveniently applied simultaneously, as exhibited below for $r=16$, $t_1=5$. (Solutions of the ENSC type are ignored. Obviously unnecessary entries are omitted from the tabulation.)

i	t_i	*A* $2t_j$	*B-I* $2t_j-t_1$	*B-II* $r+2t_j-t_1$
1	5	10	5*	
2	10			
3	15			
4	4	8	3	
5	9	18	13	
6	14			
7	3	6	1	
8	8	16	11	
9	13			
10	2	4	−1	
11	7	14	9	
12	12			
13	1	2	−3	
14	6	12*	7	
15	11			

The asterisks indicate the numbers which pass the indicated test as the middle term of the inequality. Thus, for case *A*, $j=14$, giving solutions

$(32q+22, 16q+21, 16q+5)$ for $q=1, 2, \cdots$. For case B-I, $j=1$, giving solutions $(48q+20, 32q+26, 16q+5)$ for $q=2, 4, 6, \cdots$. For case B-II, $j=13$, giving solutions $(48q+28, 32q+26, 16q+5)$ for $q=1, 3, 5, \cdots$.

The three tests together are, from the nature of their construction, exhaustive, yielding, if we include the ENSC type solutions discussed under case B, all "primitive" solutions for a given choice of r and t_1. Non-primitive solutions are obtained by adding $2kc$ to a and to b.

Editorial Note. The editors wish to thank H. D. Grossman for assistance in preparing this solution for publication.

Self-reciprocal Curves

E 666 [1945, 218]. *Proposed by N. D. Lane, St. Andrew's College, Aurora, Ont.*

For what value of k is the curve $x^m y^n = k$ self-reciprocal with respect to the circle $x^2+y^2=1$?

Solution by Francis Hall, Hunter College. Let (x_1, y_1) be any point on the curve $x^m y^n = k$. The equation of the tangent to the curve at this point is

$$my_1x + nx_1y = (m+n)x_1y_1. \tag{1}$$

Let P_2: (x_2, y_2) be the pole of (1) with respect to the circle $x^2+y^2=1$. Then the polar of P_2 with respect to the circle, namely $x_2x+y_2y=1$, must be the same line as (1). Therefore

$$my_1/x_2 = nx_1/y_2 = (m+n)x_1y_1,$$

or

$$x_1x_2 = m/(m+n), \qquad y_1y_2 = n/(m+n).$$

Using the fact that (x_1, y_1) and (x_2, y_2) lie on the curve we then find

$$k = \pm\,[m^m n^n (m+n)^{m+n}]^{1/2}.$$

Editorial Note. Using homogeneous coordinates it is easy to show that the curve $f(x, y, z)=0$ will be self-reciprocal to the circle $x^2+y^2=z^2$ if and only if $f(x, y, z)$ divides $f(\partial f/\partial x, \partial f/\partial y, -\partial f/\partial z)$. Imposing this condition on the curve $x^m y^n = kz^{m+n}$ we readily find the values of k as given above.

A Finite Series

E 670 [1945, 218]. *Proposed by C. D. Olds, Purdue University*

Sum the series

$$\sum_{r=1}^{2n-1} (-1)^{r-1} r \Big/ \binom{2n}{r}.$$

I. *Solution by Irving Kaplansky, Columbia University.* Let S be the desired sum. Multiply the identity

$$\frac{2n+2}{2n+1}\frac{1}{\binom{2n}{r}} = \frac{1}{\binom{2n+1}{r}} + \frac{1}{\binom{2n+1}{r+1}}$$

by $(-1)^{r-1}r$ and sum from $r=1$ to $r=2n$. On the left side we obtain

$$(S-2n)(2n+2)/(2n+1),$$

and on the right

$$\frac{1}{\binom{2n+1}{1}} - \frac{1}{\binom{2n+1}{2}} + \cdots + \frac{1}{\binom{2n+1}{2n-1}} - \frac{1}{\binom{2n+1}{2n}} - 2n = -2n,$$

the binomial coefficients cancelling in pairs. Solving for S we find

$$S = n/(n+1).$$

II. *Solution by J. S. Frame, Michigan State College.* Let the desired sum be denoted by S, and define the related sum S':

$$S' = \sum_{r=1}^{2n-1} (-1)^{r-1} \Big/ \binom{2n}{r}.$$

Now we have

$$(r+1)\Big/\binom{2n}{r} = (2n-r)\Big/\binom{2n}{2n-1-r},$$

and consequently

$$S + S' - 1 \equiv \sum_{r=0}^{2n-1} (-1)^{r-1}(r+1)\Big/\binom{2n}{r} = 0, \tag{1}$$

since a reversal of the order of terms in the sum replaces each term by its negative. For the same reason

$$nS' - S \equiv \sum_{r=1}^{2n-1} (-1)^{r-1}(n-r)\Big/\binom{2n}{r} = 0. \tag{2}$$

Solving the two linear equations (1) and (2) simultaneously, we find

$$S = n/(n+1), \qquad S' = 1/(n+1).$$

Also solved by Murray Barbour, Scott Crom, and E. P. Starke.

A Hobo in Distress

E 671 [1945, 274]. *Proposed by Alice Malice, Owen Sound, Ontario*

A hobo has crossed three-eighths of a high railway bridge when he hears behind him an express train coming at sixty miles per hour. He can just save his

life by running to either end of the bridge. How fast can he run? (There is no question of acceleration or deceleration; the engineer is merciless and the man gets going at once.)

I. *Solution by C. W. Foard, Eastman Kodak Company, Rochester, N. Y.* The time required for the hobo to run 5/8–3/8, or 1/4, the length of the bridge is the same as for the train to cross the entire span. Therefore the hobo's speed is 15 m.p.h.

Note that the hobo's speed is independent of the length of the bridge. If the bridge is 480 feet long, the hobo is a grade school 100 yard dash man, requiring 13.6 seconds for the run. However, if the bridge is 1.6 miles long, his best effort is indeed noteworthy—the long awaited four minute mile! Shades of Gunder Haegg!

II. *Solution by W. C. Rufus, University of Michigan.* The time on the bridge in either direction the hobo ran would be proportional to the distance he ran and inversely proportional to his speed relative to the train. Let his top speed be x m.p.h. Then $3/(60-x)=5/(60+x)$, or $x=15$ m.p.h.

Also solved by E. F. Allen, D. W. Alling, N. H. Anning, Gene Archer, LeRoy Babcock, W. G. Brady, William Bunyan, H. N. Carleton, M. L. Constable, J. H. Cross, Monte Dernham, William Douglas, R. L. Duncan, Daniel Finkel, H. G. Funkhouser, C. B. Gass, E. B. Hancock, Frank Hawthorne, Robert Hoskins, O. J. Karst, Jr., Jerry Norris, E. K. Paxton, J. Price, A. R. Rhodes, E. D. Schell, M. J. Sheehy, E. S. Smith, F. C. Smith, Margaret Sproul, E. P. Starke, J. A. Tierney, W. Unterberg, Jeanette Van Os, R. H. Wilson, Jr., and an unsigned solver. N. H. Anning remarks that essentially this problem can be found on page 329 of Part II of the High School Algebra by Birchard and Robertson, Toronto, William Briggs, 1889.

Some Special Regular Polygons

E 673 [1945, 274]. *Proposed by L. S. Shively, Ball State Teachers College*

Does there exist a regular polygon having both these properties: (a) a diagonal is equal to the sum of two other diagonals; (b) a diagonal is equal to the sum of a side and another diagonal?

Solution by E. P. Starke, Rutgers University. By considering the angles at the center of a regular polygon of n sides and radius r, we have

$$d_j = 2r \sin j\pi/n, \qquad j = 1, 2, 3, \cdots, [n/2], \tag{1}$$

where d_1 is the length of a side and d_j is the length of a diagonal which subtends j sides. We require solutions of

$$d_m = d_p + d_q, \qquad 0 < p \leqq q < m \leqq \tfrac{1}{2}n.$$

This is condition (b) when $p=1$ and condition (a) when $p \geqq 2$. By (1) we have

$$\begin{aligned} 2 \sin (m\pi/n) &= 2 \sin (p\pi/n) + 2 \sin (q\pi/n) \\ &= 4 \sin \{(p+q)\pi/2n\} \cos \{(q-p)\pi/2n\}. \end{aligned}$$

An obvious solution is obtained by putting $(p+q)\pi/2n=\pi/6$ and $(q-p)\pi/2n=\pi/2-m\pi/n$. Then

$$q = n/3 - p, \qquad m = n/3 + p.$$

Evidently n must be a multiple of 3. For conditions (a) and (b) to hold simultaneously we need at least $p=1$ and $q=2$. Thus the conditions on p, q, m, and n require that $n\geqq 12$. Finally for any regular polygon of $3k$ sides ($k\geqq 4$), both conditions are satisfied. In particular

$$\text{if } n = 12, \quad d_5 = d_1 + d_3 \text{ and } d_6 = d_2 + d_2;$$
$$\text{if } n = 15, \quad d_6 = d_1 + d_4 \text{ and } d_7 = d_2 + d_3;$$
$$\text{if } n = 18, \quad d_7 = d_1 + d_5 \text{ and } d_8 = d_2 + d_4 \text{ and also } d_9 = d_3 + d_3.$$

Also solved as above by J. B. Kelly and by the Proposer, who showed geometrically that the regular 18-gon possesses the required properties.

A Projective Invariant

E 675 [1945, 274]. *Proposed by Howard Eves, College of Puget Sound*

Show that the ratio of the curvatures of two curves in a plane at a point of contact is invariant under projection.

Solution by the Proposer. A. *Synthetic treatment.* Let the two curves have at P the common tangent PT. Let O be any fixed point not on PT and let a line through O cut the curves near P in Q and R. Let (Q) and (R) be the circles tangent to the curves at P and passing through Q and R respectively. Let the line through O cut (Q) again in q and (R) again in r. Then, as $T\to P$,

$$\begin{aligned}\lim (OQTR) &= \lim (OQ/OR)(TR/TQ)\\ &= \lim (OQ/OR)(Tq/Tr)\\ &= \lim (OQ/OR)\lim (Tq/Tr) = a/b,\end{aligned}$$

where a and b are the radii of curvature at P of the two given curves. (For circles (Q) and (R) approach the circles of curvature at P as limiting positions, and $Tq/Tr\to a/b$.)

Using bars in the projected figure we similarly find that as $\overline{T}\to\overline{P}$, $\lim (\overline{QQTR})=\bar{a}/\bar{b}$. But $(OQTR)=(\overline{OQTR})$, whence $a/b=\bar{a}/\bar{b}$, and the theorem is proved.

B. *Analytic treatment.* Cartesian coordinate systems can be chosen on two planes p and $\bar{p}$ such that any given central projection of p on $\bar{p}$ takes the form

$$\bar{x} = mx/y, \qquad \bar{y} = mn/y.$$

Letting primes indicate differentiation with respect to y and $\bar{y}$ we then find

$$\bar{x}'' = m^2nx''/\bar{y}^3.$$

Now let $x=f(y)$ and $x=g(y)$ be two curves in plane p, tangent to each other at

the point $P:(x_0, y_0)$, and let R be the ratio of the two curvatures at P. Then

$$R = \frac{-f''(y_0)}{(1+[f'(y_0)]^2)^{3/2}} \frac{(1+[g'(y_0)]^2)^{3/2}}{-g''(y_0)} = f''(y_0)/g''(y_0).$$

Similarly, the ratio of the curvatures of the two projected curves in plane $\bar{p}$ is

$$\begin{aligned}\overline{R} &= \bar{f}''(\bar{y}_0)/\bar{g}''(\bar{y}_0) \\ &= (m^2 n f''(y_0)/\bar{y}_0^3)/(m^2 n g''(y_0)/\bar{y}_0^3) \\ &= R,\end{aligned}$$

and the theorem is proved.

A. W. Tucker furnished for this theorem the following references from *Introduction à la Géométrie Projective Differentielle*, by Fubini and Čech (Paris, 1931), p. 18: "Ce rapport [des courbures euclidiennes des courbes C et Γ dans le point de contact] est donc (Smith [1] et Mehmke [1], [2]) un invariant projectif."

H. J. S. Smith, [1], London Math. Soc. 2, 1869, pp. 196–248.

R. Mehmke, [1], Zeit. für Math. 36, 1891, pp. 56–60, [2], Zeit. für Math. 36, 1891, pp. 206–213.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4185. *Proposed by B. M. Stewart, Michigan State College*

In an unexplored region known as Wild Basin, a hunter found himself lost. But he had in hand a compass, and there were visible on two distant peaks fire ranger stations, A and B, whose bearings from his own cabin, O, he knew. From one observation point, C, the hunter took bearings on A and B; walking to another observation point, D, nearby, he took bearings on A, B, and C. Somehow he felt these seven bearings ought to enable him to find the direction homeward, *i.e.*, the bearing from D to O.

Show that the hunter's problem may be solved if he has either (1) mathematical tables or (2) a straightedge, in this case using the compass card as a protractor.

Note. Suggested by a correspondent of M. H. Ingraham, University of Wisconsin.

4186. *Proposed by Fritz Herzog, Michigan State College*

It is known from the theory of probability that for x fixed, $0<x<1$, the largest of the $n+1$ terms ${}_nC_rx^r(1-x)^{n-r}$, $r=0, 1, \cdots, n$, is asymptotically equal to $[2\pi nx(1-x)]^{-1/2}$, as $n\to\infty$. Show that for all integral values of n and r with $n\geqq 1$, $0\leqq r\leqq n$ and for all values of x with $0<x<1$

$$ {}_nC_rx^r(1-x)^{n-r} < 1/[2enx(1-x)]^{1/2} $$

and that this represents the best inequality in the sense that the numerator on its right cannot be replaced by any number less than unity.

4187. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For a given tetrahedron the ratio of the distances of the Monge point and of the circumcenter to the common perpendicular to two opposite edges is equal to $\cos\theta$, where θ is the angle between the two edges.

SOLUTIONS

A Multiplication Puzzle

4135 [1944, 475]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

With the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, taken one at a time, form two numbers of five digits each such that one is a multiple of the other.

Solution by E. P. Starke, Rutgers University. Let N and aN be the two five-digit numbers. It appears there are 94 solutions corresponding to the following values of a and N:

$a=2$,		
	1'35'48, 1'38'45, 1'3'485,	18 solutions
	3'27'09, 3'29'07, 2'3'079,	12
	1'4685, 1'4865, 15'486, 16'485, 45'186, 46'185,	12
	2'0679, 2'0769, 27'069, 29'067, 267'09, 269'07;	6
$a=3$,		
	16794, 17694, 20583, 23058, 30582, 32058;	6
$a=4$,		
	15237, 17235, 17352, 21735, 23517, 23715, 20394, 20439;	8
$a=5$,		
	13458, 13584, 13854, 14538, 15384, 18534, 14586, 14658,	
	15846, 15864, 18564, 18654;	12
$a=6$		none
$a=7$,		
	14076;	1
$a=8$,		
	10459, 10469, 10537, 10579, 10592, 10674, 10679, 10742,	
	10794, 10932, 10942, 10953, 10954, 12073, 12307, 12345;	16
$a=9$		
	10638, 10647, 10836.	3

The values of N for $a=2$ are separated by the mark (′) into parts which can be rearranged without destroying the desired property. Thus 1′35′48 represents the six numbers 13548, 14835, 35148, 35481, 48135, 48351. Omitting permutations which introduce an initial zero, we have the number of solutions indicated. That there can be no value of N for $a=2$ without such a separation is evident from the following considerations: in N the only digits less than 5 are the initial digit and those which follow the mark (′); in aN the only even digits are the final digit and those which precede the mark; without a separation, N would require four digits greater than 4, while aN would require four odd digits, which is not possible.

For $a=5$, N cannot be odd, which may be seen as follows: since only the initial digit of N is 1, the digit 0 evidently cannot appear in $5N$; thus 0 is a digit of N; then the digit in the preceding place in $5N$ is either 0 or 5; since the final digit of $5N$ is 5 there is a duplication of either 0 or 5, contrary to hypothesis. Now when N is even, $\frac{1}{2}N$ is the same as $5N$ except that the digit 0 is placed at the other end. Hence all values of N for $a=5$ may be obtained from pairs N_1, $2N_1=N$, where N_1 has the initial digit 0. Such values of N_1 are obtained directly from the list above for $a=2$.

In general we have

$$10234 \leqq N \leqq 98765/a.$$

Also, since the sum of the digits of N and aN is 45, we have

$$(a+1)N \equiv 0 \bmod 9,$$

whence $N\equiv 0 \pmod 9$ for $a=3, 4, 6, 7, 9$, and $N\equiv 0 \pmod 3$ for $a=2, 5$. With these restrictions in mind it is easy to systematize the testing of possible values of N so as to detect duplication of digits in N and aN without unreasonable expenditure of time.

Solved also by Murray Barbour, Shepard Bartnoff, Monte Dernham, Irving Kaplansky, E. D. Schell, and the proposer.

Divisors of Factorials

4137 [1944, 533]. *Proposed by P. Erdös, Purdue University*

Given an integer $x\leqq n^2/4$ which has no prime factor greater than n, show that $n!\equiv 0 \bmod x$.

Solution by Irving Kaplansky, Columbia University. I shall establish the following generalization. Let $x\leqq n^k/k^k$, where k is a positive integer $\geqq 2$, and suppose that for any prime factor p of x, $p^{k-1}\leqq n$. Then $n!\equiv 0 \pmod x$. The proposed problem is the case $k=2$.

Proof. Clearly it suffices to consider the case where x is a prime power p^r with $r\geqq k$. Then $n!\equiv 0 \pmod{p^r}$ will follow a fortiori from $rp\leqq n$, or $r^kp^k\leqq n^k$. Since by hypothesis $p^rk^k\leqq n^k$, we need only prove

$$r^k p^k \leqq p^r k^k \tag{1'}$$

or

$$f(r) = \frac{\log r - \log k}{r - k} \leqq \frac{\log p}{k}. \tag{1}$$

The numerator of $f'(r)$ is

$$g(r) = (r - k)/r - (\log r - k \log k).$$

Now $g'(r) = k/r^2 - 1/r \leqq 0$ for $r \geqq k$. Hence $g(r)$ is decreasing and, since $g(k) = 0$, $g(r) \leqq 0$ for $r \geqq k$, and hence finally $f(r)$ is a decreasing function of r for $r \geqq k$.

Case I. $p > 2$. Then $f(k) = 1/k < (\log p)/k$, and (1) is true for $r = k$, hence for all $r \geqq k$.

Case II. $p = 2$. Here (1) is an equality when $r = 2k$. Since (1') is an equality for $r = k$, we need only consider the case $k < r < 2k$.

Case II*a*. $k = 2$. Then $r = 3$, $x = p^r = 8$, $n \geqq \sqrt{4x} > 4$ and 4! is divisible by 8.

Case II*b*. $k = 3$. Then $r = 4$, 5; $x = 16$, 32; $n \geqq 3x^{1/3} \geqq 6$, 8. In each case x divides $n!$

Case II*c*1. $k \geqq 4$, $r \leqq 3k/2$. We are trying to prove that 2^r divides $n!$ Now by the argument that led to (1'), we know that $2k \leqq n$. Hence it will suffice to prove that 2^r divides $(2k)!$. The power of 2 which occurs in $(2k)!$ is

$$k + [k/2] + [k/4] + \cdots \geqq 3k/2 \geqq r.$$

Case II*c*2. $k \geqq 4$, $r > 3k/2$. From $x = 2^r \leqq n^k/k^k$, we have $n \geqq k\, 2^{r/k} > 4 \cdot 2^{3/2} > 11$. Now the power of 2 dividing $n!$ is

$$[n/2] + [n/4] + \cdots \geqq 3n/4 - 2,$$

and for $n \geqq 12$, $3n/4 - 2 > 3n/\sqrt{32}$. Hence to prove that 2^r divides $n!$ it will suffice to prove $r < 3n/\sqrt{32}$. Now repetition of the arugment which led to (1) yields (2) as the inequality which we wish to prove:

$$F(r) = k(\log r - \log k) - r \log 2 + k \log (\sqrt{32}/3) \leqq 0. \tag{2}$$

$F(3k/2) = 0$, and $F'(r) = k/r - \log 2 < 2/3 - \log 2 < 0$, so that $F(r) \leqq 0$, as desired.

Solved also by D. H. Browne, J. B. Kelly and Sol Rubinow, and W. J. Robinson.

Editorial Note. The solution by Kelly and Rubinow may be put in the following form: If p is a prime factor of x, denoted by $p \mid x$; then, since $p \leqq n$, $p \mid n!$. Suppose first that the multiplicity of p in x is $2r$, $r \geqq 1$; then $2p^r \leqq n$, and it follows that $p^{2r} \mid n!$.

Suppose next that the multiplicity of p in x is $2s+1$. $s \geqq 1$, then $n \geqq 2\sqrt{p}\, p^s$. Consider *Case A* where $4p^s < n$. Here it easily follows that $p^{2s+1} \mid n!$. *Case B*, $4p^s \geqq n$. Here $4p^s \geqq n \geqq 2\sqrt{p}\, p^s$, $2 \geqq \sqrt{p}$, $2\sqrt{p} \geqq p$, $n \geqq p^{s+1}$. It now follows that $p^{2s+1} \mid n!$. This completes the proof that $x \mid n!$.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

Professor J. I. Tracey, Branch Head of the Mathematics Department of the Biarritz American University, announces the following staff: Sgt. E. F. Arndt, Frank Ayres, F. R. Bamforth, I. A. Barnett, Lt. Col. A. C. Cohen, Pfc. S. D. Conte, Cpl. J. C. Gibson, L. S. Hill, Capt. A. O. Lindstrum, R. B. McClenon, E. R. C. Miles, J. H. Neelley, A. L. Nelson, Lt. Stephen Oselinsky, C. G. Phipps, J. C. Polley, B. P. Reinsch, Capt. W. T. Scott, C. R. Sherer, Gabor Szegö, Lt. H. E. Temmer, H. S. Thurston, J. I. Tracey, Lt. Col. A. S. Turner, Major V. R. Walker, R. M. Winger.

Dr. E. E. Blanche, Branch Head of the Mathematics Department of the Florence American University, announces the following staff: Cpl. Thomas Anderson, T. A. Bancroft, H. C. Christofferson, W. H. Fagerstrom, C. G. Jaeger, T/Sgt. William Lassow, Lt. T. A. Love, Sk3C Emanuel Mehr (U.S.N.R.), Sgt. W. F. Prahl, Jr., Capt. M. C. Resick, Cpl. W. C. Stone.

Captain Aaron Bakst has been awarded the Bronze Star Medal for meritorious achievement in connection with military operations in the Southwest Pacific Area.

The University of Michigan announces: Professor C. E. Love is on leave of absence for the year 1945–46; Dr. R. C. F. Bartels has been promoted to an assistant professorship and, together with Assistant Professors G. E. Hay and Wilfred Kaplan, has returned to the University.

F. L. Celauro has been appointed to an assistant professorship at Lehigh University.

Assistant Professor Max Coral of Wayne University has been promoted to an associate professorship.

Paul Cramer has been appointed head of the physics department of Huron College, South Dakota.

Dr. F. C. Gentry has been appointed to an associate professorship at Louisiana Polytechnic Institute.

Dr. R. A. Good has been appointed to an assistant professorship at the University of Maryland.

Associate Professor E. H. Hadlock of Hastings College, Nebraska, has been promoted to a professorship.

Professor W. L. Johnson of Mississippi Southern College, Hattiesburg, has been appointed dean of men.

Dr. A. M. Peiser has been appointed head of the Statistics Research Group at the Langley Field laboratory of the National Advisory Committee for Aeronautics.

Dr. C. J. Thorne has been appointed to an assistant professorship at the University of Utah.

Lt. Col. W. H. White has resumed his duties as assistant professor of mathematics at Virginia Polytechnic Institute.

The following appointments to instructorships are announced:

Bergen Junior College, Teaneck, New Jersey: Edward Rayher

University of Kansas: Edison Greer

Vanderbilt University: J. A. Tierney

Professorship Emeritus M. J. Babb of the University of Pennsylvania died October 27, 1945.

Professor J. D. Tamarkin of Brown University died November 18, 1945.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

UNIVERSAL MILITARY SERVICE IN PEACE TIME

A REPORT OF A SUBCOMMITTEE OF THE WAR POLICY COMMITTEE*

Foreword (prepared by Professor M. H. Stone, Chairman of the War Policy Committee): The War Policy Committee of the American Mathematical Society and the Mathematical Association of America was formed to study the many questions of professional and scientific policy arising out of the war. No subject has been of greater interest or more vital concern to the Committee than the relations between scientific effectiveness on the one hand and the military requirements of the nation on the other. A most important aspect of this subject is treated in the report on Universal Military Service in Peace Time

* Prepared in July, 1945, by a subcommittee consisting of Professors W. L. Hart (Chairman) Saunders Mac Lane, and C. B. Morrey, Jr., and approved by the War Policy Committee. This report has also been approved by the Council of the American Mathematical Society and the Board of Governors of the Mathematical Association of America.

which is now made public. This report, prepared some time ago by a special subcommittee, is directed in the main at points upon which mathematicians as such are particularly qualified to express informed opinions. Whatever view may ultimately prevail concerning universal military training in peace time—and it should be emphasized that there are many citizens, mathematicians included, who doubt the wisdom of introducing such a peace-time military program—it is clearly of the first importance that no program deleterious to the scientific and technological vigor of the nation should be adopted. The report deals frankly and in detail with this vital segment of the problem now before Congress. In offering the recommendations of the report as a professional contribution to the current discussion, the War Policy Committee hopes to render a modest public service strictly within the natural sphere of its activity.

1. *Introduction.* In the Congress of the United States, bills are under consideration (S. 188 and H.R. 515, companion bills) which would prescribe a year of military service in peace time for all young men of suitable physical qualifications. These bills present the nation with a proposal for a fundamental new departure in our national life and it is appropriate that the widest discussion should occur concerning all sides of the situation which would result. It is natural that the greater part of the arguments pro or con about S. 188 in the public press should deal with military, internal political, and international aspects of the matter. Various spokesmen of the general field of education, particularly at the college level, as well as of specific fields of learning have likewise discussed the measure under consideration. It is the main purpose of this report to state opinions concerning the impact of compulsory military service in peace time on those phases of our national life which are of primary significance to mathematicians and which lie in areas where we may speak with special authority. In carrying out this aim, it is natural that at times we shall phrase remarks from a general educational viewpoint not primarily associated with the field of mathematics. A summary of the opinions advanced in the report is to be found in Section 11.

2. *Orientation of the report.* In the discussion of any measure like S. 188, we may conceive of a division of interest between two major questions. First, one may ask, "Should the United States have universal military service in peace time?" Second, "If Congress is to pass such an act, what provisions should it contain in order that the greatest possible good should be obtained for the nation as a whole and for the young men who will perform the service?" We realize that these questions are considerably interrelated but, for our purposes, we shall act as if they are relatively independent. It is probable that the decision on the first question should and will be reached mainly on the basis of significant national and international considerations of a social, economic, political, and military nature. In this connection it is likely that only small weight will be given to those points where our opinions as mathematicians or educators would be of any special importance as compared to those ideas we may express individually as mere citizens. Hence we have decided that, in regard to the first

question and the most intimately connected phases of the second question, we shall make only a single statement.

> We believe that a decision about introducing universal military service in peace time should be reached only after an investigation by a carefully selected commission, preferably appointed by the President of the United States. We recommend that this commission should include not only representatives of the armed forces but also individuals from all other important sections of our population, with a strong representation from the fields of industry, science, and technology which have been of such great importance during the present war. This commission should have the wide objective of considering what combination of universal military service, scientific development work on weapons of war, and associated industrial and educational measures would be best adapted to insure the safety of the nation.

The balance of this report will be concerned with matters particularly germane to a discussion of the second of the two major questions which we have mentioned, although we have no intention of considering all its aspects. Our remarks will be phrased under the explicit assumption that *universal military service in peace time will be adopted.* However, this viewpoint is not to be taken as evidence that we desire or expect a bill such as S. 188 to become law.

For future reference, we note that the present bill S. 188 includes the following features.

> 2.1. The bill omits specification of all subsidiary values as justification for military training and is based only on reasons relating to national security and the prevention of dissipation of training resources and experience.
> 2.2. The bill contains provision for a period of four years (ages 18 to 22) in which the individual may select the year for military training. A high school graduate with his parents' consent may volunteer to start his year of training while he is still 17 years of age.
> 2.3. The year of training will be used by the Army and Navy as a basis for selecting non-commissioned and commissioned officers.

3. *Background relating to the present war.* The importance of technical developments of new weapons and tactics during the military operations of the last few years has emphatically shown that science has made a fundamental contribution to the military power of the nation. More particularly, the abstract science of mathematics has been put to effective application in a surprising number of fields, including even the extensive use of mathematicians in operational analysis work overseas. These facts are important in the formulation of an intelligent long term policy for maintaining the nation's military strength. It can safely be said that the military potential of a country does not depend solely upon the size and character of its armed forces. This potential also depends largely upon the technical and industrial talent available and upon the vigor, ability, and training of its scientific and technical experts.

> Long range military planning, therefore, requires that able young men be encouraged both in preparing themselves in basic scientific and mathematical knowledge and in acquainting themselves with the ways in which this knowledge can be put to use in time of war or national emergency.

With the preceding viewpoint in mind, we deeply regret to note that the manpower policies of the United States during the war have not been well formulated with respect to the utilization of men trained in science and technology or potentially able to assimilate such training. Draft deferments for the purpose of obtaining certain varieties of technical training have been at times difficult and often impossible to obtain. Some valuable technical men have been wasted in non-technical activities and others would have been wasted had it not been for the energetic effort of representatives of science and technology in prodding reluctant or unenthusiastic government bureaus into action. During the war the United States has followed a sadly shortsighted and unintelligent policy with regard to science and technology by not providing for appropriate annual increments of undergraduate men to be trained in those fields. There are facts at hand which indicate that England and probably Russia have not made a similar error during the war. We agree strongly with the many opinions which have recently been expressed concerning the necessity for prompt resumption of training for the scientific professions, even before the end of the war with Japan. We believe that any additional delay in provisions for such training may have fatal effects not only on national health and the technological side of our economy but, also, on our means for future national defense. These remarks are well justified by the established facts about the present and potential shortages of physicians, dentists, Ph.D.'s in the various fields of the physical sciences, and technologists. This background leads us to the following conclusion.

> We consider it very important that universal military service in peace time should not be planned in such a way as to interfere with efforts, first, to eliminate as quickly as possible the present shortage of scientists and technologists and, second, to provide for a continuous generous supply of such essential categories of trained citizens in the future.

4. *An attitude about exemptions from military service.* It has been the traditional American viewpoint that all men should be treated as nearly alike as possible in fundamental respects whenever a call for universal military service is issued by our Government. It could be argued that this was a major cause for some of the short-sighted manpower policies of the present war period which have been referred to previously. However, we believe that it would have been possible for Selective Service to have been administered during the war without violating the specified tradition and, also, without introducing various major evils. This democratic viewpoint is so deeply seated that, regardless of whatever arguments might be presented against it, neither the general public nor the affected young men could be expected to give friendly support to measures for creating numerous exempt classes of the population in case universal military

service in peace time should be adopted. Realistically, we appreciate that any request for the exemption of mathematicians or physical scientists, for instance, could be expected to induce requests for the exemption of social scientists, and scholars in almost all other fields of learning, as well as representatives designated by innumerable pressure groups outside the field of education. The result of such requests for exemptions would undoubtedly be the creation of an ironclad policy of *no* exemptions. The requests might even develop an unsympathetic attitude among the directors of military service with respect to rational plans for differentiation of training, which we propose to emphasize in place of exemptions.

> We shall proceed, then, with the explicit premise that it would be undesirable to suggest the exemption of men with special mathematical talent and that each man, regardless of his abilities, will have to spend one year of his life in military service.

5. *A liberal interpretation of the words "military service."* The present rate of increase in the military uses of science and technology convinces one that success in any future war will be dependent on continued progress in science and on the existence of a large reservoir of trained technologists. Thus it is essential that any program for universal military service should be consistent with associated plans for the development of our scientific and technological potential. It follows that the final program should not be organized with the narrow objectives of merely disciplining masses of men and preparing them for handling existing weapons of war. Hence, the term "military service" should be interpreted in a liberal fashion.

> "Military service" should include not only the usual routine service in the Armed Forces but, also, various other highly technical forms of training which are just as essential for enhancing the military strength of the nation. These types of service should be made available for properly qualified young men under some studied system for differentiation of training in accordance with ability. In particular, the system should provide appropriate advanced types of service in the case of the small but important group having special aptitude for and training in mathematics, the physical sciences, or technology.

6. *Candidates for the most highly technical military service.* During the present war, under Selective Service the Armed Forces have made intelligent efforts to place the inductees into various enlisted classifications in accordance with the abilities shown by the men in civilian life and in various tests given by the Armed Forces. Also, similar efforts have been made to locate men with appropriate backgrounds as candidates for commissioned ranks in various branches of the Armed Forces. It should be anticipated that these commendable practices in the direction of differentiated training would continue under universal military service in peace time. The Committee then directs the main force of its recommendation for differentiated training at the cases of the exceptionally

superior young men who, let us say, have the requisite mathematical and scientific aptitude and inclination to become Ph.D.'s in mathematics or a physical science, or to advance to the highest levels of attainment in an engineering field. Under the reasonable hypothesis that less than one-third of the men in our population with such potentialities finally pursue careers in the indicated fields, we are led to estimate the number of men of this sort available in any annual age group at about 3000 men. If we should attempt to sift them out of the enormous annual group of high school graduates by any existing devices for the prediction of mathematical or scientific aptitude, the percentage of error in diagnosis would be very high. It is likely that we would have to earmark about 20,000 boys annually in order to be reasonably certain that the final group would include 90% of the 3000 boys whom we have mentioned. Such an attempt at selection of 20,000 boys would entail severe difficulties in the field of educational measurements and also many later complications if the members of the earmarked group were to be given individual attention during the next four to seven years. Our final plan will indicate a procedure for obtaining such candidates which appears more simple than an attempt to earmark them at an early age. At present we offer merely the following conclusion. Among the boys who graduate from high school in any year there is a group of about 20,000 with such high mathematical and scientific aptitude that they deserve special consideration under any plan for universal military service. This group will hereafter be referred to as the *select technical group*.

7. *Advantages of deferment of military service for the select group.* It is likely that any boy in the group would graduate from high school by the time he is 18. Regardless of considerations related to military service, the rarity of the aptitudes present in the group makes it desirable that each member should be shielded from unnatural influences which might lead him away from the fields of science and technology where his abilities would be so valuable to the nation. We believe that he should be encouraged to carry out any latent inclination to enter upon scientific or technical training. Immediately after graduation from high school, an interruption of a full year in the educational process might eliminate or dim his embryonic urge to scientific study. The interruption would also cause an undesirable break in the important preliminary stages of progress in mathematics and science. Moreover, a year of military service at age 17 or 18 might expose the boy to certain military, social, or industrial activities which, although appealing to a youth, could lead him after a few years to an undesirably low terminal point outside of advanced science or technology, with a corresponding loss to the nation.

If our attention is now fastened primarily on differentiated training at a high level under military service, we can likewise argue that a delay beyond age 18 in this service may be desirable for a boy in the select group. When he graduates from high school, he will not yet possess the requisite mathematical and scientific background to be eligible for or to obtain maximum good from many of the high grade types of differentiated service which can be made available. Before

he receives college training, his possibilities might be merely those of an exceptionally intelligent young man with latent scientific abilities and interests but no particularly useful background knowledge. He might even gain a permanently wrong orientation with respect to various aspects of military service where he should eventually become most useful to the nation in case of war.

The preceding remarks lead us to advocate the following flexible provision, similar to Item 2.2, in any final regulations governing universal military service in peace time.

> The directors of the program should have the power to approve a request for deferment of the year of service for any young man when this is deemed best for the interest of the nation. In particular, continued efficient progress with high achievement in the study of mathematics, a physical science, or technology, even through the stage of a Ph.D. degree, should be considered sufficient cause for such deferment.

Conceivable complications in the administration of the rules for deferment should not be considered a deterrent to the adoption of the preceding recommendation. It should be possible to develop a smooth routine for such a process in peace time which would be superior to the method employed with respect to draft deferments of scientists and scientific students during the war. In this connection it is pertinent to observe that a proposal for special treatment immediately under Selective Service for a select group of 20,000 was presented to Congress recently.

8. *Suggestions concerning highly differentiated service.* We proceed under the assumption that the deferment recommendation of Section 7 could be adopted. Then, we propose the following rough outlines of a plan for implementing the suggestions about differentiated service.

8.1. A board should be created to canvass the possibilities for in service training on intermediate and higher technical levels for properly qualified applicants and to supervise plans for instituting such training. This board should involve civilian representatives of the fields of mathematics, the physical sciences, and engineering, together with representatives of the Armed Forces.

8.2. The search for training locations should cover the various laboratories, research divisions, arsenals, and other technical activities of the Army and Navy. Also, the search should extend into associated private industries and related curricula and research in universities. As a mere sample of possibilities we mention the following types of training which might be given to properly selected groups each year.

8.21. An advanced course in meteorology for the future use of the Army or Navy, along the lines of the curriculum presented at five major universities in the United States during the early years of the war, and associated field experience.

8.22. In service training at the Ballistic Research Laboratory of the

Army Ordnance Department at the Aberdeen Proving Ground.

8.23. In service training in the artillery fire control division at an ordnance plant.

8.24. In service training at a laboratory of the Signal Corps, or at an aeronautical research laboratory of either the Army, the Navy, private industry, or a university.

8.25. Study of the mathematical and operational phases of cryptanalysis in the intelligence division of the Army or Navy.

8.3. For each of the training possibilities, the Board should decide upon appropriate academic prerequisites to be demanded of any man who is to qualify for the opportunity. These prerequisites should probably be much more extensive than for any similar training during the war, because in peace time interested men would have ample opportunity to obtain the necessary background. A pamphlet should then be published listing the available varieties of technical training which might be taken as "military service" so that interested boys, even early in their high school life, could begin to orient their school work with respect to the future opportunities in military service. The pamphlet should urge boys of proper aptitudes to defer their entrance to military service until they gain the prerequisites for the type or types of service which appeal to them.

8.4. If administratively possible, assignments to any one of the most exacting and desirable varieties of training should be made on the basis of applications asserting definite interest in the field involved, as well as possession of the necessary prerequisites.

8.5. In addition to the technical training experience mentioned previously, an appropriate amount of routine "boot" training should be given either simultaneously with the technical work or in a separate period of time during the year of service.

8.6. The directing Board should examine the possibility of granting commissions in the Reserve Corps of the Army or Navy for satisfactory completion of certain types of technical training which would be available to the most highly qualified young men during their year of service. Acceptance by a young man of the opportunity of such training should not commit him to accepting such later commission as might be offered in this connection.

9. *Civilian educational by-products of universal military service.* If the interesting nature of the various technical types of differentiated military service, at low or intermediate as well as at high levels, and the corresponding academic prerequisites are properly advertised, strong repercussions might occur in the field of secondary education. It is probable that boys of ability and their parents would then demand maximum mathematical and scientific opportunities in the secondary curriculum for students of proper ability. Also, regardless of the possibility of differentiated training, it is our belief that a requirement of a year of military service, with the attendant delay in the study of the professions, would place an added demand for educational efficiency on both the high

schools and colleges. The knowledge resulting from a brief exposure to substantial secondary mathematics or science is easily forgotten. On the other hand, a long exposure to these fields gives the student a body of content which, even though it should remain dormant during a year of military service, would be effective in the next year after a brief review.

Hence, if universal military service is adopted, added efforts should be made in the secondary field to expose the better students to as much substantial mathematics and physical science as possible.

10. *Educational responsibilities of the Armed Forcs resulting from universal military service.* Even without the delay which would be caused by military service, the time for completing professional training encroaches on the age of greatest productivity and ingenuity as well as on the normal age for assumption of family responsibilities. Hence, under universal military service, the Armed Forces will be presented with a challenge to compensate as much as possible for the year of delay which would be introduced. It is obvious that the handicaps suffered by a young man due to an interruption of a year in his formal education would be considerably lessened if he should systematically carry on correspondence study during the year.

If universal military service is adopted, the Armed Forces should offer to the men in training the fullest cooperation and encouragement in the extension of their formal education by means of correspondence courses. At the college level, we recommend that this be done by paying the registration fees for men who take and diligently pursue courses offered by regular college correspondence departments, rather than through such an agency as the present Armed Forces Institute.

11. *Summary.* In the development of this report we have emphasized th following points and recommendations.

11.1 A joint civilian and military commission should be appointed by the President of the United States to study the problem of universal military service in peace time before final action is taken on the matter.

11.2. An expression of opinions by the American Mathematical Society and the Mathematical Association of America should be focused on constructive suggestions about the administration of universal military service if it should be adopted and about the solution of resulting educational problems.

11.3. No outright exemptions from universal military service should be requested.

11.4. The required military service should be highly differentiated in accordance with the aptitudes and training of the young men involved, with emphasis on exceptional differentiation for those few with the greatest technical abilities.

11.5. A system for deferring the year of military service should be instituted so that gifted young men might prepare themselves for advanced varieties of differentiated service before entering the Armed Forces.

11.6. The possibilities for technical varieties of differentiated service should be canvassed by a joint civilian and military board. The resulting training programs with the corresponding academic prerequisites should be well advertised among high school boys, their parents, and the teachers and administrators in the secondary field.

11.7. If universal military service is adopted, the field of secondary education will have the added responsibility of increasing the efficiency and quantity of instruction given in mathematics and physical science to the students of better than average ability.

11.8. The Armed Forces should encourage young men to continue their education during military service by taking correspondence work through regular school channels, and the Armed Forces should pay the costs of such study if a man carries it through diligently.

STUDENT DEFERMENT

On September 18, the National Headquarters of Selective Service issued he following regulation regarding the deferment of college students.

"Any person who entered upon a course of instruction at a college or university before he became eighteen years of age and who is ordered to report for induction during a quarter or semester of such course of instruction shall, upon his request, have his induction postponed (1) until the end of such quarter or semester, or (2) until he ceases to pursue continuously and satisfactorily such course of instruction, whichever is the earlier."

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ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April 13, 1946.
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KANSAS
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LOUISIANA-MISSISSIPPI, Ruston, La., March 15 and 16, 1946
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, N. Y., May 11, 1946
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PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN, Boulder Colo., April 19–20, 1946
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DISTRIBUTIONS DERIVED FROM THE MULTINOMIAL EXPANSION

G. B. PRICE, University of Kansas

1. Introduction. This exposition describes certain probability distributions associated with the multinomial expansion. It begins with a description of the graph of the multinomial distribution. Then the following three topics are treated: (a) the probability distribution for the number of classes in a sample which contain at least one variate; (b) an elementary test for goodness of fit and remarks on the Chi-Square Test; (c) the distribution of $(y+\cdots+y)$ in terms of the distribution of y.

The binomial distribution is an important feature of a first course in statistics, but distributions associated with the multinomial expansion are not emphasized to the same extent. Many text books give a derivation of the normal probability surface from the multinomial expansion but go no further. Others refer to the "multinomial distribution, the forms of which are so many and so various as almost to defeat classification," but include only one or two simple examples. It is hoped that the three problems treated here, each of importance in itself, will give some indication of the rôle of the multinomial distribution.

2. The multinomial distribution. Consider an event characterized by a random variable y that takes on one and only one of the values $1, 2, \cdots, k$ on each trial. Let the probability that y takes the value i on a single trial of the event be p_i. Then $p_1+p_2+\cdots+p_k=1$. From well known laws of the subject, the probability that y has the value 1 f_1 times, that it has the value 2 f_2 times, $\cdots$, and that it has the value k f_k times, in some order, on n trials is

$$M(f_1, f_2, \cdots, f_k) = \frac{n!}{f_1!f_2!\cdots f_k!} p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k}, \tag{2.1}$$

where $f_1+f_2+\cdots+f_k=n$. All possible outcomes correspond to all possible terms of the form (2.1), and they are the terms in the expansion of $(p_1+p_2+\cdots+p_k)^n$. The probability distribution which consists of all the terms (2.1) is called the multinomial distribution* for obvious reasons (see [1], p. 50 and [6], pp. 58–59). The special case in which $k=2$ is the well-known binomial distribution.

It will be necessary for later purposes to have a clear understanding of the nature of the graph of the multinomial distribution. The special case $k=3$ will be examined first because it is typical, and in addition the essential features of it can be drawn explicitly and visualized in ordinary 3-space.

The general term in $(p_1+p_2+p_3)^n$ is

$$M(f_1, f_2, f_3) = \frac{n!}{f_1!f_2!f_3!} p_1^{f_1} p_2^{f_2} p_3^{f_3}, \tag{2.2}$$

* Standard notation would require the use of $x_1, x_2, \cdots, x_k$ in place of $f_1, f_2, \cdots, f_k$, but it is felt that the use of the f's, suggesting frequencies, will contribute to the ease of reading certain parts of this paper.

and it is associated with the point (f_1, f_2, f_3) in euclidean 3-space. The value of the term can be plotted as an ordinate in the fourth dimension of an (f_1, f_2, f_3, M) space, but for the purposes of this paper it will be better to consider that the term (2.2) is merely marked at the point (f_1, f_2, f_3). Then, since $f_1+f_2+f_3=n$, the terms of the multinomial distribution are marked at the points which have integral coordinates and which lie on the equilateral triangle cut out of the plane $f_1+f_2+f_3=n$ by the first octant. This multinomial distribution has the following important property: the terms marked along any line $f_1=$constant of the triangle are, except for a constant factor, the terms of a binomial distribution with respect to f_2 and f_3; similarly for terms along the lines $f_2=$constant and $f_3=$constant. In particular, the terms marked along the lines $f_1=0$, $f_2=0$, and $f_3=0$ are exactly the terms of the binomial distributions $(p_2+p_3)^n$, $(p_3+p_1)^n$, $(p_1+p_2)^n$.

In the general case, the term $M(f_1, f_2, \cdots, f_k)$ is marked at the point $(f_1, f_2, \cdots, f_k)$ in k-dimensional euclidean space. The terms $M(f_1, f_2, \cdots, f_k)$ are thus marked at the lattice points of a $(k-1)$-cell in k-space. The property stated in the special case and generalizations of it hold for the general multinomial distribution $M(f_1, f_2, \cdots, f_k)$.

The Number of Classes in a Sample which Contain at Least one Variate

3. Statement of the problem. Let the random variable y of section 2 have the value i if and only if a variate drawn at random from a certain population lies in the ith class interval of the frequency distribution. Then p_i denotes the probability that a variate drawn at random will lie in the ith class interval. If a sample of n variates is drawn, there is a certain probability $P(x; n, k)$ that the n variates lie in exactly x, $1 \leqq x \leqq k$, of the k class intervals. The problem is the following: to determine the probability distribution of x and also its properties.

4. Determination of the distribution of x in the most general case. It will be instructive to consider the special case $k=3$ before the solution of the general problem is undertaken. The term $M(f_1, f_2, f_3)$ in (2.2) is the probability that a sample of n variates will be drawn in which the classes 1, 2, 3 have frequencies f_1, f_2, f_3 respectively. The probability of $x=1$ is the probability of drawing a sample in which two of the three frequencies f_1, f_2, f_3 are zero and the third is n. An examination of the graph of $M(f_1, f_2, f_3)$, described in section 2 above, shows that the terms for which two of the three frequencies are zero are exactly those terms which are marked at the three vertices of the triangle. Thus $P(1; n, 3)$ is the sum of these three terms. The probability of $x=2$ is the probability of drawing a sample in which exactly one class frequency is zero; the corresponding terms $M(f_1, f_2, f_3)$ are those marked along the sides, but not at the vertices, of the triangle. Thus $P(2; n, 3)$ is the sum of the terms marked on the sides but not at the vertices of the triangle. Finally, each term plotted on the interior of the triangle gives the probability of some sample for which $x=3$, and $P(3; n, 3)$ is the sum of all these terms. These facts, together with the fact stated in section 2 that the binomial distributions $(p_2+p_3)^n$, $(p_3+p_1)^n$ and $(p_1+p_2)^n$ are plotted along the sides of the triangle, yield the formulas for $P(x; n, 3)$. They are:

$$(4.1)\quad \begin{cases} P(1; n, 3) = & (p_1^n + p_2^n + p_3^n), \\ P(2; n, 3) = & [(p_2+p_3)^n + (p_3+p_1)^n + (p_1+p_2)^n] - 2(p_1^n + p_2^n + p_3^n), \\ P(3; n, 3) = (p_1+p_2+p_3)^n - & [(p_2+p_3)^n + (p_3+p_1)^n + (p_1+p_2)^n] + (p_1^n + p_2^n + p_3^n). \end{cases}$$

Consider the expression for $P(2; n, 3)$. The term $[(p_2+p_3)^n+(p_3+p_1)^n+(p_1+p_2)^n]$ contains all terms plotted along the sides, including those at the vertices, of the triangle. Since each vertex belongs to two sides of the triangle, it is necessary to subtract twice the sum $(p_1^n+p_2^n+p_3^n)$ to obtain $P(2; n, 3)$. Similarly, in computing $P(3; n, 3)$, $(p_1+p_2+p_3)^n$ gives all the terms in the triangle. It is necessary to subtract $[(p_2+p_3)^n+(p_3+p_1)^n+(p_1+p_2)^n]$ in order to remove the terms on the sides; but since each vertex belongs to two sides, this step removes the terms at the vertices twice. It is thus necessary to add $(p_1^n+p_2^n+p_3^n)$ in order to obtain $P(3; n, 3)$.

The method which has been applied in the special case can be extended to the general case. Let s_j denote the sum of j of the probabilities $p_1, p_2, \cdots, p_k$, and let $S_j=\sum s_j^n$, where the summation extends over all possible combinations of j probabilities that can be selected from $p_1, p_2, \cdots, p_k$. Arguments similar to those employed in the special case lead to the following result, where the C's denote the binomial coefficients as usual:

$$(4.2)\quad \begin{cases} P(1; n, k) = & {}_{k-1}C_0\, S_1, \\ P(2; n, k) = & {}_{k-2}C_0 S_2 + (-1)\; {}_{k-1}C_1\, S_1, \\ \cdots\cdots\cdots \\ P(k-3; n, k) = & {}_3C_0 S_{k-3} - \cdots + (-1)^{k-4}{}_{k-1}C_{k-4} S_1, \\ P(k-2; n, k) = & {}_2C_0 S_{k-2} - {}_3C_1 S_{k-3} + \cdots + (-1)^{k-3}{}_{k-1}C_{k-3} S_1, \\ P(k-1; n, k) = & {}_1C_0 S_{k-1} - {}_2C_1 S_{k-2} + {}_3C_2 S_{k-3} - \cdots + (-1)^{k-2}{}_{k-1}C_{k-2} S_1, \\ P(k; n, k) = & S_k - {}_1C_1 S_{k-1} + {}_2C_2 S_{k-2} - {}_3C_3 S_{k-3} + \cdots + (-1)^{k-1}{}_{k-1}C_{k-1} S_1. \end{cases}$$

5. The special case in which all probabilities are the same. An important special case of the distribution arises when all p_i are equal, that is, when $p_i=1/k$ for $i=1, 2, \cdots, k$. If this value is inserted in the formulas (4.2), they take the following form after a certain amount of simplification:

$$(5.1)\qquad P(x; n, k) = k^{-n}{}_kC_x\Delta^x0^n, \qquad x = 1, 2, \cdots, k,$$

where

$$(5.2)\qquad \Delta^x0^n = x^n - {}_xC_1(x-1)^n + {}_xC_2(x-2)^n - \cdots (-1)^x0^n.$$

The distribution is well known in the form (5.1), and tables are available to assist in the calculations (see [2], pp. 19–20 and 60–61).

The sum of the probabilities in any probability distribution is 1; hence,

$$(5.3)\qquad \sum_{x=1}^{k} P(x; n, k) = 1.$$

Let the mean and standard deviation of the distribution $P(x; n, k)$, for fixed n and k, be denoted by $\bar{x}_k$ and σ_k respectively. From (5.1) we have

$$(k - x)P(x; n, k) = k(1 - 1/k)^n P(x; n, k - 1), \tag{5.4}$$

$$x(k - x)P(x; n, k) = xk(1 - 1/k)^n P(x; n, k - 1). \tag{5.5}$$

Summation of these equations with respect to x, using (5.3), gives

$$k - \bar{x}_k = k(1 - 1/k)^n,$$

$$k\bar{x}_k - (\bar{x}_k^2 + \sigma_k^2) = k(1 - 1/k)^n \bar{x}_{k-1} = (k - \bar{x}_k)\bar{x}_{k-1}.$$

Solution of these equations for $\bar{x}_k$ and σ_k^2 gives

$$\bar{x}_k = k[1 - (1 - 1/k)^n], \tag{5.6}$$

$$\sigma_k^2 = (k - \bar{x}_k)(\bar{x}_k - \bar{x}_{k-1}). \tag{5.7}$$

It follows that

$$\lim_{n\to\infty} \bar{x}_k = k; \qquad \lim_{n\to\infty} \sigma_k^2 = 0; \tag{5.8}$$

$$\begin{aligned} \bar{x}_k = \bar{x}_{k-1} &= 1, \qquad n = 1, \\ \bar{x}_k &> \bar{x}_{k-1}, \qquad n \geqq 2. \end{aligned} \tag{5.9}$$

Furthermore,

$$\begin{aligned} \lim_{n\to\infty} P(x; n, k) &= 0, \qquad 1 \leqq x \leqq k - 1, \\ &= 1, \qquad x = k. \end{aligned} \tag{5.10}$$

6. A second derivation and a recurrence relation. A second derivation can be given for the probability distribution (5.1), and it will be included here to bring out an important recurrence relation. First, it will be necessary to recall the following properties of the "differences of zero":

$$\Delta^x 0^n = 0, \quad x > n; \qquad \Delta^n 0^n = n!; \tag{6.1}$$

$$\Delta^x 0^{n+1} = x\Delta^{x-1} 0^n + x\Delta^x 0^n. \tag{6.2}$$

Proofs of these relations will be found in [3], p. 36.

Consider now the determination of $P(x; n, k)$ in the special case $p_i = 1/k$, $i = 1, 2, \cdots, k$. It is clear that $P(1; 1, k) = 1$, and that $P(x; 1, k) = 0$ for $x > 1$ since only one class can contain a variate if only one is drawn. Consider next the situation when $n = 2$. Now $P(1; 2, k) = 1/k$, for the two variates belong to the same class only under the following conditions: the first variate belongs to an arbitrary class, but the second variate belongs to the same class as the first one. The probability of the first event is 1, and that of the second is $1/k$; the probability that both events happen is thus their product $1/k$. Next, $P(2; 2, k) = (k-1)/k$ by the same argument, for the two variates belong to different classes only if the first belongs to one class and the second to some one of the remaining $(k-1)$ classes. As before, $P(x; 2, k) = 0$ for $x > 2$.

Suppose $P(x; n-1, k)$ has been determined for $x=1, 2, \cdots, k$. It will be shown that the probability distribution $P(x; n, k)$ also can be determined. The n variates of a sample occupy exactly x classes if and only if one of the following two situations occurs: (a) the first $(n-1)$ variates drawn occupy $(x-1)$ classes and the last variate drawn falls in one of the remaining $k-(x-1)$ classes; (b) the first $(n-1)$ variates occupy x classes, and the last variate falls in one of these same x classes. It follows that

$$(6.3) \quad P(x; n, k) = \frac{k-x+1}{k} P(x-1; n-1, k) + \frac{x}{k} P(x; n-1, k),$$

where it is convenient to define $P(x-1; n-1, k)=0$ for $x=1$. A complete induction can now be carried through to show that the probability distribution $P(x; n, k)$, $x=1, 2, \cdots, k$, can be calculated for every value of n.

Finally, it can be shown that the distribution is given by the formula (5.1). First, it can be shown from (5.2) and (6.1) that the formula (5.1) yields the values that have been calculated for $P(x; 1, k)$ and $P(x; 2, k)$. Assume next that the formula (5.1) holds for the distribution $P(x; n-1, k)$. Substitute from (5.1) in (6.3) and simplify by means of (6.2); it follows that the formula (5.1) holds also for the distribution $P(x; n, k)$. A complete induction then establishes the validity of (5.1) for all x, n, and k.

The second derivation of the probability distribution $P(x; n, k)$, based on the fundamental recurrence relation (6.3), is thus complete.

It may be observed that (5.4) is satisfied by the distributions $P(x; n, k)$.

The nature of the distribution and also the variation of the distribution as n increases can be observed by plotting the distributions for $n=1, 2, \cdots, 15$ from the values given in Table I.

TABLE I. Values of $P(x; n, 10)$ for $n=1, 2, \cdots, 15$
$p_i=1/10$, $i=1, 2, \cdots, 10$

n	x									
	1	2	3	4	5	6	7	8	9	10
1	1.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2	.1000	.9000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
3	.0100	.2700	.7200	.0000	.0000	.0000	.0000	.0000	.0000	.0000
4	.0010	.0630	.4320	.5040	.0000	.0000	.0000	.0000	.0000	.0000
5	.0001	.0135	.1800	.5040	.3024	.0000	.0000	.0000	.0000	.0000
6	.0000	.0028	.0648	.3276	.4536	.1512	.0000	.0000	.0000	.0000
7	.0000	.0006	.0217	.1764	.4234	.3175	.0605	.0000	.0000	.0000
8	.0000	.0001	.0070	.0857	.3175	.4022	.1693	.0181	.0000	.0000
9	.0000	.0000	.0022	.0392	.2102	.4001	.2794	.0653	.0036	.0000
10	.0000	.0000	.0007	.0172	.1286	.3451	.3556	.1361	.0163	.0004
11	.0000	.0000	.0002	.0073	.0746	.2714	.3870	.2156	.0419	.0020
12	.0000	.0000	.0001	.0031	.0417	.2001	.3794	.2885	.0808	.0062
13	.0000	.0000	.0000	.0013	.0227	.1409	.3457	.3447	.1305	.0143
14	.0000	.0000	.0000	.0005	.0121	.0959	.2983	.3794	.1863	.0273
15	.0000	.0000	.0000	.0002	.0064	.0636	.2472	.3930	.2436	.0460

7. Relation of sample size to the number of class intervals occupied. In certain applications it is necessary to know how many variates must be distributed in the k classes in order that, on the average, a certain proportion of them will contain at least one variate. The answer is obtained from (5.6). Set $\bar{x}=tk$, $0<t<1$. Then

$$n = \frac{\log (1 - t)}{\log (1 - 1/k)}. \tag{7.1}$$

If k is large, then a first approximation gives

$$n \sim - k \log (1 - t). \tag{7.2}$$

Thus, when k is large, the number of variates that must be distributed among the k classes so that, on the average, a certain proportion t of the classes shall contain at least one variate is approximately proportional to k. This result and the values in Table II are important in certain of the applications.

TABLE II. The Approximate Value of n for which $\bar{x}=tk$

t	.25	.50	.75	.90	.95	.99
n	$.29k$	$.69k$	$1.39k$	$2.30k$	$3.00k$	$4.61k$

An Elementary Test for Goodness of Fit and Remarks on the Chi-Square Test

8. Statement of the problem and remarks. Consider again the situation described in section 3 above: a sample of n variates is drawn under conditions such that the probability that each variate will belong to the ith class interval of the sample is p_i, $i=1, 2, \cdots, k$. The problem is to obtain a criterion in terms of which it can be determined whether the sample actually drawn is a likely or unlikely one on the basis of chance. It will be in order to make this very general statement of the problem more precise.

The probability that a sample with class frequencies $f_1, f_2, \cdots, f_k$ will be drawn is given by the expression $M(f_1, f_2, \cdots, f_k)$ in (2.1), but this information in itself does not solve the problem. The solution must involve in some form a comparison of the probabilities of all possible samples. The first step will be to determine the most likely sample and in general to order all samples in terms of the magnitudes of their probabilities. Then when a sample S is drawn, the following procedure will give an estimate of whether it is likely or unlikely. Divide the class of all samples into two classes A and B as follows: in class A place all samples with a greater probability than the given sample S, and in B place all samples with the same or a smaller probability than S. A comparison of the sum of the probabilities of all samples in class A with the sum of the probabilities of all samples in class B is then the criterion for determining whether the sample S is likely on the basis of chance. If the former sum is large in comparison with the latter (the definition of "large" is the specification of the significance level), we say the sample S is unlikely on the basis of chance.

One final remark is in order. The steps outlined involve only simple arithmetic, and they can be carried out without difficulty in case both the number of classes k and the size n of the sample S are small. It is at this point that the Chi-Square Test comes in: it supplies a means for approximating the classes A and B and the probabilities in question with ease, sufficient accuracy, and speed so that the test becomes a workable one in the applications.

9. The most probable sample. The class frequencies of that sample of n variates which is most likely, that is, which has a greater probability than any other single sample, are the exponents of the largest term in the expansion of $(p_1+p_2+\cdots+p_k)^n$. In certain cases at least it is possible to predict the result entirely on the basis of probability considerations. It is to be expected that, when $p_1, p_2, \cdots, p_k$ and n have values such that $np_1, np_2, \cdots, np_k$ are integers, the most likely sample is the one for which $f_1=np_1, f_2=np_2, \cdots, f_k=np_k$. The result is strictly an arithmetic one and an arithmetic proof can be given. Let

$$\frac{n!}{m_1!m_2!\cdots m_k!}p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k} \tag{9.1}$$

be the maximum term in the expansion of $(p_1+p_2+\cdots+p_k)^n$ or one of the maximum terms in case there are several. Consider the terms

$$\frac{n!}{f_1!f_2!m_3!\cdots m_k!}p_1^{f_1}p_2^{f_2}p_3^{m_3}\cdots p_k^{m_k}. \tag{9.2}$$

They are the terms of a binomial expansion multiplied by a constant, and they will have a maximum for $f_1=m_1$, $f_2=m_2$. Since the term (9.1) is equal to or greater than the two terms adjacent to it in the expansion (9.2), it follows that m_1 and m_2 are solutions of the inequalities $p_2(f_1+1)\geqq p_1f_2$, $p_1(f_2+1)\geqq p_2f_1$. Similar considerations apply to each pair of exponents. Thus a necessary condition that (9.1) be a maximum term is that $f_1=m_1, \cdots, f_k=m_k$ be a solution of

$$\begin{cases} p_j(f_i+1)\geqq p_if_j, \\ p_i(f_j+1)\geqq p_jf_i, \end{cases} \tag{9.3}$$

where i, j take on each distinct pair of values in $1, 2, \cdots, k$.

Now suppose that $np_1, np_2, \cdots, np_k$ are integers. Since their sum is n, they are the exponents of one of the terms in the expansion. Furthermore, $f_i=np_i$, $f_j=np_j$ is a solution of the inequalities (9.3). Finally, the necessary condition is also sufficient in this case. Let the term

$$\frac{n!}{(np_1)!(np_2)!\cdots(np_k)!}p_1^{np_1}p_2^{np_2}\cdots p_k^{np_k} \tag{9.4}$$

be divided by any other term in the expansion $(p_1+p_2+\cdots+p_k)^n$, the latter term being conveniently written in the form

$$\frac{n!}{(np_1+t_1)!(np_2+t_2)!\cdots(np_k+t_k)!}p_1^{np_1+t_1}p_2^{np_2+t_2}\cdots p_k^{np_k+t_k} \tag{9.5}$$

where $t_1+t_2+\cdots+t_k=0$. Inspection shows that a quotient greater than 1 is obtained. The proof is thus complete that (9.4) is the maximum term, and that $np_1, np_2, \cdots, np_k$ are the class frequencies of the most likely sample.

It should be observed that the term (9.4) is the term which occurs at the mean of the multinomial distribution $M(f_1, f_2, \cdots, f_k)$ (see section 2 above).

The situation in case $np_1, np_2, \cdots, np_k$ are not all integers is not easily described. The maximum term still occurs near the mean of the distribution, but there may be several equal terms. We shall not explore these cases further.

10. An elementary test of goodness of fit. The remainder of the discussion will be restricted to the special case in which $p_i=1/k$ for $i=1, 2, \cdots, k$. It will be instructive to consider first the case $k=3$, $n=6$. The numbers are small enough in this case so that the probability of each possible sample can be calculated. The results are summarized in Table III. The first column gives all the

TABLE III. Probabilities of Samples and the Order of their Likelihood in the Case $k=3$, $n=6$

Frequencies f_1, f_2, f_3	Probability* of Drawing the Sample f_1, f_2, f_3	Probability* of Drawing a Sample with Frequencies f_1, f_2, f_3 in Some Order	Order of Likelihood	$(f_1-2)^2+(f_2-2)^2+(f_3-2)^2$
2, 2, 2	90	90	1	0
3, 2, 1	60	360	2	2
4, 1, 1	30	90	3	6
3, 3, 0	20	60	4	6
4, 2, 0	15	90	5	8
5, 1, 0	6	36	6	14
6, 0, 0	1	3	7	24

* Each entry in this column is to be divided by $3^6=729$.

distinguishable partitions, apart from order, of the integer 6 into three positive or zero integers. The sets of numbers in this column, together with all of their distinguishable permutations, represent all of the possible samples. The second column gives the probability that a sample will be drawn in which the frequencies in the class intervals 1, 2, 3 are the numbers f_1, f_2, f_3 respectively in the first column. The third column is obtained by multiplying the second column by the number of samples which have the frequencies f_1, f_2, f_3 in some order, that is, by the number of distinguishable permutations of the integers f_1, f_2, f_3. The fourth column gives the order of likelihood of the samples with frequencies f_1, f_2, f_3, namely, the order in the sense of decreasing magnitude of the numbers in the second column. Finally, the last column gives what in the general case would be $\sum_1^k(f_i-n/k)^2$.

It will be observed that the sample with the greatest probability is the one with class frequencies $np_i=n/k=2$ as shown in the last section.

Next, one's attention is caught by the striking fact that the numbers in the last column can almost be used to obtain the order of likelihood of the samples. This result is much less surprising if a plot of this multinomial expansion of the kind described in section 2 is examined. Consider an expanding sphere with center at the mean (and maximum) of the distribution, and recall that along each line f_i = constant of the triangle the distribution is a symmetrical binomial distribution multiplied by a constant. If the expanding sphere is followed as it passes through the successive lattice points of the grid, each new lattice point included being examined in relation to the last ones included, it will be observed that as $(f_1-2)^2+(f_2-2)^2+(f_3-2)^2$ increases, the probability of the sample f_1, f_2, f_3 decreases. It is nevertheless true that in one case there are samples with different probabilities in spite of the fact that they all have the same value of $(f_1-2)^2+(f_2-2)^2+(f_3-2)^2$.

Unfortunately the relations described in the last paragraph do not hold in general. The case of the binomial distribution, however, is altogether special, for a sample f_1, f_2 is more likely than a sample f_1', f_2' if and only if $(f_1-n/2)^2+(f_2-n/2)^2<(f_1'-n/2)^2+(f_2'-n/2)^2$ for every n. For $k=3$, however, the situation changes even for $n=12$. Table IV summarizes the results in this case.

TABLE IV. Probabilities of Samples and the Order of their Likelihood in the Case $k=3$, $n=12$

Frequencies f_1, f_2, f_3	Probability* of Drawing the Sample f_1, f_2, f_3	Probability* of Drawing a Sample with Frequencies f_1, f_2, f_3 in Some Order	Order of Likelihood	$(f_1-4)^2+(f_2-4)^2+(f_3-4)^2$
4, 4, 4	34,650	34,650	1	0
5, 4, 3	27,720	166,320	2	2
6, 3, 3	18,480	55,440	3	6
5, 5, 2	16,632	49,896	4	6
6, 4, 2	13,860	83,160	5	8
7, 3, 2	7,920	47,520	6	14
6, 5, 1	5,544	33,264	7	14
7, 4, 1	3,960	23,760	8	18
8, 2, 2	2,970	8,910	9	24
8, 3, 1	1,980	11,880	10	26
6, 6, 0	924	2,772	11	24
7, 5, 0	792	4,752	12	26
9, 2, 1	660	3,960	13	38
8, 4, 0	495	2,970	14	32
9, 3, 0	220	1,320	15	42
10, 1, 1	132	396	16	54
10, 2, 0	66	396	17	56
11, 1, 0	12	72	18	74
12, 0, 0	1	3	19	96

* Each entry in this column is to be divided by $3^{12}=531,441$.

It may be emphasized finally that a table such as Table IV provides a solution to the problem stated in section 8. Suppose, that to test the hypothesis $p_i = 1/3$ for $i = 1, 2, 3$, a sample of 12 variates is drawn, and that frequencies of 10, 1, 1 are obtained in the three classes. Reference to Table IV shows that this sample is 16th in the order of likelihood. The sum of the last four numbers in the third column of Table IV is 867. Thus, a sample as unlikely as the one drawn (the occurrence of an event in class B of section 8) will be expected only 867 times out of 531,441 trials. On any of the usual significance levels, the drawing of a sample with frequencies 10, 1, 1 is so unlikely on the basis of chance that the hypothesis $p_i = 1/3$, $i = 1, 2, 3$, would certainly be rejected.

The third and fourth columns of Table IV define a probability distribution. The random variable takes on the values listed in the fourth column, and the probability that it takes on any value is given by the corresponding entry in the third column. This probability distribution has been obtained by combining the terms of the multinomial distribution in a certain way. The probability distribution $P(x; n, k)$ also was obtained from the multinomial distribution, but by combining its terms in a different manner.

It may be observed that the probability of drawing the most likely sample is small. Apply the test in the case in which class A contains only the most likely sample, and class B contains all the rest. Table IV shows that the most likely sample would be drawn only 34,650 times out of 531,441 samples. Furthermore, an application of Stirling's Formula to (9.4) shows that the probability of the most likely sample has the limit zero as n tends to infinity; that is, the probability of drawing a sample in class B tends to 1.

The construction of tables such as Table IV becomes laborious for larger values of n and k, but they can be constructed when required. Their chief usefulness would be in the cases of small n and k for which the Chi-Square Test does not give good results. The labor involved in the computation would be immensely greater in case the probabilities $p_1, p_2, \cdots, p_k$ are not all equal. Many of the problems encountered in testing the goodness of fit of a curve can be reduced, however, to the case of equal probabilities by adjusting the class intervals so that the probabilities become equal.

11. The Chi-Square Test. After everything has been said, however, it must be emphasized that it would be impractical to construct any set of tables such as Tables III and IV that would serve for all purposes. Fortunately, however, an approximate solution of the problem was found long ago that is adequate for most purposes. Tables which give the distribution of a quantity known as Chi-Square supply the answer.

Briefly, the Chi-Square solution is the following. A quantity χ^2 is defined by

$$\sum_{i=1}^{k} \frac{(f_i - np_i)^2}{np_i} = \chi^2. \tag{11.1}$$

It should be observed that χ^2 is proportional to the quantity tabulated in the

last column of Tables III and IV when $p_1=p_2=\cdots=p_k=1/k$. Next, the multinomial distribution is approximated by a continuous probability distribution. The procedure in testing whether a given sample is likely on the basis of chance is the following: The value of χ^2 for the given sample is determined from (11.1). The totality of all possible samples is divided into two classes A' and B'; those samples with a value of χ^2 smaller than the given sample are placed in class A', and the remainder are placed in class B'. Reference to the tables of χ^2 then gives an approximate value for the probability that a sample chosen at random will fall in class A' or B'.

It will be observed that the Chi-Square Test approximates the solution which was given in section 10 to the problem stated in section 8. The first difference arises from the fact that the classes A' and B' in the Chi-Square Test are not the same as the classes A and B defined in section 8. The results in section 10, however, indicate that the classes A', B' differ but slightly from the classes A, B. Further approximations arise from replacing the discrete multinomial distribution by a continuous distribution. A full discussion of the latter approximations will be found in [4].

The Distribution of $(y+\cdots+y)$ in Terms of the Distribution of y

12. Statement of the problem. Consider again a random variable y (see section 2), but assume this time that it takes on the values $0, 1, 2, \cdots, k$ with probabilities $p_0, p_1, \cdots, p_k$. The problem is to determine the distribution of the sum of the values y for n variates drawn at random, and also the properties of this distribution.

Consider the probability of drawing n variates for which the sum of the y's is x. The term

$$M(f_0, f_1, \cdots, f_k) = \frac{n!}{f_0!f_1!\cdots f_k!} p_0^{f_0} p_1^{f_1} \cdots p_k^{f_k}, \tag{12.1}$$

in which $f_0+f_1+\cdots+f_k=n$ and $0f_0+1f_1+\cdots+kf_k=x$, is the probability of one of the ways in which a sum x can be obtained. The term $M(f_0, f_1, \cdots, f_k)$ is a term in the multinomial expansion $(p_0+p_1+\cdots+p_k)^n$. Furthermore, the total probability that the n variates drawn will have the sum of the y values equal to x is the sum of all the multinomial terms (12.1) for which $0f_0+1f_1+\cdots+kf_k=x$. It is well known that this probability is exactly the coefficient of t^x in the expansion of $(p_0+p_1t+p_2t^2+\cdots+p_kt^k)^n$, the function $p_0+p_1t+\cdots+p_kt^k$ being the generating function of probability (see [5], pp. 16–20, 55–56 and [6], p. 60).

The problem thus reduces to the determination and study of the coefficients in $(p_0+p_1t+\cdots+p_kt^k)^n$ when this function is expressed as a polynomial in t. It is desirable to generalize the problem as follows: determine the coefficients in $(a_0+a_1t+\cdots+a_kt^k)^n$ and $(a_0+a_1t+\cdots+a_kt^k+\cdots)^n$ when these functions are expressed as a polynomial in t and a power series in t respectively. There are no restrictions on the coefficients $a_0, a_1, \cdots, a_k, \cdots$.

13. Determination of the coefficients. Set

$$(a_0 + a_1 t + \cdots)^n = \sum_{x=1}^{\infty} C(x; n; a_0, a_1, \cdots) t^x. \tag{13.1}$$

The first of three methods that can be applied for the determination and study of $C(x; n; a_0, a_1, \cdots)$ is based on combinatorial considerations. The coefficient $C(x; n; a_0, a_1, \cdots)$ is the sum of all products formed by taking n symbols from the set $a_0, a_1, \cdots$ in all possible orders, repetitions being allowed, but such that the sum of the subscripts equals x. The coefficient of $a_0^{n_0} a_1^{n_1} a_2^{n_2} \cdots$ is the multinomial coefficient

$$\binom{n}{n_1, n_2, \cdots} = \frac{n!}{n_0! n_1! n_2! \cdots}, \qquad n_0 + n_1 + n_2 + \cdots = n. \tag{13.2}$$

Then

$$C(x; n; a_0, a_1, \cdots) = \sum_{s=x} \binom{n}{n_1, n_2, \cdots} a_0^{n_0} a_1^{n_1} a_2^{n_2} \cdots, \tag{13.3}$$

where the summation extends over all values of $n_0, n_1, n_2, \cdots$ such that $s \equiv 0n_0 + 1n_1 + 2n_2 + \cdots = x$ and $n_0 + n_1 + n_2 + \cdots = n$. For example,

$$\begin{aligned} C(4; n; a_0, a_1, \cdots) = {} & \binom{n}{4} a_0^{n-4} a_1^4 + \binom{n}{2, 1} a_0^{n-3} a_1 a_2 \\ & + \binom{n}{1, 1} a_0^{n-2} a_1 a_3 + \binom{n}{2} a_0^{n-2} a_2^2 \\ & + \binom{n}{1} a_0^{n-1} a_4. \end{aligned} \tag{13.4}$$

The second method that can be used to determine $C(x; n; a_0, a_1, \cdots)$ is based on the theory of functions. If $a_0 + a_1 t + \cdots$ represents an analytic function $f(t)$, then

$$C(x; n; a_0, a_1, \cdots) = \frac{1}{x!} \frac{d^x [f(t)]^n}{dt^x}\bigg|_{t=0}. \tag{13.5}$$

This result is true formally in any case. In many cases this method of determining $C(x; n; a_0, a_1, \cdots)$ is very powerful. For example, if y is distributed according to the Poisson Distribution $e^{-m}[1 + m + m^2/2! + \cdots]$, then an application of (13.5) to the generating function $e^{-m}[1 + mt + m^2 t^2/2! + \cdots] = e^{-m+mt}$ shows that $y + \cdots + y$ (n summands) is distributed according to the Poisson Distribution $e^{-mn}[1 + mn + m^2 n^2/2! + \cdots]$. A similar application of (13.5) to $1 + t + t^2 + \cdots = 1/(1-t)$ shows that

$$C(x; n; 1_0, 1_1, \cdots) = \frac{n(n+1)\cdots(n+x-1)}{x!}. \tag{13.6}$$

The third method for the determination and study of the coefficients $C(x; n; a_0, a_1, \cdots)$ is based on recurrence relations and difference equations. Since $(a_0+a_1t+\cdots)_n=[\sum_1^\infty C(x; n-1; a_0, a_1, \cdots)t^x](a_0+a_1t+\cdots)$, then

$$C(x; n; a_0, a_1, \cdots) = \sum_{i=0}^{x} a_i C(x-i; n-1; a_0, a_1, \cdots). \tag{13.7}$$

This relation is the generalization of the relation that holds in the Pascal Triangle. A generalized Pascal Triangle can be constructed step-wise; the rows give the coefficients in $(a_0+a_1t+\cdots)^n$ successively for $n=1, 2, \cdots$:

$$\begin{cases} C(0; 1; a_0, a_1, \cdots) & C(1; 1; a_0, a_1, \cdots) & \cdots \\ C(0; 2; a_0, a_1, \cdots) & C(1; 2; a_0, a_1, \cdots) & \cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots \end{cases} \tag{13.8}$$

From another point of view, (13.7) is a set of difference equations by means of which the terms in the nth row of the triangle can be obtained successively. For $x=0$, (13.7) is $C(0; n; a_0, a_1, \cdots)=a_0C(0; n-1; a_0, a_1, \cdots)$. The solution of this difference equation in n, subject to the initial condition $C(0; 1; a_0, a_1, \cdots)=a_0$, is $C(0; n; a_1, a_2, \cdots)=a_0^n$. For $x=1$, (13.7) is $C(1; n; a_0, a_1, \cdots)-a_0C(1; n-1; a_0, a_1, \cdots)=a_1C(0; n-1; a_0, a_1, \cdots)$, a difference equation in n with the right-hand member a known function. The solution which satisfies the initial condition $C(1; 1; a_0, a_1, \cdots)=a_1$ is $C(1; n; a_0, a_1, \cdots)=na_0^{n-1}a_1$. If we assume that $C(0; n; a_0, a_1, \cdots), \cdots, C(x-1; n; a_0, a_1, \cdots)$ have been determined, then (13.7) gives the difference equation in n

$$\begin{aligned} &C(x; n; a_0, a_1, \cdots) - a_0C(x; n-1; a_0, a_1, \cdots) \\ &\qquad = \sum_{i=1}^{x} a_i C(x-i; n-1; a_0, a_1, \cdots), \end{aligned} \tag{13.9}$$

in which the right-hand member is a known function, for the determination of $C(x; n; a_0, a_1, \cdots)$. A complete induction then shows that all coefficients $C(x; n; a_0, a_1, \cdots)$ can be determined by solving successively the difference equations (13.9) subject to the initial condition $C(x; 1; a_0, a_1, \cdots)=a_x$.

14. Properties of the coefficients in the finite case. If $a_k\neq 0$, $a_i=0$ for $i>k$, then $C(x; n; a_0, \cdots, a_k)=0$ for $x>nk$. More precisely, it can be shown that $C(x; n; a_0, \cdots, a_k)$ has the factor n for $x>0$, the factor $n(n-1)$ for $x>k$, the factor $n(n-1)(n-2)$ for $x>2k$, and so on.

If the relation

$$C(x; n; a_0, \cdots, a_k) = C(nk-x; n; a_0, \cdots, a_k), \quad x=0, 1, \cdots, nk, \tag{14.1}$$

holds for $n=1$, that is, if $a_x=a_{k-x}$ for $x=0, 1, \cdots, k$, then the same relation holds for every value of n. For, because of the assumed symmetry, $[t^k(a_0+a_1t^{-1}+a_2t^{-2}+\cdots+a_kt^{-k})]^n$ equals $t^{nk}\sum_0^{nk}C(x; n; a_0, \cdots, a_k)t^{-x}$ $=\sum_0^{nk}C(x; n; a_0, \cdots, a_k)t^{nk-x}$ on the one hand and $\sum_0^{nk}C(x; n; a_0, \cdots, a_k)t^x$

on the other. Equation (14.1) then follows by equating coefficients of t^x in the last two expressions written.

By multiplying the expansions of $(a_0+a_1t+\cdots+a_kt^k)^m$ and $(a_0+a_1t+\cdots+a_kt^k)^n$, it is found that

$$C(x; m+n; a_0, \cdots, a_k) = \sum_{i+j=x} C(i; m; a_0, \cdots, a_k)C(j; n; a_0, \cdots, a_k). \tag{14.2}$$

If $m=n$, $x=nk$, and the coefficients satisfy the symmetry relation (14.1), then

$$C(nk; 2n; a_0, \cdots, a_k) = \sum_{x=0}^{nk} [C(x; n; a_0, \cdots, a_k)]^2. \tag{14.3}$$

This equation in the case $k=1$ states a familiar property of the binomial coefficients.

15. Some special cases. The most interesting special case results from setting $a_i=1$, $i=0, 1, \cdots, k$, and $a_i=0$ for $i>k$. The values of $C(x; n; 1_0, \cdots, 1_k)$ for $k=1, 2, 3$ and $x=0, \cdots, 6$ are given in Table V, where the symbols denote binomial or multinomial coefficients as explained by (13.2).

TABLE V. Values of $C(x; n; 1_0, \cdots, 1_k)$ for $k=1, 2, 3$ and $x=0, 1, \cdots, 6$

x	$k=1$	$k=2$	$k=3$
0	$\binom{n}{0}$	$\binom{n}{0}$	$\binom{n}{0}$
1	$\binom{n}{1}$	$\binom{n}{1}$	$\binom{n}{1}$
2	$\binom{n}{2}$	$\binom{n}{2}+\binom{n}{1}$	$\binom{n}{2}+\binom{n}{1}$
3	$\binom{n}{3}$	$\binom{n}{3}+\binom{n}{1,1}$	$\binom{n}{3}+\binom{n}{1,1}+\binom{n}{1}$
4	$\binom{n}{4}$	$\binom{n}{4}+\binom{n}{2,1}+\binom{n}{2}$	$\binom{n}{4}+\binom{n}{2,1}+\binom{n}{2}+\binom{n}{1,1}$
5	$\binom{n}{5}$	$\binom{n}{5}+\binom{n}{3,1}+\binom{n}{1,2}$	$\binom{n}{5}+\binom{n}{3,1}+\binom{n}{1,1}+\binom{n}{2,1}+\binom{n}{1,1}$
6	$\binom{n}{6}$	$\binom{n}{6}+\binom{n}{4,1}+\binom{n}{2,2}+\binom{n}{3}$	$\binom{n}{6}+\binom{n}{4,1}+\binom{n}{2,2}+\binom{n}{3}+\binom{n}{1,1,1}+\binom{n}{3,1}+\binom{n}{2}$

For the case $k=1$ the triangle (13.8) is the familiar Pascal Triangle. The generalized Pascal Triangles shown in Table VI result in the cases $k=2$ and $k=3$.

TABLE VI. Generalized Pascal Triangles for $C(x; n; 1_0, \cdots, 1_k)$ in the Cases $k=2, 3$

n	$k=2$	$k=3$
0	1	1
1	1 1 1	1 1 1 1
2	1 2 3 2 1	1 2 3 4 3 2 1
3	1 3 6 7 6 3 1	1 3 6 10 12 12 10 6 3 1
4	1 4 10 16 19 16 10 4 1	1 4 10 20 31 40 44 40 31 20 10 4 1
.	. . . . .	. . . . .

The recurrence relations (13.7) supply a simple means of writing out these triangles: for $k=2$ ($k=3$) any number in the triangle is equal to the sum of three (four) numbers in the row above it, namely, the one above it and the two (three) to its left.

By examining the solutions of the difference equations (13.9), or by applying properly the argument that led to (13.6), it can be shown that

$$C(x; n; 1_0, \cdots, 1_k) = \frac{n(n+1)\cdots(n+x-1)}{x!}, \qquad x = 0, 1, \cdots, k.$$

Equation (13.1) with $t=1$ gives $(k+1)^n = \sum_0^{nk} C(x; n; 1_0, \cdots, 1_k)$.

16. Applications to probability distributions. The solution of the problem stated in section 12 can now be described. If y is a random variable which takes on the values $0, 1, \cdots, k$ with probabilities $p_0, p_1, \cdots, p_k$, then $x = y + \cdots + y$ (n summands) is a random variable which takes on the values $0, 1, \cdots, nk$, and the probability that it takes on the value x is $C(x; n; p_0, \cdots, p_k)$. Let $\bar{x}_n$ and σ_n denote, for fixed k, the mean and standard deviation of the distribution $C(x; n)$, the latter being written for simplicity in place of $C(x; n; p_0, \cdots, p_k)$. Then*

$$\bar{x}_n = n\bar{x}_1 \tag{16.1}$$

$$\sigma_n^2 = n\sigma_1^2. \tag{16.2}$$

From the definition of $\bar{x}_n$ and (14.2) it follows that

$$\begin{aligned}\bar{x}_n = \sum_{x=0}^{nk} xC(x; n) &= \sum_{x=0}^{nk} \sum_{i+j=x} (i+j)p_i C(j; n-1) \\ &= \sum_{i=0}^{k} ip_i \sum_{j=0}^{(n-1)k} C(j; n-1) + \sum_{i=0}^{k} p_i \sum_{j=0}^{(n-1)k} jC(j; n-1) \\ &= \bar{x}_1 \cdot 1 + 1 \cdot \bar{x}_{n-1}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sigma_n^2 + \bar{x}_n^2 = \sum_{x=0}^{nk} x^2 C(x; n) &= \sum_{x=0}^{nk} \sum_{i+j=x} (i+j)^2 p_i C(j; n-1) \\ &= \sum i^2 p_i \sum C(j; n-1) + 2\sum ip_i \sum jC(j; n-1) \\ &\quad + \sum p_i \sum j^2 C(j; n-1) \\ &= (\sigma_1^2 + \bar{x}_1^2)\cdot 1 + 2\bar{x}_1 \cdot \bar{x}_{n-1} + 1\cdot(\sigma_{n-1}^2 + \bar{x}_{n-1}^2); \\ \sigma_n^2 &= \sigma_1^2 + \sigma_{n-1}^2.\end{aligned}$$

From these relations (16.1) and (16.2) follow by induction.

Consider an example. Suppose that n symmetrical dice are thrown, each die having $k+1$ faces numbered $0, 1, \cdots, k$. The problem is to find the probability

* The statement concerning the standard deviation was added at the suggestion of the referee.

distribution of the sum x of the values which appear on the n dice. In this case $p_i=1/(k+1)$. From (13.3) it follows that

$$C(x; n; p_0, \cdots, p_k) = \frac{C(x; n; 1_0, \cdots, 1_k)}{(k+1)^n}, \qquad p_i = \frac{1}{k+1},$$

for $x=0, 1, \cdots, nk$. The results in section 14 show that the probability distribution of x is symmetrical. According to the results in section 15, the distribution is similar in many respects to the binomial distribution. It follows both from the symmetry and from (16.1) that $\bar{x}=nk/2$; and from (16.2) that $\sigma^2=nk(k+2)/12$.

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THE FERMAT AND HESSIAN POINTS OF A TRIANGLE

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1. Introduction. The properties of the triangle which are presented here are proved analytically in the Morley *Inversive Geometry.** The properties seem so noteworthy as to demand straightforward proofs by the methods of modern geometry, since, to the author's knowledge, the relations are neither mentioned nor proved in any of the standard works on the subject. The paper will confine itself to the discussion of those properties which are less familiar, it being assumed that the reader is acquainted with certain terms and theorems which are standard topics in modern geometry.

2. The Fermat points. The first points to be considered are those known as the Fermat points.† These may be defined in the following manner:

Consider a triangle $A_1A_2A_3$ with equilateral triangles $A_2A_3P_1$, $A_3A_1P_2$, $A_1A_2P_3$ described externally on its sides. Then it is known that the lines A_1P_1, A_2P_2, A_3P_3 are concurrent at a point F, which is known as a Fermat point of the triangle. Similarly, if equilateral triangles $A_2A_3P_1'$, $A_3A_1P_2'$, $A_1A_2P_3'$ are constructed internally on the sides of $A_1A_2A_3$ ($A_1A_2A_3$ being, in this case, non-equilateral) the lines A_1P_1', A_2P_2', A_3P_3' are concurrent at a second Fermat point, F'.

* Morley, Frank, and Morley, F. V.: Inversive Geometry, Ginn and Co., 1933.

† Also known as *isogonic centers*. Johnson, R. A.: Modern Geometry. Houghton Mifflin Co., 1929.

The Fermat points are characterized by the fact that the segments A_iP_i are equal and intersect at angles of 60°, and the segments A_iP_i' are equal and intersect at angles of 60°. The points are named from the fact that F is (for triangles having no angle exceeding 120°) the solution to Fermat's problem: to find a point the sum of whose distances from the vertices of a triangle is a minimum.

Let M_1 be the midpoint of A_2A_3 and let N_1 and N_1' be the centers of the equilateral triangles $A_2A_3P_1$ and $A_2A_3P_1'$ respectively. The centroid, G, of $A_1A_2A_3$ lies on A_1M_1, and $A_1G=2GM_1$. Then, since M_1 is the midpoint of P_1P_1', it is clear that G is also the centroid of $A_1P_1P_1'$, and therefore divides the median of this triangle drawn from P_1 in the ratio 2:1. Thus since $P_1N_1' = 2N_1'P_1'$, GN_1' is parallel to A_1P_1'. Similarly it may be shown that GN_1 is parallel to A_1P_1.

Let N_1G and $N_1'G$ be produced to intersect A_1P_1' and A_1P_1 at R_1 and R_1' respectively. Then, since G is the centroid of $A_1P_1P_1'$, N_1R_1 and $N_1'R_1'$ are bisected at G, and circles may be described with centers at G and diameters N_1R_1 and $N_1'R_1'$. Let these circles respectively intersect A_1P_1' and A_1P_1 again in points E' and E. Now, since $N_1'E$ is perpendicular to A_1P_1 and N_1E' is perpendicular to A_1P_1', it follows that the circles circumscribing triangles $A_2A_3P_1$ and $A_2A_3P_1'$ pass, respectively, through E and E'. From previous considerations, therefore, we see that E and E' coincide with the Fermat points F and F' of $A_1A_2A_3$. In addition, since GN_1 and GN_1' are parallel to A_1F and A_1F', it follows that

$$\sphericalangle FA_1F' = \sphericalangle GFA_1 = \sphericalangle A_1F'G = \tfrac{1}{3} \sphericalangle FGF', \pmod{\pi}.$$

Summarizing we have: *GN_i and GN_i' are parallel to A_iP_i and A_iP_i' respectively; the points N_i and N_i' are the vertices of two equilateral triangles (with sides perpendicular to A_iF and A_iF') inscribed in the circles with centers at G and radii equal to GF' and GF respectively; and*

$$\sphericalangle FA_iF' = \sphericalangle GFA_i = \sphericalangle A_iF'G = \tfrac{1}{3} \sphericalangle FGF', \pmod{\pi}.$$

3. The Hessian points. Closely associated with the Fermat points are their isogonal conjugates, known as the Hessian points.* Let these points be designated by H and H', H being the isogonal conjugate of F. The sides of the pedal triangle $H_1H_2H_3$ of H are perpendicular to A_1F, A_2F, A_3F; the pedal triangle is therefore equilateral, its center being the midpoint, Q, of HF. Similarly, the pedal triangle $H_1'H_2'H_3'$ of H' is equilateral, its sides being perpendicular to A_1F', A_2F', A_3F', and its center at Q', the midpoint of $H'F'$.

Since the sides of the equilateral triangles $N_1N_2N_3$ and $H_1H_2H_3$ are respectively perpendicular to A_1F, A_2F, A_3F, the two triangles are homothetic. Let K be the center of similitude. Since, in the similitude, the center G of $N_1N_2N_3$ corresponds to the center Q of $H_1H_2H_3$ and the point H (the point of concurrence of perpendiculars at H_1, H_2, H_3 to the sides of $A_1A_2A_3$) corresponds to the circumcenter, O, of $A_1A_2A_3$, it follows that K must lie on OH and GQ, and that HF is parallel to the Euler line OG. Furthermore, since the distances of N_1, N_2, N_3 from the corresponding sides of $A_1A_2A_3$ are proportional to these sides, the

* Also known as *isodynamic points.* Johnson, *ibid.*

distances from K to the sides of $A_1A_2A_3$ are also proportional to these sides. K is therefore the symmedian point of $A_1A_2A_3$.* Similar considerations show that $H_1'H_2'H_3'$ is homothetic to $N_1'N_2'N_3'$, the center of similitude again being the symmedian point K. Thus K lies on OH' and on GQ', and $H'F'$ is parallel to OG. Now it is known that the points O, H, K, H' of any triangle form a harmonic set, so that G, Q, K, Q' must also be harmonic, from which it is easy to see that G, F, H' are collinear, as also are G, F', H and F, F', K. Summarizing the results we have:

In any triangle the line joining a Fermat point with the corresponding Hessian point is parallel to the Euler line; the line joining a Fermat point with the non-corresponding Hessian point passes through the centroid; the line joining the Fermat points intersects the line joining the Hessian points at the symmedian point.

4. Distance relations. Let a designate the distance GF' and let b designate the distance GF. Then

$$A_1P_1 = A_2P_2 = A_3P_3 = 3GN_1 = 3a,$$
$$A_1P_1' = A_2P_2' = A_3P_3' = 3GN_1' = 3b.$$

Applying the law of cosines to triangle $A_1A_2P_1$ we find

$$\begin{aligned} 9a^2 &= a_1^2 + a_3^2 - 2a_1a_3 \cos(\alpha_2 + \pi/3) \\ &= a_1^2 + a_3^2 - a_1a_3 \cos\alpha_2 + a_1a_3\sqrt{3} \sin\alpha_2 \\ &= \tfrac{1}{2}(2a_1^2 + 2a_3^2 - 2a_1a_3 \cos\alpha_2) + a_1a_3\sqrt{3} \sin\alpha_2 \\ &= \tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2) + 2\sqrt{3}\,\Delta, \end{aligned}$$

where Δ is the area of $A_1A_2A_3$. Similarly

$$9b^2 = \tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2) - 2\sqrt{3}\,\Delta.$$

Now, let K_1 be the foot of the perpendicular from K to A_2A_3. Then†

$$KK_1 = 2a_1\Delta/(a_1^2 + a_2^2 + a_3^2).$$

Also

$$M_1N_1 = a_1\sqrt{3}/6.$$

Thus, if R is the ratio of similitude of $H_1H_2H_3$ to $N_1N_2N_3$,

$$\begin{aligned} R &= KK_1/(M_1N_1 + KK_1) \\ &= [2\Delta/(a_1^2 + a_2^2 + a_3^2)]/[\sqrt{3}/6 + 2\Delta/(a_1^2 + a_2^2 + a_3^2)] \\ &= 2\sqrt{3}\,\Delta/[\tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2) + 2\sqrt{3}\,\Delta] \\ &= 2\sqrt{3}\,\Delta/9a^2. \end{aligned}$$

* Johnson, *ibid.*, Arts. 342, 490. K must be either the symmedian point or an exsymmedian point; but no one of the exsymmedian points lies on OHH'.

† Johnson, *ibid.*, Art. 342.

Similarly, if R' is the ratio of similitude of $H_1' H_2' H_3'$ to $N_1' N_2' N_3'$,

$$R' = -2\sqrt{3}\,\Delta/9b^2.$$

Then, from similar triangles,

$$GH/GF' = HQ/H'Q' = HK/H'K = (HK/OK)/(H'K/OK) = R/R' = b^2/a^2,$$

whence

$$GH = b^2/a.$$

Similarly

$$GH' = a^2/b.$$

Therefore: *H is the inverse of F' with respect to the circle circumscribing $N_1' N_2' N_3'$ and H' is the inverse of F with respect to the circle circumscribing $N_1N_2N_3$.*

Since H and F are isogonal conjugates

$$\sphericalangle A_3A_1H = \sphericalangle P_1A_1A_2.$$

and

$$\sphericalangle A_1A_3H = \sphericalangle FA_3A_2 = \sphericalangle A_1P_1A_2,$$

so that triangles A_1HA_3 and $A_1A_2P_1$ are similar, and $A_1H/a_2 = a_3/3a$, whence

$$A_1H = a_2a_3/3a.$$

For like reasons

$$A_1H' = a_2a_3/3b,$$

and it follows that

$$A_1H/A_1H' = b/a.$$

Since the same reasoning holds for the other vertices we have: *The circumcircle is the locus of a point whose distances from the Hessian points are in the constant ratio b/a.*

And since

$$GH/GH' = b^3/a^3$$

we have: *The ratio of the distances of the Hessian points to the centroid equals the cube of the ratio of the distances from the Hessian points to any vertex.*

Also, $\sphericalangle HA_iH' = \sphericalangle FA_iF'$, whence: *The angle subtended at the Hessian points by any vertex equals one third the angle (mod π) subtended at the Hessian points by the centroid.*

These relations are noteworthy since they lead to a geometric method for the solution of a cubic equation.*

* Morley and Morley, *ibid.*, Art. 42.

The distance between the Hessian points may be expressed in terms of the sides of the triangle by considering the triangles $N_1N_1'G$ and $H'HA_1$. These triangles are similar since $\sphericalangle N_1'GN_1 = \sphericalangle HA_1H'$, and $A_1H/A_1H' = b/a = GN_1'/GN_1$. Therefore

$$HH'/N_1N_1' = A_1H/b \quad \text{and} \quad HH'/N_1N_1' = A_1H'/a,$$

whence

$$(HH')^2/(N_1N_1')^2 = (A_1H)(A_1H')/ab = a_2^2a_3^2/9a^2b^2,$$

or

$$HH'/N_1N_1' = a_2a_3/3ab.$$

Hence, since $N_1N_1' = a_1\sqrt{3}/3$,

$$HH' = a_1a_2a_3\sqrt{3}/9ab.$$

Substituting

$$3a = [\tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2) + 2\sqrt{3}\,\Delta]^{1/2},$$
$$3b = [\tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2) - 2\sqrt{3}\,\Delta]^{1/2},$$

we finally have

$$HH' = 2a_1a_2a_3\sqrt{3}/[a_1^2 + a_2^2 + a_3^2 - 48\Delta^2]^{1/2}.$$

If $\sphericalangle HGH' = 3\phi$, then $\sphericalangle N_i'GN_i = \phi$, (mod $\pi/3$). We may always so letter the given triangle such that

$$\sphericalangle N_1'GN_1 = \phi, \qquad \sphericalangle N_2'GN_2 = \pi/3 - \phi, \qquad \sphericalangle N_3'GN_3 = 2\pi/3 - \phi.$$

Then, by applying the law of cosines to the triangles $N_i'GN_i$, we find

$$\cos^2\phi = (a_2^2 + a_3^2 - 2a_1^2)^2/(a_1^2 + a_2^2 + a_3^2 - 48\Delta^2),$$
$$\cos^2(\pi/3 - \phi) = (a_3^2 + a_1^2 - 2a_2^2)^2/(a_1^2 + a_2^2 + a_3^2 - 48\Delta^2),$$
$$\cos^2(2\pi/3 - \phi) = (a_1^2 + a_2^2 - 2a_3^2)^2/(a_1^2 + a_2^2 + a_3^2 - 48\Delta^2).$$

In a similar manner, distances between other points in the figure may be expressed in terms of a, b, and the angle ϕ, and consequently in terms of the sides of the triangle itself, if desired.

A little joke (?) by the late David Eugene Smith. Ever so often a kind fate intervenes to make some error, to which all humans are prone, result in something amusing. In David Eugene Smith's monumental *Rara Arithmetica*, under the date 1537, Professor Smith listed as one of the arithmetical publications of that year the following: "L. Culman, 'Wie iung und alte Leut petten sollen,' Nurnberg, 8°; ib., 1537." The British Museum Catalogue of Printed Books gives this title, with one letter changed, "betten" (to pray), and the operation of "praying" is far from the American meaning of "petting" or "petten," and neither one is strictly arithmetical.—*L. C. Karpinski.*

A VECTOR INTERPRETATION OF THE DERIVATIVE CIRCLE

H. R. PYLE AND BEVERLY M. BARKER, Whittier College

1. Real functions. Let $w(x, y)$ represent a real function of x and y, continuous and with continuous first partial derivatives in a given region. Then $w=c$ is a contour line which we shall call the curve C (Fig. 1). Let R be a point on C whose coordinates are x and y. Then $z=x+iy$ may be considered the position vector of R.

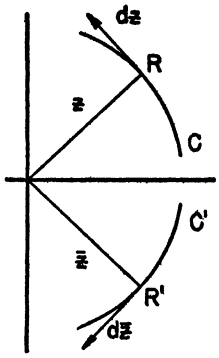


Fig. 1

Let $w'=w(x, -y)$ represent a second real function related to w, which is also continuous and whose partial derivatives are continuous in the region under consideration. Then $w'=c'$ is a contour line for w' which we shall call the curve C'. Let R', whose coordinates are x and $-y$, be the point on C' corresponding to R on C. Then $\bar{z}=x-iy$ is the position vector of R'. C' is symmetric to C with respect to the x-axis. If C is symmetric with respect to the x-axis, C' coincides with C.

The derivative circle for dw/dz may be found by the well-known process worked out by Kasner [1]. As noted by Hedrick [4],

$$\frac{dw}{dz} = D + Pe^{-2i\theta}$$

where

$$D = \frac{\partial w}{\partial z} = \tfrac{1}{2}(w_x - iw_y), \qquad P = \frac{\partial w}{\partial \bar{z}} = \tfrac{1}{2}(w_x + iw_y).$$

Here D represents the vector from the origin to the center of the derivative circle and $Pe^{-2i\theta}$ represents the vector from the center to any point on the circle. Hence the magnitude of P is the radius of the circle. The coordinates of the center are $(w_x/2, -w_y/2)$ and the radius is

$$|P| = \tfrac{1}{2}\sqrt{w_x^2 + w_y^2}\,.$$

In the terminology of vector analysis the gradient of w at the point R is written $\nabla w = lw_x + mw_y$ where l and m are unit vectors along the x and y axes respectively. In the notation of complex variables $\nabla w = w_x + iw_y$. Thus $P = \frac{1}{2}\nabla w$.

THEOREM 1. *When w is a real function of x and y, $P = (\partial w/\partial \bar{z}) = \frac{1}{2}\nabla w$ is a vector along the gradient of the curve $w(x, y) = C$ whose magnitude is half that of the gradient.*

Similarly, the gradient of w' at R' is $\nabla w' = lw_x' + mw_y'$. But at R', $w_x' = w_x$, $w_y' = -w_y$. Therefore $\nabla w' = lw_x - mw_y$, or $\nabla w' = w_x - iw_y$. Hence $D = \frac{1}{2}\nabla w'$, and is represented by a vector along the gradient of C' whose magnitude is half that of the gradient.

THEOREM 2. *The position vector of the center of the derivative circle of dw/dz has a magnitude equal to half the magnitude of the gradient of w, and is parallel to the gradient of $w' = w(x, -y)$; i.e., it is normal to the curve $w' = c'$ at R'. The diameter of the circle is equal to the magnitude of the gradient.*

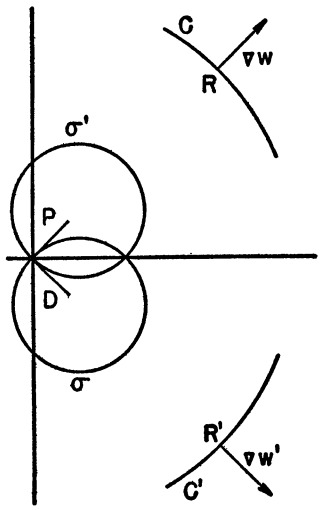


Fig. 2

In the same manner we find that

$$\frac{dw}{d\bar{z}} = P + De^{2i\theta}.$$

Therefore P is the position vector of the center of the circle and the magnitude of D is the radius. The coordinates of the center are $(w_x/2, w_y/2)$ and the radius is

$$|D| = \tfrac{1}{2}\sqrt{w_x^2 + w_y^2}.$$

Hence the position vector of the center has a magnitude equal to half the magnitude of the gradient and is parallel to the gradient of w; *i.e.*, it is normal to C at R (Fig. 2). The diameter is the same as the diameter of the circle representing dw/dz. The two circles are symmetrically placed relative to the x-axis and they both go through the origin. We shall call them conjugate derivative circles. In the figure, σ is the circle representing dw/dz; σ' represents $dw/d\bar{z}$.

$$D = \tfrac{1}{2}\nabla w', \qquad P = \tfrac{1}{2}\nabla w.$$

2. Complex functions. Let $F=u+iv$ be a function whose components $u(x, y)$ and $v(x, y)$ are real functions, continuous and with continuous first partial derivatives in the region under consideration. In 1927, Kasner [1] introduced the term *polygenic function* to denote a general complex function whose components do not satisfy, necessarily, the Cauchy-Riemann equations. Then $u=c_1$, $v=c_2$ are contour lines for u and v respectively. Call $u=c_1$ the curve C_1 and $v=c_2$ the curve C_2. Let the point R, with coordinates x and y, be their point of intersection. Then $z=x+iy$ is the position vector of R.

Consider the function $F'=u'+iv'$, where $u'=u(x, -y)$, $v'=v(x, -y)$, and call $u'=c_1'$ the curve C_1' and $v'=c_2'$ the curve C_2'. R', with coordinates x and $-y$, is the point of intersection of C_1' and C_2'. $\bar{z}=x-iy$ is its position vector.

Since

$$\begin{aligned}
D(u) &= \tfrac{1}{2}(u_x - iu_y), \qquad D(v) = \tfrac{1}{2}(v_x - iv_y)\\
P(u) &= \tfrac{1}{2}(u_x + iu_y), \qquad P(v) = \tfrac{1}{2}(v_x + iv_y),\\
D(u) + iD(v) &= \tfrac{1}{2}[(u_x + v_y) + i(-u_y + v_x)]\\
P(u) + iP(v) &= \tfrac{1}{2}[(u_x - v_y) + i(u_y + v_x)].
\end{aligned}$$

Referring to Hedrick's paper [4], we see that

$$D(F) = \frac{\partial F}{\partial z} = D(u) + iD(v), \qquad P(F) = \frac{\partial F}{\partial \bar{z}} = P(u) + iP(v).$$

Kasner [2] calls $D(F)$ the *mean derivative* and $P(F)$ the *phase derivative.*

We have seen that $D(u)$ is a vector along the gradient of C_1' whose length is half the magnitude of the gradient of u and $D(v)$ is a vector along the gradient of C_2' whose length is half the magnitude of the gradient of v. If we interpret multiplication by i as counter-clockwise rotation through a right angle, $iD(v)$ is a vector tangent to C_2'.

THEOREM 3. *The mean derivative $D(F)=D(u)+iD(v)$, where $D(u)$ is one-half the gradient of C_1' (defined by $u(x, -y)=c_1'$), and $iD(v)$ is a vector tangent to C_2' (defined by $v(x, -y)=c_2'$), whose magnitude is equal to one-half the gradient of C_2'.*

The same interpretation holds for $P(F)$, referred to the contour lines C_1 and C_2.

$D(F)$ represents the position vector of the center of the derivative circle for dF/dz, and the magnitude of $P(F)$ is the radius (Fig. 3). Likewise $P(F)$ repre-

sents the position vector of the center of the circle for $dF/d\bar{z}$ and the magnitude of $D(F)$ is its radius. For the real function w, $|D(w)| = |P(w)|$. This makes the radii of the two circles σ and σ' equal, and causes both to pass through the origin. The equality does not hold for $|D(F)|$ and $|P(F)|$.

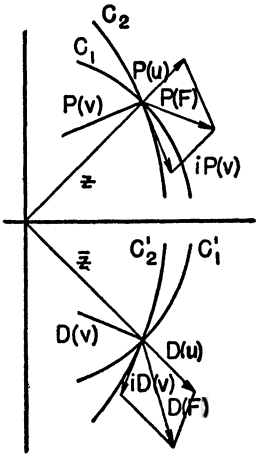


Fig. 3

When F is an analytic function of z, it is well-known that the contour lines C_1 and C_2 are orthogonal and the magnitudes of the gradients of u and v are equal. Furthermore it can be proved that the angle from the positive gradient of C_1 to the positive gradient of C_2 is counter-clockwise.

The angle from the positive gradient of $u'=u(x, -y)$ to the positive gradient of $v'=v(x, -y)$ is clockwise and equal to a right angle when $F=u+iv$ is an analytic function of z. The magnitudes of the gradients of u' and v' are equal.

Since $P(F)=P(u)+iP(v)$ and $P(u)$ and $iP(v)$ are collinear, of equal length, and oppositely directed when F is an analytic function of z, $P(F)=0$. Furthermore, since $D(F)=D(u)+iD(v)$, and $D(u)$ and $iD(v)$ are collinear, of equal length, and similarly directed, $D(F)=2D(u)=\nabla u'$.

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"I had forgotten that on the moon, with only an eighth part of the earth's mass and a quarter of its diameter, my weight was barely a sixth of what it was on earth. But now that fact insisted on being remembered." H. G. Wells, *The First Men in the Moon* (Error noted by MacKay in *Odd Numbers*).—*E. D. Schell.*

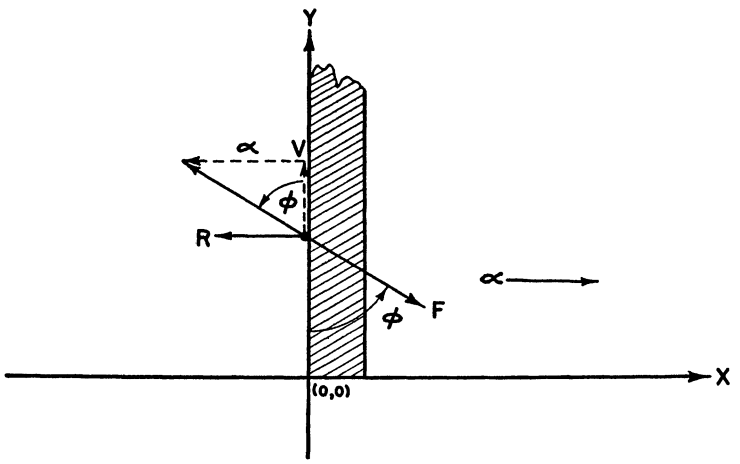
A PROBLEM IN FRICTION

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The following elementary problem in mechanics has some pedagogical interest as well as physical interest. The author has asked various people to hazard a guess as to what the solution might be. In most cases the controversy aroused seemed to me to be of as much interest as the problem itself. For that reason the problem is stated as it originated in one of my classes in mechanics. The mathematical treatment is given first. A brief physical interpretation of the results is then given. The application mentioned at the end of the article was called to my attention by Prof. W. Mason.

Problem: A *rough* horizontal floor moves in a direction perpendicular to a *smooth* vertical wall, disappearing under the base of the wall. The velocity of the floor is a constant, α. A particle is projected with an initial velocity v_0 along the line of intersection of the floor and the wall. Determine the distance traveled by the particle, and the time required to travel this distance. (Assume both wall and floor to be of infinite extent.)

Solution: Choose a rectangular coordinate system (x, y, z) so that the line of intersection of the floor and wall coincides with the y-axis, and so that the vertical wall coincides with the (y, z) plane. Suppose the motion of the floor is in the direction of the positive x-axis. If $[x(t), y(t), 0]$ are the coordinates of the particle p, the initial conditions may be expressed in the form $x(0)=0$, $y(0)=0$, $x'(0)=0, y'(0)=v_0>0$.



The force of friction F of the floor acting on the particle p is opposite to the direction of the velocity of p *relative* to the floor. Let ϕ be the acute angle which this direction makes with the y-axis. The reaction of the wall is denoted by R. The following diagram illustrates the motion in the (x, y)-plane.

Furthermore $|F| = \mu Mg$, where μ = coefficient of friction, and M = mass of p. The equations of motion of p are

$$(1)\qquad \begin{aligned} M\frac{d^2x}{dt^2} &= -R+\mu Mg\sin\phi \equiv 0,\\ M\frac{d^2y}{dt^2} &= -\mu Mg\cos\phi. \end{aligned}$$

Clearly $x''(t)\equiv 0$, for if there exists an interval $t_1\leqq t\leqq t_2$ such that $x(t_1)=0$, $x'(t_1)=0$, $x''(t)<0$, $(t_1<t\leqq t_2)$, we would have $x(t)<0$ and $x'(t)<0$ for $(t_1<t\leqq t_2)$. However, in this case $R\equiv 0$, $(t_1<t\leqq t_2)$, which would imply the contradictory result $x''(t)>0$, $(t_1<t\leqq t_2)$. Since $x(t)$ cannot be positive because of the wall, it follows that $x''(t)\equiv 0$.

Since

$$\cos\phi = \frac{y'}{\sqrt{y'^2+\alpha^2}}$$

the second of equations (1) has the form

$$(2)\qquad \frac{d^2y}{dt^2} = \frac{-\mu g y'}{\sqrt{y'^2+\alpha^2}}$$

where $y'=dy/dt$. This is a special case of the differential equation $y''=f(y')$. For our purpose it is advisable to use a parametric solution of the form

$$t-t_0=\int_{v_0}^{v}\frac{dp}{f(p)},\qquad y-y_0=\int_{v_0}^{v}\frac{p\,dp}{f(p)},\qquad f(p)\neq 0,$$

where $y'=v$, and $y''=(dv/dy)v$. Equation (2) then has the parametric solution

$$\mu g t=-\int_{v_0}^{v}\frac{\sqrt{p^2+\alpha^2}}{p}\,dp,\qquad \mu g y=-\int_{v_0}^{v}\sqrt{p^2+\alpha^2}\,dp,\qquad v\neq 0.$$

These are expressible in the form

$$(3)\qquad \begin{aligned} \mu g t &= \sqrt{v_0^2+\alpha^2}-\sqrt{v^2+\alpha^2}+\alpha\log\left(\frac{\sqrt{v_0^2+\alpha^2}-\alpha}{v_0}\cdot\frac{v}{\sqrt{v^2+\alpha^2}-\alpha}\right),\\ \mu g y &= \frac{1}{2}\left(v_0\sqrt{v_0^2+\alpha^2}-v\sqrt{v^2+\alpha^2}\right)+\frac{\alpha^2}{2}\log\left(\frac{v_0+\sqrt{v_0^2+\alpha^2}}{v+\sqrt{v^2+\alpha^2}}\right). \end{aligned}$$

Observe that both t and y are defined for $0<v\leqq v_0$. Since

$$\begin{aligned} \lim_{v\to 0}\frac{v}{\sqrt{v^2+\alpha^2}-\alpha} &= \infty,\\ \lim_{v\to 0}\frac{1}{v^2+\sqrt{v^2+\alpha^2}} &= \frac{1}{\alpha}, \end{aligned}\qquad \alpha\neq 0,$$

equations (3) imply that for $\alpha\neq 0$,

$$\lim_{v\to 0} t = \infty,$$

$$\lim_{v\to 0} y = \frac{1}{2\mu g}\left[v_0\sqrt{v_0^2+\alpha^2} + \alpha^2 \log\left(\frac{v_0+\sqrt{v_0^2+\alpha^2}}{\alpha}\right)\right]. \tag{4}$$

This *solution* of the problem is based on the assumption that the usual laws of friction hold.

It is interesting to observe that certain of these results can be obtained by an application of general existence theorems for differential equations.* Equation (2) can be written in the form

$$\frac{dv}{dt} = -\mu g \frac{v}{\sqrt{v^2+\alpha^2}} \equiv f(v), \qquad v(0) = v_0 > 0. \tag{5}$$

When $\alpha = 0$, dv/dt is a negative constant in which case $v(t)\to 0$ for a finite value of t. When $\alpha \neq 0$, the boundedness of $|f(v)|$ for *all values* of v implies, by virtue of the existence theorems, that $v(t)$ is defined for all values of t. The solution $v(t)$ is decreasing. It cannot vanish for a *finite* value of t, otherwise the solution would not be *unique*, since $v(t)\equiv 0$ is also a solution. Moreover, if $v(t) \geqq c > 0$ for all t, then (5) implies that $dv/dt \leqq -\mu g c(v_0^2+\alpha^2)^{-1/2}$. But this implies that $v(t)$ is tending to zero for a *finite* value of t, which is impossible by virtue of the preceding arguments. Hence $v(t)\to 0$ as $t\to\infty$, when $\alpha \neq 0$.

The physical significance of the above results appears plausible when one considers the fact, $\tan\phi = \alpha/v$. As $v\to 0$, the angle $\phi\to\pi/2$. Hence the vertical component of F, $F_1 \equiv |F|\cos\phi$, approaches zero as $v\to 0$, since $|F|$ is constant.

This type of problem arises in other places, and the principle is also put to practical use. In the checking of pressure gages, a dead weight tester† is often used. In this instrument a cylindrical piston which supports standardized weights is rotated about its axis of symmetry in order to reduce the *vertical* component of wall friction, thus allowing easier vertical movement. This is not solely a matter of static friction. Furthermore observe in the problem discussed here, that if a particle is at rest at the base of the wall, it can be set into motion more easily, if the floor is in motion. The larger the velocity α, the easier is the effort.

* This verification was suggested to me by the referee.

† Moyer, J. A., Power Plant Testing, 1913. McGraw-Hill, p. 17.

Tolstoy's modesty. "Had I been told when I was at the university that others understood the integral calculus while I did not, that would have touched my pride." Tolstoy.—*P. L. Chessin.*

Wolves. "Mathematicians are like lovers: grant a mathematician the least principle, and he will draw from it a consequence which you must also grant him, and from this consequence another." Fontenell.—*P. L. Chessin.*

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A REFERENCE

J. H. M. WEDDERBURN, Princeton University

Since writing my note *On Pascal's Theorem*, which appeared in the August-September issue of this MONTHLY, I have noticed that equation (1) of the lemma used is given by L. N. M. Carnot, *Géométrie de Position*, Paris, 1803, §236, p. 293.

I might have pointed out that the proof given applies to Pappus' Theorem and gives what seems to be the most elementary proof of it since it only involves Menelaus' Theorem. The case in which AP_1, BP_2, CP_3 meet in a point then leads to a number of interesting exercises in euclidean geometry. It is worth noticing also that the proof only requires four points on each line although the figure degenerates if there are fewer than five.

A PSYCHOLOGICAL GAME

N. S. MENDELSOHN, Queen's University

This contribution is the outcome of an informal discussion with Dr. I. Kaplansky last June at the Canadian Mathematical Congress.

The following game is considered. Players A and B each choose a positive integer simultaneously; the player whose number is the smaller scores one point, unless the other player chooses a number exactly one greater, in which case the latter scores two points. I have called this a psychological game because in practice each player attempts to guess his opponent's next move, and usually, after a short while, one of the players gains a psychological ascendancy and piles up a huge score, in spite of the fact that the game is equitable.

This game is subject to mathematical analysis. It can be shown that the best strategy for a player A is to choose numbers 1, 2, 3, 4, 5, with frequencies 1, 5, 4, 5, 1; the choice being made in random order. The strategy is best in the following sense; (a) there is no strategy which can beat it in the long run; (b) corresponding to any other strategy, a winning counter strategy can be devised. In particular, any strategy which incorporated the number six or higher numbers would be a losing strategy.

Firstly, a small generalization. Assume that the second player scores n points ($n>1$, but is not necessarily an integer) instead of two points in the case where he chooses an integer one greater than that of the first player. In connection with this game some of the problems of interest are: (1) for a given n, determine

if it exists, the number of integers necessary to incorporate into a best strategy (in the original game this is five); (2) determine the frequency with which the various integers should be played; (3) determine the largest number n for which a winning strategy can be devised using only the first $2r+1$ integers.

The problem is solved as follows. First, form A's matrix. This is the matrix (a_{uv}) where a_{uv} is the number of points A scores when A chooses u and B chooses v. (We keep score for A alone. If B wins we subtract the points from A's score.)

$$(a_{uv}) = \begin{pmatrix} 0 & -n & 1 & 1 & 1 & 1\cdots 1 & 1\cdots \\ n & 0 & -n & 1 & 1 & 1\cdots 1 & 1\cdots \\ -1 & n & 0 & -n & 1 & 1\cdots 1 & 1\cdots \\ -1 & -1 & n & 0 & -n & 1\cdots 1 & 1\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1\cdots & & & & \end{pmatrix}.$$

Suppose that a best strategy can be devised which incorporates only the first $2r+1$ integers. Let $p(i, r)$ be the relative frequency with which i should be played using this strategy. We are interested in the relative magnitude of the numbers $p(1, r), p(2, r), p(3, r), \cdots, p(2r+1, r)$. It can be seen from symmetry that $p(1, r)=p(2r+1, r)$, $p(2, r)=p(2r, r), \cdots, p(j, r)=p(2r+2-j, r)$. For convenience, we will replace $p(j, r)$ by $p(2r+2-j, r)$ whenever $j>r+1$. Since the game is equitable A's average score should be 0, whenever B plays any of $1, 2, \cdots, (2r+1)$; (if A could have an average score greater than 0, B by using the same strategy as A could also obtain an average score greater than 0, a contradiction). The condition that A's average score is 0, if B plays $1, 2, \cdots, (2r+1)$, leads to the following $(r+1)$ equations (actually $2r+1$ equations are obtained of which r are repeated once).

$$-p(1, r) + (n-1)p(2, r) - 2p(3, r) - \cdots - 2p(r, r) - p(r+1, r) = 0$$
$$-(n+1)p(1, r) - p(2, r) + (n-1)p(3, r) - 2p(4, r) - \cdots - p(r+1, r) = 0$$
$$0p(1, r) - (n+1)p(2, r) - p(3, r) + (n-1)p(4, r) - \cdots - p(r+1, r) = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$0p(1, r) + 0p(2, r) + \cdots - (n+1)p(r-1, r) - p(r, r) + np(r+1, r) = 0.$$

Furthermore, if a best strategy is actually attained, A's score should be positive if B plays $2r+2$ or higher. If B plays $2r+2$, one obtains as a condition for a positive score the inequality

$$(n-1)p(1, r) - 2p(2, r) - \cdots - 2p(r, r) - p(r+1, r) < 0.$$

If B plays $2r+3$ or more, A's average score is positive without condition.

The above condition gives the maximum n for which a best strategy using $1, 2, 3, \cdots, (2r+1)$ is possible. A lower bound for n is obtained by assuming

that no best strategy can be based on fewer than the first $2r+1$ integers. The number $p(i, r)$ is $(-1)^{i+1}$ times the determinant obtained by removing the ith column from the following r-rowed, $(r+1)$-columned matrix:

$$\begin{vmatrix} -1 & (n-1) & -2 & -2 & -2 & -2 \cdots -2 & -1 \\ -(n+1) & -1 & (n-1) & -2 & -2 & -2 \cdots -2 & -1 \\ 0 & -(n+1) & -1 & (n-1) & -2 & -2 \cdots -2 & -1 \\ 0 & 0 & -(n+1) & -1 & (n-1) & -2 \cdots -2 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots & & & -(n+1) \; -1 & n \end{vmatrix}.$$

By expanding $p(i, r)$ along the top row, difference equations for $p(i, r)$ are obtained, *e.g.*:

$$\begin{aligned} p(1, r) &= (n-1)p(1, r-1) - 2p(2, r-1) - \cdots \\ &\quad - 2p(r-1, r-1) - p(r, r-1); \\ p(2, r) &= p(1, r-1) + 2(n+1)p(1, r-2) + \cdots \\ &\quad + 2(n+1)p(r-2, r-2) + (n+1)p(r-1, r-2); \\ p(3, r) &= p(2, r-1) + (n^2-1)p(1, r-2) - 2(n+1)^2 p(1, r-3) - \cdots \\ &\quad - 2(n+1)^2 p(r-3, r-3) - (n+1)^2 p(r-2, r-3); \textit{ etc.} \end{aligned}$$

The first of these relations yields a very elegant result. On examining the condition for a best strategy based on the first $(2r+1)$ integers, it is seen that this reduces to $p(1, r+1) < 0$. This sets a least upper bound for the value of n. The following short table gives results for the first few cases:

r	*Condition on* n	$p(i, r)$
1	$n^2-n-1<0$	$p(1, 1)=n$, $p(2, 1)=1$
2	$n^3-3n^2-4n-1<0$	$p(1, 2)=n^2-n-1$, $p(2, 2)=2n+1$, $p(3, 2)=n^2$
3	$n(n^3-6n^2-9n-3)<0$	$p(1, 3)=n^3-3n^2-4n-1$, $p(2, 3)=n(3n+2)$, $p(3, 3)=n(n^2-n-1)$, $p(4, 3)=4n^2+4n+1$.

The game can be generalized in other ways (*e.g.*, by a more generalized scoring system, the score alloted being a function of the difference between the numbers chosen by the players). The mathematical treatment of these generalizations would follow the same lines. There is one point, however, worth noting. The author has avoided the case of strategies based on the use of an even number of integers. This case is fundamentally more difficult. The difficulty arises from the fact that a skew-symmetric matrix is always of even rank. The author throws open this problem to any who may be interested.

SOME SPHERES ASSOCIATED WITH A TETRAHEDRON*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. A known property. In a triangle ABC the lines joining the vertices A, B, C to the points of contact D, E, F of the inscribed circle with the sides BC, CA, AB respectively, are concurrent at the *Gergonne point* of the triangle, and the circles inscribed in the triangles BAD and CAD touch each other on the line AD, the circles inscribed in the triangles CBE and ABE touch each other on the line BE, and the circles inscribed in the triangles ACF and BCF touch each other on the line CF.

2. A space analogy. In any tetrahedron $T \equiv ABCD$, the point P whose barycentric coordinates are proportional to the trigonometric tangents of the half-angles at the vertices A, B, C, D of the cones of revolution inscribed in the trihedral angles (A), (B), (C), (D) is comparable to the Gergonne point of a triangle.†

Let A_1, B_1, C_1, D_1 be the points of contact of the inscribed sphere (I) of T on the faces BCD, CDA, DAB, ABC. If we mark points β, γ, δ on A_1B, A_1C, A_1D such that $A_1\beta = A_1\gamma = A_1\delta = s$, say, and let the perpendiculars at β, γ, δ to the plane BCD cut IB, IC, ID in ω_b, ω_c, ω_d, then *the spheres with centers* ω_b, ω_c, ω_d *and radii* $\omega_b\beta$, $\omega_c\gamma$, $\omega_d\delta$ *have for radical axis the fixed line* A_1P, *no matter what be the length* s.‡ Therefore, if one joins the Gergonne point P of T to the contacts A_1, B_1, C_1, D_1 of the inscribed sphere, the planes bisecting the angles PA_1B, PA_1C, PA_1D respectively cut IB, IC, ID at the centers ω_b', ω_c', ω_d' of three spheres inscribed in the trihedral angles (B), (C), (D) and touching the line PA_1 in the same point Q, for, by an easy indirect argument, PA_1 is the radical axis of the spheres (ω_b'), (ω_c'), (ω_d'). The same property applies to the triples of spheres inscribed in the trihedral angles $[(C), (D), (A)]$, $[(D), (A), (B)]$, $[(A), (B), (C)]$ which touch the lines PB_1, PC_1, PD_1 respectively at common points. Thus one finds a solution to the problem: *In three of the trihedral angles of a tetrahedron* $ABCD$, *to inscribe spheres tangent to a common line at a common point.*

In an isogonic tetrahedron T, the lines PA_1, PB_1, PC_1, PD_1 pass through the vertices A, B, C, D and the triples of spheres (ω_b'), (ω_c'), (ω_d'), $\cdots$, touch the lines AA_1, BB_1, CC_1, DD_1 respectively at common points.

Note. The normal coordinates of the point P defined by means of the barycentric coordinates of R. Bouvaist (*loc. cit.*) are proportional to the radii of the circles circumscribing the faces of T, and the point P is also called the *second Lémoine point of* T.§

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* Translated from the French by Howard Eves.

† R. Bouvaist, supplement by V. Thébault, Mathesis, 1941, p. 19.

‡ R. Bouvaist and V. Thébault, Comptes-Rendus des séances de l'Acadèmie des sciences de Paris, 1940, p. 378.

§ V. Thébault, Annals de la Soc. Scient. de Bruxelles, 1922, pp. 173–177.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1944–45

Mathematics Society, Brooklyn College

The purpose of the Society is to create and maintain an interest in mathematics. Special activities, such as the publication of the *Math Mirror* (a student magazine on mathematics with an issue of 1,000 copies) and the Integration Contests, assist us in furthering this aim. But it is at the regular meetings, where the faculty and student members lecture, that we attain our goal. Lectures during the Fall semester 1944 were as follows:

Euler squares and Latin squares, by Professor Harris MacNeish
Methods of summation of finite series, by Jack Melnick
Systems of linear equations, by Julius Jackson
Approximations to the $\sqrt{2}$, by Professor James Singer
An algebra of plane geometry, by Prosessor Walter Prenowitz
Probability, by Bernard Seckler
Self-inductive theorems, by Professor Roger Johnson
The E function, by Ezra Krendel.

During the Spring semester, lectures were delivered as follows:

The hyperboloid of one sheet, by Professor Harris MacNeish
The Gamma function and Dirichlet integrals, by Jack Melnick
History and concepts of the calculus, by Professor Carl Boyer
Crytography and cryptanalysis, by Professor Jack M. Wolfe
Applications of mathematics to chemistry, by Professor Martin Meyer.

The twenty-ninth semi-annual Integration Contest, held in the Fall of 1944, resulted in victory for Professor Johnson's team. Individual honors went to Violet Bushwick. The thirtieth Contest, held in the Spring of 1945, resulted in victory for Professor Merle Bishop's first team. Individual honors went to Abigail Wishneff. During the Spring semester, the Mathematics Society published its thirteenth annual issue of the *Math Mirror*, edited by Jack Melnick, Editor-in-chief, and Bernard Seckler and Jerome M. Rehr, Associate Editors.

The officers of the Fall semester were: President, Julius Jackson; Vice-President, Ezra Krendel; Secretary, Marion Eisen; Treasurer, Jerome Rehr. The officers for the Spring semester were: President, Jack Melnick; Vice-President, Bernard Seckler; Secretary, Marion Eisen; Treasurer, Jerome Rehr. The Faculty Adviser for the Mathematics Society was Professor James Singer. The Faculty Adviser for the *Math Mirror* was Professor Harris MacNeish.

Kappa Mu Epsilon, Kansas State Teachers College, Pittsburg

Kansas Alpha Chapter held open meetings in the homes of the departmental staff members. At each meeting two or three students and one faculty member gave papers or talks on mathematical topics. Music numbers or readings were given by members or invited guests. Refreshments were served by the hostess. Student topics were:

Mathematical symbols, by Jack Marshall
Mathematics in the schools of England, by Dorothy Bradshaw
Proofs of the Pythagorean theorem, by Ralph Newman
Euclid, by Christine Carpenter
Mathematical recreations, by Miss Billie Shultz
Mathematics in the air age, by Bill Seal
Mathematics of the early American Indians, by Ann Benny
Mathematical societies and publications, by Bob Besser
David Eugene Smith, by Hattie Highfill
Surveying in ancient times, by Jack Hedgecock.

Papers presented by the faculty were:

Mathematical contests, by Dr. R. G. Smith
Quadrature of plane areas, by Professor J. A. G. Shirk
Addition and subtraction logarithms, by Professor F. C. German
United States land surveys, by Professor L. E. Curfman
Electronics and radar, by Professor W. H. Matthews of the Physics Department.

Three initiation services were held during the year for seventy-five students. Mrs. Dorothy Bradshaw and Dorothy Campion received *Kappa Mu Epsilon* pins for having the highest scholarship records of seniors majoring in mathematics. Chapter officers for 1944–45 were: President, Mrs. Dorothy Bradshaw; Vice-President, Dorothy Bernhardt; Secretary, Nancy Horton; Treasurer, Ann Benny; Corresponding Secretary, Professor W. H. Hill; Sponsor, Professor J. A. G. Shirk.

Pi Mu Epilson, University of Kentucky

During the year 1944–45 the activities of the Club were as follows:

Vectors and complex numbers, by Professor C. G. Latimer
Parametric representation, by Professor H. H. Downing
Gauss' theorem and an application, by Miss Mary Ann Macke, graduate student.

The annual picnic and initiation was held at Castlewood Barn. Eighteen members and friends were in attendance, and Mr. T. M. Hahn, a physics major, was initiated.

During the year the club presented to the mathematics library over fifty dollars worth of books.

Officers for 1945–46 are as follows: Director, Professor Flora LeStourgeon; Vice-Director, Professor H. H. Downing; Secretary-treasurer, Mrs. Lydia Fischer; Librarian, Professor M. C. Brown.

Mathematics Club, University of Buffalo

The Mathematics Club of the University of Buffalo tries to make its meetings interesting not only to those people who are in the field or who are about to enter it, but also to those who are comparatively unfamiliar with it. During the year 1944–45, as always, we have attempted to fulfill our purpose by presenting diversified programs showing new views of mathematics, and also relating it to other fields. For instance, talks were given on the following topics:

Mathematics and photography, by Robert Lockie and John Kenney

Soap films, by Lois Dodge and Betty Bushnell

Weights computing, by Irene Fisher

The planimeter, by Mary Jane Gill.

As often as possible the program chairman chooses speakers who have had interesting experiences concerning the topic. For example, Miss Fisher, who talked on the method used in finding the weight of certain parts of an airplane, had actually worked as a weights computer; and Mr. Lockie and Mr. Kenney are very proficient with the camera.

We have found that mathematics can be used for recreation as well as for practical applications. One talk that proved to be very relaxing and amusing was on the subject:

Mathematical games, by Phyllis Valentine.

Each year the Math Club invites seniors from the local high schools to a meeting in order to acquaint them with the University campus and the organization. During this meeting, the audience was surprised and delighted to hear a talk entitled:

Geometrical exercises in paper folding, by Phyllis Valentine and Mrs. Rosemary Trautman.

The Club also held two parties during the year, the annual Christmas party for the faculty and a St. Patrick's Day party, and a closing picnic sponsored by the faculty.

Elections were held at the St. Patrick's Day party, and the following people were chosen as officers: President, Irene Fisher; Vice-President, Jane Noller; Treasurer, Robert Lockie; Secretary, Ruth Cohen.

Mathematics Club, Illinois Institute of Technology

The I. T. Math Club met bi-monthly during the Fall semester, 1944, to hear the following papers:

Impossibility of trisection, by Samuel Karlin

Bernoulli's numbers, by Harley Flanders

Fractions, by Dr. L. R. Ford

Algebraic and geometric inequalities, by Dr. H. Wall

A numerical check in the solution of linear equations, by Dr. H. Rheingold

Certain Diophantine equations, by Norbert Kaufman

The Lorentz transformation, by Willard Skolnik.

Officers of the club were: President, Harley Flanders; Secretary, Norbert Kaufman; Faculty Sponsor, Dr. L. R. Ford.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

The Meaning of Relativity. By Albert Einstein. Princeton University Press, 1945. 135 pages. $2.00.

For a modern mathematician the most natural approach to a physical theory appears to be through the postulational method; we write down certain equations, develop their consequences, and after establishing a correspondence between some quantities in the equations and quantities measured by the physicist we check these consequences with the results of his experiments; if they agree we accept the theory. A mathematician may be satisfied with this situation, and for him there may be no question of further meaning. However he would not deny that there are other legitimate questions, *e.g.*, the question of how a physicist arrives at the postulates or equations which lie at the basis of the successful theory. A considerable portion of the book is devoted to discussions of this character.

The theory is developed in three stages in each of which the mathematical apparatus is introduced in close contact with physical considerations. Pre-relativity physics is first treated by means of a restricted form of three-dimensional tensor analysis, which as the author emphasizes automatically furnishes equations invariant under rotations; in the words of the author it embodies "the principle of relativity with respect to rotations." Relativity with respect to the state of motion appears in the second stage motivated by the results of the Michelson-Morley experiments; these results suggest that the laws of physics should possess invariance under Lorentz transformations—the analogues of the rotations of the first stage, and in keeping with this tensor analysis is extended by increasing the number of dimensions (and introducing imaginary components). The 1946 reader will be especially interested in the mass-energy relation which is a consequence of this development, and this interest will be enhanced by a footnote mentioning in this connection radio-active decomposition (of course, these things were first stated by Einstein back in 1905). Again in the third stage the introduction of new conceptions is done in close touch with physics; the development is suggested by the fundamental fact of equivalence between inertial and gravitational mass; the mathematical theory which embodies this suggestion is built on the model of the theory of surfaces and involves a further extension of tensor analysis and the idea of invariance. As before, the reader is not assumed to have previous familiarity with the mathe-

matical tools—they are developed—with mastery and elegance—in the body of the book. In spite of this the book should appeal not so much to a beginner as to one who has acquired the technique of calculations but has failed to develop complete inner conviction; it supplies the connection between the mathematical apparatus and the physical conceptions—in this sense it justifies its title—*The Meaning of Relativity.*

The fact that this book first published in 1922 is now reprinted without changes (except for the addition of an appendix) seems to indicate that the period of the last twenty or twenty-five years was a period of stagnation in the development of the theory which showed such an exuberant growth in the first years of its existence. This is more or less true; although the number of papers published on the theory of relativity during these years has been considerable it is difficult to name many points in which conspicuous success has been achieved; from this point of view the republication without revision may be to a great extent justified. However during this period, largely under the influence of the theory of relativity itself, our attitudes have undergone certain changes, our thinking has become so to say more radical; one may feel sorry that this attitude is not reflected to full extent in the book.

As an example we may bring up the cosmological question to which, however, much space is devoted (about one-fourth of the book, counting the appendix devoted entirely to this question and written in 1945). If the principle of relativity may be formulated by saying that no time direction is preferred as far as the *laws* of physics are concerned, it is very tempting to assume that the same idea applies to *distribution of matter.* Put another way, it seems inconsistent to assume invariance (under Lorentz transformations) of the differential equations and not to assume the same invariance for the initial conditions. Surprising as it is, the last assumption is *not* made—and it would have considerable bearing on the cosmological problem even in special relativity. Instead of uniform distribution of points in three-space we would have to consider uniform distribution of straight lines (representing matter) with respect to pseudo-euclidean metric in space time. This point of view advanced by E. A. Milne seems to be entirely neglected. Another logical consequence of the general point of view, the equivalence of all photons, or what may be called relativity of color, is also not brought out. A reader brought up on general relativity theory will also find it difficult to understand the author's point of view when he speaks of the *size* of particles.

In spite of all that has been said, and although it is not conceded in general that the best way to learn a theory is to travel the road that originally was the path of discovery, these discussions coming from the master who originated both the special and the general theory are highly interesting. It may be said that some of them permit us actually a glimpse into the ways of working of the great author.

G. Y. RAINICH

Plane and Spherical Trigonometry. Second Edition. By H. A. Simmons. New York, John Wiley and Sons, 1945. 11+387 pages. $2.25.

Plane and Spherical Trigonometry, Second Edition, by H. A. Simmons, is a rewriting of *Plane Trigonometry* by Simmons and Gore. Marked innovations appear throughout the text, as to arrangement, revision of content, modification of definitions, extension of scope and the contrast of heavy type with lighter printing (to be much commended). The author prefers to begin with the general angle rather than with the positive acute angle as was done in the earlier volume.

The titles of the chapters follow: I The Trigonometric Functions of any Angle, II Solution of Right Triangles by Natural Functions, III Trigonometric Identities, IV The Reduction Formulas, V Radian and Mil Measure, VI Line Values, Graphs, and Periods of the Functions, VII The Addition Formulas and Related Topics, VIII Logarithms, IX Solution of Triangles by Logarithms, X Inverse Trigonometric Functions, XI Trigonometric Equations. These chapters constitute the treatment of Plane Trigonometry.

As a foundation for Spherical Trigonometry, the chapter on Selected Material from Solid Geometry is introduced, including just that content which should afford the reader an intelligent understanding of Spherical Trigonometry and some important applications. The material is well chosen and well arranged, with no excess impedimenta.

The remaining chapters are XIII Spherical Trigonometry, treated quite in detail, and a chapter on Applications to Navigation and Astronomy. There are included three appendices, namely A. Complex Numbers, B. The Slide Rule and C. Important Formulas.

In chapter one are found basic definitions, rectangular and polar coordinates, vectors and the six fundamental trigonometric functions plus the three special functions. The student is thus made acquainted with the above basics earlier rather than delaying some of them till later. This chapter is not too long but does give the learner an excellent foothold. In chapter two, the six general functions are specialized for solution of the right triangle, using three place tables of natural functions. Chapters three, four, five and six are treated in the traditional method, but in lucid style. These six chapters mark the text as an excellent self-teaching as well as a good classroom instrument.

Chapter seven treats of addition formulas, the half-angle in terms of cosine alone, and sine and cosine, the product formulas for sines and cosines. The half-angle formulas in terms of sides are delayed till chapter nine. Logarithms are treated traditionally, but with quickening variety in the definitions and exercises. In the chapter on solution of triangles by logarithms there is included a brief treatment of triangulation with examples.

Selections marked with the star may be omitted, but a good class should include them or nearly all. The author holds that every student should own a slide rule, and begin early to use it. The author has made a worthy contribution to the list of trigonometry texts.

RICHARD MORRIS

NEW BOOKS RECEIVED

Algebra. A Second Course. By R. O. Cornett. New York and London, McGraw-Hill Book Co., Inc., 1945. 13+313 pages. $2.00.

Analytic Geometry. By R. R. Middlemiss. New York and London, McGraw-Hill Book Co., Inc., 1945. 7+306 pages. $2.75.

Lectures on the Theory of Functions. By J. E. Littlewood. Oxford University Press, 1944. 243 pages. $5.50.

Table of Arc Sin x. Prepared by Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1945. 19+124 pages. $3.50.

Tables of Associated Legendre Functions. Prepared by Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1945. 46+306 pages. $5.00.

The Development of Mathematics. Second Edition. By E. T. Bell. New York and London, McGraw-Hill Book Co., Inc., 1945. 13+637 pages. $5.00.

The Social Impact of Science: A Select Bibliography with a Section on Atomic Power. By the Subcommittee on War Mobilization of the Senate Military Affairs Committee. Washington, D. C., Superintendent of Documents, 1945. 5+51 pages. $0.15.

Victoria Through the Looking-glass. The Life of Lewis Carroll. By F. B. Lennon. New York, Simon and Schuster, 1945. 15+387 pages. $3.50.

Lectures on Probability and Statistics. By E. L. Dodd. Austin, University of Texas Press, 1945. 44 pages.

Higher Algebra for Schools. By W. L. Ferrar. Oxford University Press, 1945. 8+214 pages. $3.75.

College Algebra. Second Edition. By P. K. Rees and F. W. Sparks. New York and London, McGraw-Hill Book Co., Inc., 1945. 13+403 pages. $2.50.

The Isoperimetric Problem. By H. Schwerdtfeger. University of Adelaide, 1945. 1+14 pages.

Introduction to Non-Euclidean Geometry. By H. E. Wolfe. New York, The Dryden Press, 1945. 13+247 pages. $4.50.

The law of diminishing returns. If a female slave sixteen years of age, brings thirty-two (*nischas*), what will one aged twenty cost? Answer: $25\frac{3}{5}$ *nischas.* Bhascara (12th century), *Li'la' vati'*, 76. Colebrooke's translation, 1817.—*Contributed.*

* * * *

Tell all. Most learned algebraist! tell various pairs of integer numbers, the difference of which is a square, and the sum of their squares a cube. Bhascara, *Vi'ja Gan'ita*, 182. Colebrooke's translation.—*Contributed.*

* * * *

How to shine at a faculty tea. As the sun obscures the stars, so does the proficient eclipse the glory of other astronomers in an assembly of people, by the recital of algebraic problems, and still more by their solution. Brahmegupta (sixth century), *Problems*, 102. Colebrooke's translation, 1817.—*Contributed.*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 706. *Proposed by D. H. Browne, Buffalo, N. Y.*

For what values of n does $(a+b)^n$ yield only odd coefficients?

E 707. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Construct a quadrangle having given the lengths of the sides and of the line joining the midpoints of the diagonals.

E 708. *Proposed by M. P. de Regt, Todd Pacific Shipyards, Tacoma, Washington*

Is there a succession of digits which represents a prime in all systems of notation of radix r, $2 \leqq r \leqq 10$? If so, find the smallest one.

E 709. *Proposed by N. A. Court, University of Oklahoma*

In a given sphere to inscribe a tetrahedron so that three concurrent edges shall pass through three given points and the plane of the three remaining edges shall be parallel to the plane determined by the three given points.

E 710. *Proposed by D. H. Lehmer, Aberdeen Proving Ground, Md.*

Find the inverse of the symmetric matrix of the nth order: $A = \|a_{ij}\|$, in which $a_{ij} = i/j$ for $i \leqq j$.

SOLUTIONS

Duodenary Cyclic Numbers

E 672 [1945, 274]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find the cyclic numbers in the duodenary scale. (A number is said to be *cyclic* if every cyclic permutation of its digits produces a multiple of the number.)

Solution by E. P. Starke, Rutgers University. Let S be the radix of any scale of notation, and let q be prime to S. It is known that: if S^k is the lowest power of S such that $S^k - 1$ is divisible by q, then $(S^k - 1)/q = C$ is a cyclic number; if r_i is the remainder when S^i is divided by q, $0 < i \leqq k$, then the effect of multiplying C by r_i is to transfer i digits cyclically from the left of C to the right;

if q exceeds S, C must be built up to k digits by adding zeros at the left. (See S. Guttman, *On Cyclic Numbers*, this MONTHLY [1934, 159–166]. See also R. E. Moritz, *On Products Whose Digits Are Cyclical Permutations of the Digits of the Multiplicand*, this MONTHLY [1927, 33–37].)

Conversely, let N be a number of k digits in the scale of S, and let N_m be the number obtained by transferring the first m digits of N cyclically to the right. If H_m is the number represented by the first m digits of N, then it is easy to see that

$$N_m = NS^m - H_m(S^k - 1).$$

If N_m is a cyclic number so that for all values of m it is a multiple of N, say $N_m = a_m N$, we have

$$S^m - a_m = H_m(S^k - 1)/N.$$

Thus $(S^k-1)/N$ is an integer, say q, relatively prime to S, or it is q divided by a common factor, f, of $M_1, M_2, \cdots, M_{k-1}, N$. Hence

$$N = f(S^k - 1)/q.$$

Now the conditions on f require that f be a divisor of every digit of N, and hence every digit of $(S^k-1)/q$ must be smaller than S/f. If S^k is not the smallest power of S such that S^k-1 is divisible by q, the effect is merely to take the former number C and write its digits two or more times in succession to form a single number, which is obviously cyclic if C is.

We obtain cyclic numbers, C, in the scale of twelve corresponding to each q relatively prime to twelve. There are infinitely many of them (written here in the scale of twelve with X for the digit ten and E for the digit eleven):

$$(10^1 - 1)/E = 1, \qquad (10^4 - 1)/5 = 2497, \qquad (10^6 - 1)/7 = 186X35,$$
$$(10^2 - 1)/EE = 01, \qquad (10^{14} - 1)/15 = 08579214E36429X7, \quad \textit{etc.}$$

The Proposer probably expects to exclude numbers with initial digit zero, the number 1, and also the trivial extensions like 24972497. The significant results are then 2497 and $186X35$.

Also solved by the Proposer, by using the fact that the cyclic numbers contained in the system to the base S are the periods of the fractions $1/m$, where m is a whole number less than, and prime to, S. In the duodenary system we have $m = 5$, 7, and 11, the last case being trivial. For $m = 5$ and 7 we find the required cyclic numbers 2497 and $186X35$ respectively.

Magic Squares That Are Zero Determinants

E 674 [1945, 274]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that a pandiagonal magic square of order 4, when regarded as a determinant, has the value zero.

Solution by E. D. Schell, Arlington, Va. Following Kraitchik's Mathematical Recreations (p. 186), we designate the general square as follows:

$$\begin{array}{cccc} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \end{array}$$

Let S be the common sum of each of the rows, columns, and diagonals, including the broken diagonals such as A, N, K, H. These sixteen sums yield twelve independent relations, making it possible to eliminate all but four of the unknowns. This elimination yields the following square:

$$\begin{array}{cccc} A & B & C & S-A-B-C \\ E & S-A-B-E & A-C+E & B+C-E \\ S/2-C & A+B+C-S/2 & S/2-A & S/2-B \\ S/2-A+C-E & S/2-B-C+E & S/2-E & A+B+E-S/2 \end{array}$$

Now, subtracting the first column from the third, the second column from the fourth, and factoring out $C-A$ and $S-A-2B-C$, leaves columns two and four equal. Hence the determinant of the square is zero.

Also solved by D. W. Alling and the Proposer.

Editorial Note. On p. 212 of the 11th edition of Rouse Ball's *Mathematical Recreations and Essays* we find Berghot's general form for any magic square of the fourth order, along with the conditions which render it pandiagonal. If we eliminate a, b, c by means of these conditions, and then, on the resulting array, add the fourth row to the first, the third row to the second, then subtract the first column from the third and the second column from the fourth, we obtain an array of obviously zero determinant.

The Proposer was originally seeking magic squares having zero determinants. He conjectures that a minimum necessary, though not sufficient, requirement seems to be that the square be pandiagonally magic. Of a number of trials he found no pandiagonal magic square of order five which did not have a zero determinant, nor any of order seven which did have a zero determinant. Several questions are thus raised. In particular, do all fifth order pandiagonal magic squares have zero determinants?

The First Day of a Century

E 676 [1945, 340]. *Proposed by T. R. Running, University of Michigan*

Show that in using the Gregorian calendar the first day of a century cannot occur on a Sunday, a Wednesday, or a Friday.

I. *Solution by L. S. Shively, Ball State Teachers College.* The present century, which began on Jan. 1, 1901, has in it the year 2000. Hence it contains 25 leap years, and it has 36525 days. Each of the next three centuries has 36524 days.

Jan. 1, 1901 was a Tuesday, and since $36525 \equiv 6 \pmod 7$, the last day of the present century will be a Sunday. Consequently, Jan. 1, 2001 will be a Monday.

And since $36524 \equiv 5 \pmod 7$ it follows that Jan. 1, 2101 will be a Saturday, Jan. 1, 2201 a Thursday, and Jan. 1, 2301 a Tuesday.

Each succeeding cycle of 400 years beginning with Jan. 1, 2301 will repeat the same days of the week as the 400 years beginning Jan. 1, 1901. Therefore the first day of a century cannot be a Sunday, a Wednesday, or a Friday.

In case a calendar of 1901 is not at hand to verify the fact that Jan. 1 of that year fell on Tuesday, we note that the number of days of the first 44 years of the century is $(365)(44)+11$, a number which is congruent to 6 (mod 7). Since Jan. 1, 1945 was a Monday, Jan. 1, 1901 was a Tuesday.

II. *Solution by G. W. Petrie, USNR Midshipmen's School, Notre Dame.* It is easily shown that the dth day of year Y is the Dth day of the week (starting with Sunday $=1$), where

$$D \equiv Y + \left[\frac{Y-1}{4}\right] - \left[\frac{Y-1}{100}\right] + \left[\frac{Y-1}{400}\right] + d, \bmod 7,$$

where $[x]$ indicates the greatest integer in x. For the first day of a century we have

$$d = 1 \quad \text{and} \quad Y = 100C + 1.$$

Hence

$$D \equiv 2(1-C) + \left[\frac{C}{4}\right], \bmod 7.$$

D is seen to be of period 4 and to take on the values 2, 0, 5, and 3 in succession. Hence the first day of a century cannot be a Sunday, a Wednesday, or a Friday.

Also solved by Colin Blyth, Betty Boyd, D. H. Browne, Monte Dernham, F. W. Saunders, E. D. Schell, and the Proposer. Gordon Pall pointed out that this problem occurs as puzzle 416 in H. E. Dudeney's *Amusements in Mathematics.*

Editorial Note. This, and most elementary calendar problems, can be solved by use of the calendar formulae of Gauss and Zeller. For the former see *Mathematical Recreations* by Kraitchik, for the latter see *Elementary Number Theory* by Uspensky and Heaslet; both may be found in the earlier editions of *Mathematical Recreations and Essays* by Ball.

A Triakis-Tetrahedron

E 677 [1945, 340]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let the altitudes of a tetrahedron $ABCD$ meet the circumsphere again in the points A', B', C', D'. Show that the volume of the solid $ABCDA'B'C'D'$ is three times that of the given tetrahedron.

Solution by L. M. Kelly, U. S. Coast Guard Academy. Let M, G, O be the Monge point, centroid, and circumcenter of $ABCD$ and let M_a, G_a, O_a, H_a be the feet of the perpendiculars dropped from M, G, O, and A on the face BCD.

Let a perpendicular from O to AA' meet AA' in point P. Then $AP=PA'$. Also, from elementary geometry of the tetrahedron, $\frac{1}{2}(MM_a+OO_a)=GG_a=\frac{1}{4}AH_a$. Therefore

$$\begin{aligned} MM_a &= \tfrac{1}{2}AH_a - OO_a = \tfrac{1}{2}(AP + OO_a) - OO_a \\ &= \tfrac{1}{2}(AP - OO_a) = \tfrac{1}{2}(PA' - PH_a) = \tfrac{1}{2}H_aA'. \end{aligned}$$

Hence volume of $A'BCD$ is twice the volume of $MBCD$, with similar relations for the other faces. This proves the proposition.

Also solved by the Proposer.

Circular Permutations

E 678 [1945, 340]. *Proposed by H. D. Grossman, New York City*

Prove that the number of circular permutations of $n=\sum_{i=1}^{k}a_i$ objects of which a_1 are alike, a_2 are alike, *etc.*, is

$$\frac{1}{n}\sum_{d}\phi(d)\,\frac{(n/d)!}{(a_1/d)!(a_2/d)!\cdots(a_k/d)!},$$

where d ranges over all divisors of the g.c.d. of $a_1, a_2, \cdots, a_k$.

Solution by H. S. Grant, Rutgers University. A formula for the number of circular permutations P referred to in this problem was given in the solution to problem No. 519 in the April 1944 issue of the *National Mathematics Magazine*. By using the Möbius function $\mu(s)$ with its well known properties, we have, in terms of the original notation of problem No. 519,

$$P_{d'} = \sum_{s\,|\,d/d'}\mu(s)\left[\frac{\left(\sum_{i=1}^{n}(x_i/d's)\right)!}{\prod_{i=1}^{n}(x_i/d's)!}\right] \div \sum_{i=1}^{n}(x_i/d'),$$

where s ranges over the $v(d/d')$ divisors of d/d', $v(x)$ signifying the number of divisors of x. The required number of circular permutations P is then given by

$$P = \sum_{d'\,|\,d} P_{d'},$$

where d' ranges over the $v(d)$ divisors of d.

In the notation of Grossman's proposal we have

$$P_{d'} = \sum_{s\,|\,D/d'}\mu(s)\left[\frac{\left(\sum_{i=1}^{k}(a_i/d's)\right)!}{\prod_{i=1}^{k}(a_i/d's)!}\right] \div \sum_{i=1}^{k}(a_i/d'),$$

where we have written $D=(a_1, a_2, \cdots, a_k)$, and s ranges over the $v(D/d')$ divisors of D/d'. Since $\sum_{i=1}^{k}a_i=n$, this readily reduces to

$$P_{d'} = \frac{1}{n} \sum_{s|D/d'} d'\mu(s) \left[\frac{(n/d's)!}{\prod_{i=1}^{k} (a_i/d's)!} \right],$$

whence

$$P = \sum_{d'|D} \left\{ \frac{1}{n} \sum_{s|D/d'} d'\mu(s) \left[\frac{(n/d's)!}{\prod_{i=1}^{k} (a_i/d's)!} \right] \right\}.$$

For each of the $v(D)$ values of d' there are $v(D/d')$ values of s, so that the double summation consists of a totality of $\sum_{d'|D} v(D/d')$ terms. Because of the symmetry in the values assumed by d' and s we have

$$P = \frac{1}{n} \sum_{d's|D} \sum d'\mu(s) \left[\frac{(n/d's)!}{\prod_{i=1}^{k} (a_i/d's)!} \right]$$

summed over the $\sum_{d'|D} v(D/d') = \sum_{d'|D} v(d')$ pairs of values of d' and s for which $d's$ divides D. These pairs of values of d' and s may be obtained by securing the $\sum_{d|D} v(d)$ pairs of values satisfying $d's = d$, d any divisor of D. Whence

$$P = \frac{1}{n} \sum_{d|D} \sum_{d's=d} d'\mu(s) \left[\frac{(n/d)!}{\prod_{i=1}^{k} (a_i/d)!} \right].$$

Since $\sum_{d's=d} d'\mu(s) = \phi(d)$ (Hardy and Wright, *Theory of Numbers*, p. 234),

$$P = \frac{1}{n} \sum_{d|D} \phi(d) \left[\frac{(n/d)!}{\prod_{i=1}^{k} (a_i/d)!} \right].$$

A Spherical Helix

E 679 [1945, 341]. *Proposed by Marshall Naul, Cumberland, Md.*

Compute the length of the curve of intersection of the unit sphere around the origin with the right helicoid $z = \theta$.

Editorial Note. We readily find the following parametric representation for the curve of intersection,

$$x = (1 - \theta^2)^{1/2} \sin \theta, \qquad y = (1 - \theta^2)^{1/2} \cos \theta, \qquad z = \theta.$$

Then, letting s be the required arc length and letting primes indicate differentiation with respect to θ, we have

$$s = 4 \int_0^1 (x'^2 + y'^2 + z'^2)^{1/2} d\theta = 4 \int_0^1 \left[\frac{\theta^4 - 2\theta^2 + 2}{1 - \theta^2} \right]^{1/2} d\theta.$$

The numerical value of the integral may be approximated by several methods.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4189. *Proposed by Albert Wilansky, Brown University*

Prove that

$$\sum_{r=0}^{m}(-1)^r\binom{m+k+\alpha}{r+k+\alpha}\binom{r+k}{k}=\binom{m+\alpha-1}{m}, \qquad \alpha \geqq 0.$$

4190. *Proposed by Norman Anning, University of Michigan*

If a, b, c, R are integers such that $a^2+b^2+c^2=R^2$, solve in integers the simultaneous equations

$$x^2+y^2+z^2=R^2$$
$$ax+by+cz=0.$$

There are at least four solutions.

4191. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$\frac{1}{\cosh(\pi/2)}-\frac{1}{3\cosh(3\pi/2)}+\frac{1}{5\cosh(5\pi/2)}-\cdots=\frac{\pi}{8}.$$

4192. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For a given tetrahedron $A_1A_2A_3A_4$, if P is the centroid of its antipedal tetrahedron $ABCD$, its barycentric coordinates are inversely proportional to the squares of its distances to A_1, A_2, A_3, A_4; and conversely.

4169 [1945, 400]. *Corrected. Proposed by Victor Thébault, Tennie, Sarthe, France*

The tangents at the vertices A, B, C of a given triangle to its Feuerbach hyperbola form a triangle whose conjugate circle is tangent to the ninepoint circle of ABC at its Feuerbach point.

SOLUTIONS

A Trigonometric Equation

4136 [1944, 533]. *Proposed by H. S. M. Coxeter, University of Toronto*

The equation

$$\cos \pi x + \cos \pi y + \cos \pi z = 0, \qquad 0 \leqq x \leqq \tfrac{1}{2} \leqq y \leqq z \leqq 1,$$

has the trivial solutions $y=\frac{1}{2}$, $z=1-x$, and $y=2/3-x$, $z=2/3+x$. It has also the non-trivial solution $x=1/5$, $y=3/5$, $z=2/3$. Prove that it has no other rational solutions.

Solution by W. J. R. Crosby, Saskatoon, Can. It will be shown* that the only solutions of the equation

$$\cos 2\pi r_1 + \cos 2\pi r_2 + \cos 2\pi r_3 = 0,$$

where r_1, r_2, r_3 are non-negative rational numbers less than unity, are of one of the following types:

(a) $$\cos \tfrac{1}{2}\pi + \cos 2\pi r + \cos 2\pi(\pm \tfrac{1}{2} \pm r) = 0,$$

(b) $$\cos 2\pi r + \cos 2\pi(r + \tfrac{1}{3}) + \cos 2\pi(r + \tfrac{2}{3}) = 0,$$

(c) $$\cos (2\pi/3) + \cos (\pi/5) + \cos (3\pi/5) = 0,$$

and equivalent solutions obtained from these by means of the relations

$$\cos 2\pi r = \cos 2\pi(1 - r) \quad \text{and} \quad -\cos 2\pi r = \cos 2\pi(\pm \tfrac{1}{2} \pm r),$$

where r is a non-negative rational less than unity. This will certainly establish the statement made in the problem.

Let $r_k = n_k/d_k$, where $0 \leqq n < d$ and n_k, d_k are mutually prime if $n_k \neq 0$. Let p be the largest prime which is a divisor of d_1, d_2, or d_3. Then we can find numbers δ_k, l_k, c_k, ν_k such that

$$d = \delta_k p^{l_k}, \quad \text{and} \quad n_k = c_k\delta_k + \nu_k p^{l_k},$$

where δ_k is prime to p, $0 \leqq c < p^{l_k}$, $c_k = 0$ if $l_k = 0$ but otherwise c_k is prime to p. Therefore, if we write f_k for ν_k/δ_k, we shall have

$$r_k = n_k/d_k = \nu_k/\delta_k + c_k/p^{l_k} = f_k + c/p^{l_k}.$$

We shall assume that $l_1 \geqq l_2 \geqq l_3$.
Now let

$$g_k(x) = \tfrac{1}{2}(e^{2\pi i f_k} x^{c_k p^{l_1 - l_k}} + e^{-2\pi i f_k} x^{p^{l_1}} - c_k p^{l_1 - l_k} \quad \text{if} \quad c_k \neq 0,$$
$$= \cos 2\pi r_k \quad \text{if} \quad c_k = 0,$$

and let

$$U(x) = \sum_{k=1}^{3} g_k(x).$$

Then

$$g_k(e^{2\pi i/p^{l_1}}) = \cos 2\pi r_k,$$

* The method is an adaptation of a method employed to solve a similar problem by P. Gordan: Math. Ann., Vol. 12, 1877, p. 29.

and consequently, if r_1, r_2, r_3 satisfy the given equation,

$$U(e^{(2\pi i/p^{l_1})}) = 0.$$

L. Kronecker* has proved that the polynomial

$$P(x) = 1 + x^{p^{l_1-1}} + x^{2p^{l_1-1}} + \cdots + x^{(p-1)p^{l_1-1}}$$

is not the product of two polynomials of smaller degree of which the coefficients are rational functions of sth roots of unity, if s is not a multiple of p. From this result we first deduce two lemmas.

LEMMA 1. *The polynomial $U(x)$ is divisible by the polynomial $P(x)$.*

Proof: The polynomials $U(x)$ and $P(x)$ are not prime to each other since both vanish for $x=\exp(2\pi i/p^{l_1})$. The greatest common divisor of these polynomials has coefficients which are rational functions of the coefficients of $U(x)$ and $P(x)$. Therefore, by Kronecker's theorem, the greatest common divisor must be $P(x)$.

LEMMA 2. *If we express $U(x)$ as a sum of polynomials $\sum_t U_t(x)$, where $U_t(x)$ contains those terms of $U(x)$ of the form bx^c with $c\equiv t$ modulo p^{l_1-1}, then $U(x)$ is divisible by $P(x)$, and consequently $U(\exp(2\pi i/p^{l_1}))=0$.*

Proof: If we divide $U_t(x)$ by $P(x)$, we obtain a relation of the form $U_t(x)=P(x)Q_t(x)+R_t(x)$, where the degree of $R_t(x)$ is less than that of $P(x)$, and the index of every power of x in $Q_t(x)$ and $R_t(x)$ is congruent to t modulo p^{l_1-1}. From this we deduce that

$$U(x) = \sum_t U_t(x) = P(x)\sum_t Q_t(x) + \sum_t R_t(x),$$

and therefore that $\sum_t R_t(x)=0$, by *Lemma* 1. Consequently $R_t(x)=0$ for each t.

We now divide our discussion of possible solutions of the equation into a number of special cases.

1. $l_1=1\geqq l_2\geqq l_3$. By *Lemma* 1, since the degree of $U(x)$ is less than p, and the degree of $P(x)$ is $p-1$, we must have $U(x)=mP(x)$ where m is a constant.

1.1 If $l_1=l_2=l_3=1$, then $U(0)=0$, and consequently $m=0$ since $P(0)=1$. Therefore $U(x)$ is zero for every x. If any one of the functions g_1, g_2, g_3, vanishes identically, it must contain only one power of x. If the sum of two terms representing two of these functions vanishes, it is easily seen that the sum of the two functions is zero, and therefore that the remaining function vanishes identically. Therefore either one of the functions g_1, g_2, g_3 contains only one power of x, or else the three functions each have the same two powers of x.

1.11 If one of the functions contains only one power of x, then for the corresponding value of k we find that $c_k=p-c_k$, and therefore $2c_k=p$, and we conclude that $p=2$ and $c_k=1$. Then, since p is the largest prime in d_1, d_2, d_3, we

* L. Kronecker: Journal de Math., Vol. 19, 1854, p. 177–192.

must have $d_1=d_2=d_3=2$. But this provides no solution for the problem and is therefore of no interest.

1.12. If the three functions g_1, g_2, g_3 each contain the same two powers of x, let b_1, b_2, b_3 be the three coefficients of one of the powers of x. Then the three coefficients of the other power of x are $1/b_1$, $1/b_2$, $1/b_3$, and $b_1+b_2+b_3=0$, $1/b_1+1/b_2+1/b_3=0$, *i.e.*, $b_2b_3+b_3b_1+b_1b_2=0$. Therefore b_1, b_2, b_3 are the three roots of the equation

$$z^3 - e^{2\pi i(3s)} = 0, \quad \text{where} \quad b_1b_2b_3 = e^{2\pi i(3s)}.$$

Consequently we may take $b_1=\exp(2\pi is)$, $b_2=\exp(2\pi i(s+1/3))$, $b_3=\exp(2\pi i(s+2/3))$. The corresponding solution is found to be

$$\cos 2\pi(s + c/p) + \cos 2\pi(s + 1/3 + c/p) + \cos 2\pi(s + 2/3 + c/p) = 0,$$

which is of type (b).

1.2. If $l_1=l_2=1$, $l_3=0$, then $m=\cos 2\pi r_3$.

1.21. If further $\cos 2\pi r_3=0$, then $\cos 2\pi r_1+\cos 2\pi r_2=0$, and consequently $r_2=\pm\frac{1}{2}\pm r_1$. We thus obtain a solution which is evidently of type (a).

1.22. If $\cos 2\pi r_3\neq 0$, then $U(x)=P(x)\cos 2\pi r_3$. Now the number of different powers of x in $P(x)$ is p, and in $U(x)$, 2, 3 or 5. Hence p is one of the numbers 2, 3, 5. But if $p=2$, the two functions g_1, g_2 contain only one power of x, and this case has been treated before in 1.11.

1.221. If $p=3$, then comparing coefficients we find that

$$\tfrac{1}{2}e^{2\pi if_1} + \tfrac{1}{2}e^{2\pi if_2} = \tfrac{1}{2}e^{-2\pi if_1} + \tfrac{1}{2}e^{-2\pi if_2} = \cos 2\pi r_3,$$

or

$$\tfrac{1}{2}e^{2\pi if_1} + \tfrac{1}{2}e^{-2\pi if_2} = \tfrac{1}{2}e^{-2\pi if_1} + \tfrac{1}{2}e^{2\pi if_2} = \cos 2\pi r_3.$$

In the first case $f_1=-f_2=\pm r_3$, and in the second $f_1=f_2=\pm r_3$. The corresponding solution in each case is of type (b).

1.222. If $p=5$, then

$$\tfrac{1}{2}e^{2\pi if_1} = \tfrac{1}{2}e^{-2\pi if_1} = \tfrac{1}{2}e^{2\pi if_2} = \tfrac{1}{2}e^{2\pi if_2} = \cos 2\pi r.$$

Hence $f_1=f_2=0$ and $\cos 2\pi r_3=\frac{1}{2}$, *i.e.* $r_3=1/6$, or $f_1=f_2=\frac{1}{2}$ and $\cos 2\pi r_3=-\frac{1}{2}$, *i.e.* $r_3=1/3$. If now c_1, c_2, c_3 are chosen in such a way that $U(x)$ has five different powers of x, solutions are obtained which are of type (c).

1.3. If $l_1=1$, $l_2=l_3=0$, then

$$U(x) = (\cos 2\pi r_2 + \cos 2\pi r_3)P(x).$$

1.31. If further $\cos 2\pi r_2+\cos 2\pi r_3=0$, then $\cos 2\pi r_1=0$, and the solution must be of type (a).

1.32. If $\cos 2\pi r_2+\cos 2\pi r_3\neq 0$, then, since $U(x)$ has two or three different powers of x, $p=2$ or 3. But since the case $p=2$ has been considered in 1.11, we assume that $p=3$. Then

$$\tfrac{1}{2}e^{2\pi i f_1} = \tfrac{1}{2}e^{-2\pi i f_1} = \cos 2\pi r_2 + \cos 2\pi r_3,$$

and each of these quantities must evidently be equal to $\pm\frac{1}{2}$. Hence $f_1=0$ and $\cos 2\pi r_2+\cos 2\pi r_3=\frac{1}{2}$, or $f_1=\frac{1}{2}$ and $\cos 2\pi r_2+\cos 2\pi r_3=-\frac{1}{2}$, where the denominators of r_2 and r_3 have no prime factor greater than 2. But this is obviously impossible.

2. Let us next assume that $l_1\geqq 2$, $l_1\geqq l_2\geqq l_3$. The indices of the powers of x appearing in g_1, g_2, g_3 are respectively c_1, $p^{l_1}-c$; $c_2p^{l_1-l_2}$, $p^{l_1}-c_2p^{l_1-l_2}$; $c_3p^{l_1-l_3}$, $p^{l_1}-cp^{l_1-l_3}$. When two of these indices are congruent modulo p^{l_1-1}, we shall say that the two corresponding terms of $U(x)$ are similar. Our conclusions in each case will be derived from *Lemma* 2.

2.1. If $l_1>l_2\geqq l_3$, the terms of g_1 are dissimilar to the terms of g_2 and g_3 and are similar to each other only if $p=l_1=2$ and $c_1=1$.

2.11. If $p=l_1=2$, then $g_1(\exp(2\pi i/4))=\cos 2\pi r_1=0$. We thus obtain a solution of type (a).

2.12. If $p\neq 2$, then $P(x)$ divides each term of $g_1(x)$. This is evidently impossible.

2.2. If $l_1=l_2\geqq l_3$, then the terms of g_3 are dissimilar to the terms of g_1 and g_2, and similar to each other only if $l_3=0$ or 1, or if $l_3=p=2$.

2.21. If $l_3=0$, or if $l_3=p=2$, then $\cos 2\pi r_3=0$, and we obtain a solution of type (a).

2.23. If $l_3=1$, then again $\cos 2\pi r_3=0$, and consequently $p=2$ and $l_3=2$. This is a contradiction.

2.24. If the two terms of g_3 are dissimilar to each other, then $P(x)$ divides both these terms; this is impossible.

2.3. If $l_1=l_2=l_3$, the two terms of each function g_1, g_3, g_3 are dissimilar to each other unless $p=l_1=l_2=l_3=2$, and therefore, except in this special case, there are at most three similar terms in $U(x)$.

2.31. If $p=l_1=l_2=l_3=2$, then the corresponding solution is of type (a).

2.32. If there are two pairs of three similar terms in $U(x)$, then the sum of each group of terms must vanish for $x=\exp(2\pi i/p^{l_1})$. The two equations obtained in this way have exactly the same form as the two equations for b_1, b_2, b_3 in 1.12. The corresponding solution is therefore of type (b).

Derangements

4146 [1945, 47]. *Proposed by Gaines N. Lang, Emory University*

Suppose that n men check their hats, and then each man takes a hat at random. Let $g(n)$ denote the probability that no one gets his own hat. Find the value of $g(n)$ and its limit for $n\to\infty$.

Solution by G. B. Huff, Southern Methodist University. This is the famous example of De Montmort, worked on page 24 of Coolidge's *Probability*. A reference is given there to *Essai d'analyse sur les jeux de hasard*, 2nd ed. Paris, 1713, p. 132. The following solution may be of interest.

The number of satisfactory arrangements $a(n)$ is subject to the recursion relation

$$a(n) = (n-1)[a(n-2) + a(n-1)]. \tag{1}$$

Indeed, the man numbered n must have the hat of man i, $1 \leqq i \leqq n-1$, and the latter does or does not have the hat numbered n. The first occurs in $a(n-2)$ ways and the second in $a(n-1)$ ways.

Now $g(n) = a(n)/n!$ and this with (1) yields the relation

$$g(n) - g(n-1) = -[g(n-1) - g(n-2)]/n. \tag{2}$$

Since $g(2) - g(1) = 1/2!$, it follows that $g(n) - g(n-1) = (-1)^n/n!$, and

$$g(n) = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}. \tag{3}$$

Clearly,

$$\lim_{n\to\infty} g(n) = e^{-1}.$$

Solved also by D. W. Alling, Murray Barbour, Colin Blyth, D. H. Browne, Howard Eves, R. E. Greenwood, C. O. Oakley, G. W. Petrie, Joseph Rosenbaum, Henry Scheffé, E. D. Schell, E. P. Starke, Alan Wayne, and the proposer.

Editorial Note. In addition to the above the following other references were given

(a) T. W. Anderson, Jr., *Annals of Math. Stat.* vol. 14, 1943, pp. 426 *etc.*
(b) Ball and Coxeter, *Math. Recreations and Essays*, 11th ed., p. 46.
(c) Murray Barbour, *School Science and Math.*, Prob. 883, 1925, p. 869; Prob. 938, 1927, p. 92.
(d) Chrystal, *Algebra*, vol. 2, pp. 22–25.
(e) Hall and Knight, *Higher Algebra*, 1942 ed., p. 404, ex. 3.
(f) Irving Kaplansky, *On a generalization of the Probleme des Rencontres*, this MONTHLY, 1939, 159.
(g) Levy and Roth, *Elements of Probability*, p. 97.
(h) Todhunter, *A History of the Mathematical Theory of Probability*, chap. 8, p. 93.
(i) J. V. Uspensky, *Introduction to Mathematical Probability*, pp. 168. *Elementary Number Theory*, 1939, pp. 105–107.
(j) Whitworth, *Choice and Chance*, chap. 2.

Several solutions are similar to the above and the remaining solutions are similar to special cases of the applications below of two types of inversion which will be stated. Starke is the only solver who mentions one of the general inversion formulas, and one of his solutions is based on the formula in the second reference of (i). The inversion formulas lead to a known generalization of the problem. One set of the inversion formulas follows.

Given the function $\phi(r-i)$ where i, r are integers, $0 \leqq i \leqq r$, set

$$\psi(r) = \sum_{i=0}^{r} {}_rC_i\phi(r-i). \tag{1}$$

Then

$$\phi(r) = \sum_{i=0}^{r} {}_rC_i(-1)^i\psi(r-i), \tag{2}$$

and conversely.

There are several proofs. It will be shown that (1) follows from (2). The right member of (1) becomes

$$\sum_{i=0}^{r} {}_rC_i \sum_{j=0}^{r-i} {}_{r-i}C_j(-1)^j\psi(r-i-j) = \sum_{i=0}^{r} {}_rC_i \sum_{t=0}^{r-i} {}_{r-i}C_i(-1)^{r-i-t}\psi(t)$$

$$= \sum_{t=0}^{r} {}_rC_i(-1)^{r-t}\psi(t) \sum_{i=0}^{r-t} {}_{r-t}C_i(-1)^i.$$

For $t<r$ the second sum in the last expression is $(1-1)^{r-t}=0$; for $t=r$ it is unity; and this completes the proof of this part. The proof that (2) follows from (1) is similar; or we may use the fact that the determinant for the set of equations (1) is not zero.

A second proof uses the operator $\delta_r=1-U_r^{-1}$ as follows;

$$\delta_r\psi(r) = \phi(r) + \sum_{j=1}^{r-1} ({}_rC_j - {}_{r-1}C_{j-1})\phi(r-j) = \sum_{j=0}^{r-1} {}_{r-1}C_j\phi(r-j).$$

Hence

$$\delta_r^r\psi(r) = \phi(r) = (1 - U_r^{-1})^r\psi(r) = \sum_{i=0}^{r} {}_rC_i(-1)^i\psi(r-i).$$

For the converse use $\bar{\delta}_r=1+U_r^{-1}$.

Another form of this inversion is

$$F(n, r) = \sum_{i=0}^{r} {}_rC_if(n-i, r-i), \tag{3}$$

$$f(n, r) = \sum_{i=0}^{r} {}_rC_i(-1)^iF(n-i, r-i), \tag{4}$$

where i, r, n are integrs $0\leqq i\leqq r\leqq n$.

An application. Consider the permutations, P's, of the elements $1, 2, \cdots, n$ in this order, and select a particular set of r of these elements, say the first r. A particular subset of i of these r elements is chosen, and denote by $\phi(r-i)$ the number of P's for which each of the i elements is in its original position and no one of the remaining $r-i$ elements is in its original place. Then (1) gives $\psi(r)=n!$ and $\psi(r-i)=(n-i)!$. We then have from (2) the number of P's, $\phi(r)$, for which no one of the chosen r elements is in its original place

$$\phi(r) = \sum_{i=0}^{r} {}_rC_i(-1)^i(n-i)!. \tag{5}$$

In the problem $r=n$ and we have

$$\phi(n) = n!\sum_{i=0}^{n} \frac{(-1)^i}{i!} = s(n), \qquad s(0) = 1, \qquad s(1) = 0, \tag{6}$$

$$s(n) = ns(n-1) + (-1)^n.$$

Some writers call the function $s(n)$ "subfactorial n."

In the second inversion formulas we consider N objects and n properties numbered $1, 2, \cdots, n$. A set of r of these properties is arbitrarily chosen, say the first r. Denote by $M^{(r)}_{a_1a_2\ldots a_i}$ the number of objects each having the indicated i properties selected from the chosen r and none of the remaining $r-i$; for $i=0$ we write $M^{(r)}$ which means the number of objects having none of the r properties. Denote by $N^{(r)}_{a_1a_2\ldots a_i}$ the number of objects with the i indicated properties chosen from the r and none or one or more of the remaining $r-i$; $M^{(r)}_{12\ldots r} = N^{(r)}_{12\ldots r}$, and for $i=0$ we write $N^{(r)}=N$. Then obviously

$$N = \sum_{i=0}^{r} \sum M^{(r)}_{a_1a_2\cdots a_i}, \tag{7}$$

where the second sum has ${}_rC_i$ terms. Then the previous inversion suggests that

$$M^{(r)} = \sum_{i=0}^{r} (-1)^i \sum N^{(r)}_{a_1a_2\cdots a_i}. \tag{8}$$

This is also suggested by the first two or three cases $r=1, 2, 3$, which are easily computed in turn. The simplest proof seems to be the following. Let a particular object have the m properties $a_1, a_2, \cdots, a_m$ chosen from the r and also the properties $b_1, b_2, \cdots, b_k$ chosen from the $n-r$, *i.e.*, excluding the r set, and no other properties, $0<m\leqq r$, and $0\leqq k\leqq n-r$. It will be shown that this object is eliminated in the counting in the right member of (8), and this will prove (8). For $i=0$ this object is counted once; for $i>m$ it is not considered; and for $i\leqq m$ it is counted $(-1)^i\,{}_mC_i$ times. In all it is counted a number of times equal to

$$\sum_{i=0}^{m} {}_mC_i(-1)^i = (1-1)^m = 0;$$

and this completes the proof.

An application. For the N objects we take the $n!$ permutations, P's, of the elements $1, 2, \cdots, n$ in this order. The property i is that the element i is in its original place, and we select r of the elements, say the first r. Then $M^{(r)}$ denotes the number of P's for which no one of the chosen r is in its original place; $N^{(r)}_{a_1a_2\ldots a_i}=(n-i)!$; and (8) becomes the formula (5) where $\phi(r)$ is replaced by $M^{(r)}$.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

The University of Wisconsin in Milwaukee wishes to exchange (or sell) some duplicate volumes of mathematical books and American and foreign periodicals. Any library or individual interested should write Professor Morris Marden, 623 W. State Street, Milwaukee, Wisconsin.

Dr. J. J. Barron of Brown University has been appointed to an assistant professorship at Marquette University.

Associate Dean Walter Bartky of the University of Chicago has been made dean.

Dr. Lipman Bers has been appointed to an assistant professorship at Syracuse University.

Associate Professors D. G. Bourgin and J. L. Doob of the University of Illinois have been promoted to professorships.

Professor Emeritus L. L. Dines of the Carnegie Institute of Technology has been appointed to a visiting professorship at the University of Saskatchewan.

Assistant Professor Churchill Eisenhart of the University of Wisconsin has been promoted to an associate professorship.

Mary L. Elveback has been appointed to an assistant professorship at Rockford College.

Dr. P. E. Guenther of the Case School of Applied Science has been promoted to an assistant professorship.

Assistant Professor Israel Halperin of Queen's University, Kingston, Ontario, has been promoted to an associate professorship.

Dr. E. D. Hellinger of Northwestern University has been appointed to a professorship.

Dr. H. A. Jordan of Harvard University has been appointed to an associate professorship at Mount Holyoke College.

Associate Professor S. C. Kleene of Amherst College has been appointed to an associate professorship at the University of Wisconsin.

Dr. C. C. Lin of the California Institute of Technology has been appointed to an assistant professorship at Brown University.

Dr. H. B. Mann of Brown University has been appointed to an associate professorship at Ohio State University.

Dr. Margaret E. Mauch of Michigan State College has been appointed to an assistant professorship at the University of Akron.

Dr. Josephine M. Mitchell of Connecticut College has been appointed to an associate professorship at Winthrop College.

Dr. R. S. Pate has been appointed to an associate professorship at the University of South Carolina.

Assistant Professor R. F. Rinehart of the Case School of Applied Science has been promoted to an associate professorship.

Assistant Professor O. F. G. Schilling of the University of Chicago has been promoted to an associate professorship.

Dr. Seymour Sherman has been appointed to an assistant professorship at the College of the University of Chicago.

Dr. C. L. Siegel has been appointed to a professorship at the Institute for Advanced Study.

Dr. Ernst Snapper of Princeton University has been appointed to an assistant professorship at the University of Southern California.

Assistant Professor N. E. Steenrod of the University of Michigan has been promoted to an associate professorship.

Dr. W. R. Transue has been appointed to an associate professorship at Kenyon College.

Dr. Bernard Vinograde of Tulane University has been appointed to an assistant professorship at Iowa State College.

Dr. E. L. Welker of the University of Illinois has been promoted to an assistant professorship.

Dr. Jacob Wolfowitz has been appointed to an associate professorship at the University of North Carolina.

Professor Emeritus H. F. Blichfeldt of Stanford University died November 16, 1945.

Assistant Professor Annie W. Fleming of Iowa State College died September 19, 1945.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

NEW SELECTIVE SERVICE POLICY PERTAINING TO SCIENTISTS

The National Headquarters of Selective Service issued Local Board Memorandum No. 115-M on November 26, 1945. Because it is of importance to mathematicians it is reproduced below.

PART I

1. *General.*—The demands of the armed forces and industry during the emergency have resulted in a curtailment in advanced studies for men having high technical and scientific qualifications. Since the cessation of active fighting, the demands of long range national interest require a resumption of these studies in order to fully develop the technical and scientific skills which have been acquired and to provide adequate teaching facilities for returning veterans who desire to resume their studies in these fields.

2. *Reconversion Working Committee on Deferment and Selective Release.*—(a) Pursuant to a request from the Director of War Mobilization and Reconversion, the Reconversion Working Committee on Deferment and Selective Release has been established to assist in the accomplishment of this purpose. The Committee is composed of representatives of the (1) Office of Scientific Research and Development; (2) Civilian Production Administration; (3) Office of Rubber Reserve; (4) Petroleum Administration for War; (5) War Department; (6) Navy Department; (7) United States Employment Service; (8) Office of Education; (9) National Roster of Scientific and Specialized Personnel; and (10) the Selective Service System. The Chairman of the Committee will be the Director of War Mobilization and Reconversion, or a staff member designated by him. The functions of the Committee are to:

(1) Indicate to the Director of War Mobilization and Reconversion the specific occupations in which shortages of personnel threaten to interfere with the national health, safety, or interest.
(2) Formulate the specific standards indicating that a man is qualified to engage in a selected occupation.
(3) Certify to the Director of the Selective Service System those individuals meeting the standards established by the Committee.
(4) Indicate to the War and Navy Departments the categories of occupations in which shortages detrimental to the national interest could be relieved by release of men from the armed forces.

PART II

1. *Operation of plan.*—The Director of the Office of War Mobilization and Reconversion will examine all proposals by the Committee relating to deferment and release, and shall transmit to the Selective Service System and the War and Navy Departments all approved proposals. The Director will transmit the recommendations for deferment to local boards through the appropriate State Directors.

2. *Classification policies.*—Pursuant to the provisions of this memorandum and under the general authority contained in paragraph 5 of Part II of Local Board Memorandum No. 115, as amended, local boards will give serious consideration to the occupational deferment of registrants engaged in the technical and scientific fields set forth in Part III of this memorandum.

PART III

1. *Advanced studies in the physical sciences or engineering.*—(a) Any registrant who is accepted by an accredited college or university as a candidate for a Master's or Doctor's degree in the physical sciences or engineering may be certified by the Office of War Mobilization and Reconversion to the Director of Selective Service as essential to the national interest in a civilian capacity.

(b) The fact that a candidate for a Master's or a Doctor's degree may engage in part-time employment or other activities will not affect his certification under this paragraph so long as his academic standing is satisfactory.

2. *University teaching in the physical sciences or engineering.*—Any registrant who is to be employed by an accredited college or university as a teacher of physical sciences or engineering may be certified by the Office of War Mobilization and Reconversion to the Director of Selective Service as essential to the national interest in a civilian capacity.

3. *University research in the physical sciences or engineering.*—(a) Any registrant (1) who is to be employed by or attached to the staff of an accredited college or university for research in the physical sciences or engineering and (2) who signifies his intention to engage in such an activity may be certified to the Director of Selective Service as essential to the national interest in a civilian capacity.

(b) Such a registrant will be certified only if (1) the research to be undertaken by the registrant contributes significantly to the national interest, and (2) inability of the individual registrant to undertake the research will result in its delay.

4. *Submission of information in certain cases.*—Any registrant who wishes to be certified under the provisions of paragraphs 1, 2, and 3 above, must present to the Office of War Mobilization and Reconversion, Washington, D. C., the following documents in triplicate:

(a) A notarized statement of his intention:

(1) To undertake graduate studies leading to a Master's or Doctor's degree in the physical sciences or engineering; or

(2) To engage in the teaching of physical sciences or engineering; or

(3) To undertake advanced research in the physical sciences or engineering on the staff of or attached to a college or university.

(b) A statement from an accredited college or university signed by a responsible official of the college or university, indicating that the registrant:

(1) Has been accepted as a candidate for a Master's or Doctor's degree in the physical sciences or engineering; or

(2) Has been accepted as a teacher of physical sciences or engineering; or

(3) Is to be employed by or attached to the staff of the college or university for research in the physical sciences or engineering, together with a statement indicating in detail the nature of the research to be performed by the registrant, the scope of the registrant's responsibilities for the research, and the necessity for securing the individual registrant for the work.

(c) A statement showing the registrant's name, address, age, Selective Service local board number and address, classification, and educational qualifications.

5. *Study leading to a B.S. or B.A. degree in physical sciences or engineering.*— (a) Any registrant who has satisfactorily completed at least three years of work leading to a Bachelor's degree in the physical sciences or engineering may be certified by the Office of War Mobilization and Reconversion to the Director of Selective Service as essential to the national interest in a civilian capacity, provided such registrant has served for a period of not less than two years in a project directly connected with the war effort.

(b) Any registrant who wishes to be certified under the provisions of this paragraph must present to the Office of War Mobilization and Reconversion, Washington, D. C., the following documents in triplicate:

(1) A notarized statement of his intention to continue undergraduate studies leading to a B.S. or B.A. degree in the physical sciences or engineering.

(2) A statement from an accredited college or university, signed by a responsible official of the college or university, indicating that the registrant has been accepted for the fourth year of study in a course leading to a B.S. or B.A. degree in the physical sciences or engineering.

(3) A statement showing the registrant's name, address, age, Selective Service local board number and address, classification, and educational qualifications.

(4) A statement from the registrant's employer that he has been engaged for at least two years in scientific war work.

COLLEGE MATHEMATICS CREDIT AND VOCATIONAL ADVICE FOR THE VETERAN

C. C. HURD, Allegheny College*

1. Introduction. Veterans are returning to college in increasing numbers. Departments of Mathematics will therefore undoubtedly wish to devote careful attention to the administrative problems of: (a) Granting appropriate credit for any in-service mathematical experience which the veteran may have had; (b) Disseminating educational and vocational advice to those veterans who may be interested in mathematics as a career.

These problems are here stated and discussed in terms of the veteran. However, it is clear that certain analogous problems exist with respect to the college student who is not a veteran.

2. The Granting of Credit. Many institutions and professional associations have accepted in principle the policy urged by the American Council on Education[1] that credit for military experience be granted "in terms of demonstrated educational competence." In order that this policy may be implemented with respect to granting credit in mathematics the Subcommittee on Examinations, War Policy Committee of the A.M.S. and M.A.A., has prepared USAFI multiple choice type subject examinations in College Algebra, Plane Trigonometry, Analytical Geometry, and Calculus.[2] The Cooperative Test Service is standardizing these examinations and the Subcommittee has stated its expectancy that various schools will desire to determine supplementary data on Form B of these tests for their own information and use. These examinations, whose topic content was determined "on results of extensive sampling of the opinions of college departments of mathematics," and which were prepared by men competent in mathematics and skilled in administration, appear to be unusually good multiple choice type examinations.

Two related questions immediately arise however. Is any multiple choice type examination, however good, adequate for the evaluation problem at hand? How shall the examination results be used with respect to the individual veteran?

Veterans will have obtained in-service mathematical experience through three principal channels: In-Service Training Schools,[3] USAFI Correspondence Courses,[4] and Off-Duty Educational Programs.[5] The questions raised above with

* Sections 1 and 2 were written while the author was on active duty at the United States Coast Guard Academy.

[1] Sound Educational Credit for Military Experience, American Council on Education, Washington, D. C., 1943, p. 2.

[2] See Activities of the Committee on Examinations this MONTHLY, March, 1945, pp. 173–175.

[3] See A Guide to the Evaluation of Educational Experiences in the Armed Services, American Council on Education, Washington, D. C., 1944.

[4] See USAFI courses in college mathematics, this MONTHLY, May, 1945, pp. 287–289.

[5] See McGrath, E. J., Off-duty education programs of the U. S. Navy, Educational Record, January, 1944, pp. 35–47.

respect to the adequacy of and the manner of using the USAFI examinations for evaluation will be discussed for:

a. The veteran who will study no mathematics in college and offers in-service mathematical experience for "terminal" or non-prerequisite credit.

b. The veteran who offers in-service mathematical experience for "pre-requisite credit" for more advanced mathematics courses and who is not a prospective mathematics major.

c. The veteran who expects to major in mathematics.

a. *Terminal Credit.* For the purpose at hand the principal advantages of using multiple choice examinations appear to lie in speedy and uniform scoring methods.[6] A department which expects large numbers of applicants for terminal credit might therefore elect to use a USAFI subject examination alone if it believes that *in fact* the examination measures the objectives which the department has established for the course in question. In the contrary case the department might choose to supplement or indeed to replace the USAFI examination by an examination of its own construction. For example, a department whose "general mathematics course" has, among others, the objectives of giving the student an acquaintance with the history of mathematics, an appreciation of mathematics as a fruitful example of logic, and an understanding of the significant place which mathematics occupies in our cultural and intellectual life, would certainly not grant a veteran credit for this course on the basis of USAFI examination results alone.

Whatever device may be adopted for evaluating terminal credit it is important that the "rate of forgetting" of unused concepts and skills be recognized and compensated for. Instructors in the colleges and in the Armed Forces have been startled, for example, by the trigonometric incompetence in the year of 1942, say, of a college man who "earned" an *A* or *B* grade in Trigonometry in a good college in the year of 1939. Since there is every reason for treating the veteran fairly and since there is no conceivable reason for discriminating against him, it is suggested that any individual departments which provide no pre-examination review adjust their terminal credit evaluation standards for time lag. Therefore, a veteran who studied Trigonometry in 1942 by correspondence and who offers this experience for terminal credit in 1945 would be expected to know no more about trigonometry than one of "our own" students who has been away from trigonometry for three years.

b. *Pre-requisite Credit.* The USAFI subject examinations are comprehensive in their coverage of skills and concepts. If a department offers beginning college mathematics in the traditional fashion (algebra, trigonometry, analytical geometry, calculus), a satisfactory total score on a USAFI subject test, coupled with satisfactory scores on each of the subtests, should indicate the veteran's ability to enter the next course in the sequence. Consequently, credit should be

[6] The possibility of using the results of these tests for standardization between departments of mathematics poses an educational question which should be considered independently of the veteran problem. See McGrath, E. J., "The procrustean bed of higher education," School and Society, vol. 61, 1945, pp. 81–84.

given for the pre-requisite course. Here "satisfactory" must be defined by the individual department on the basis of its experience with the test and on the basis of its own standards.

The department which teaches "unified mathematics" on the freshman and/or sophomore levels may find the USAFI sub-test scores of value if it decides that the granting of partial credit in these courses is practicable.

c. *Credit for Majors.* There are men in the Armed Forces and in war industry today who are high in quantitative intelligence, who are interested in mathematics, and who would be concentrating in mathematics on the undergraduate or graduate level had not the war intervened. Some of these men have been able to continue their study of mathematics and other subjects not only through the in-service opportunities mentioned above, but through correspondence with former teachers, independent reading, conversation, and travel. In view of the increasingly critical shortages of professional mathematicians in education, industry, and government it is imperative that educational institutions adopt the most enlightened means of discovering these men and of discerning their present level of mathematical maturity and general educational competence.

For this purpose a department or institution might wish to include some or all of the following measures: (1) A scholastic aptitude test, (2) USAFI college level tests of General Educational Development (Correctness and Effectiveness of Expression, Interpretation of Reading Material in the Social Studies, Interpretation of Reading Material in the Natural Sciences, Interpretation of Literary Materials), (3) USAFI subject tests in mathematics and in any other subjects in which the veteran believes himself to be competent, (4) The Graduate Record Examination[7] profile tests and advanced mathematics test, (5) Short written papers, (6) An informal oral examination, (7) An appraisal of the candidate's formal educational record.

If such a survey reveals that the veteran is relatively mature mathematically and intellectually, and that his formal credits do not measure up to his achievement, it is urged that he be placed immediately in the advanced courses of which he is capable and that appropriate pre-requisite credit be granted.

3. Disseminating Educational and Vocational Advice. Acting under Public Law 346, Servicemen's Readjustment Act of 1944, and under Public Law 16, the Vocational Rehabilitation Program, the Veterans Administration released on March 5, 1945,[8] the names of the first fifty colleges and universities in which Veterans Guidance Centers have been established. All disabled veterans who desire vocational rehabilitation under Public Law 16 will report to these centers for testing and educational and vocational counseling. Veterans who undertake educational courses under Public Law 346 are urged, though not required, to

[7] Published by the Carnegie foundation for the Advancement of Teaching. The word "Graduate" should not imply that these tests are useful only for graduate students. Norms are available for students completing their sophomore year in college. Indeed the writer found the GRE results to be a valuable aid in deciding which fourth class (freshman) cadets at the U. S. Coast Guard Academy were best qualified for an "honors program" in mathematics.

[8] See OWI 4127, Office of War Information, Washington 25, D. C.

take advantage of the services of these centers. Do these Veterans Guidance Centers have adequate information regarding the vocational possibilities of the study of mathematics? The "Dictionary of Occupational Titles,"[9] usually an important aid to counselors, is certainly not too revealing from the vocational standpoint when it states "Mathematician (education). A term applied to a worker who has attained eminence in some field of mathematics and is a recognized authority in that field."

Numerous as the veterans will be in educational institutions in the next few years, their number is relatively small compared with the thirteen or more million civilian students who will undoubtedly enter college in the next ten years. Therefore another important question is: Do the hundreds of educational and vocational counselors in high schools and colleges of the country have adequate information regarding the vocational possibilities of the study of mathematics?

The answer to the above questions must be "NO!" Even though there are some publications which provide important information on the problem[10] it nevertheless appears to be true that in no set of easily accessible documents can the counselor or the student obtain answers to such important questions as: What special abilities and interests are required for success in mathematics; how can these abilities be measured? What special education and training are required; where is it obtainable; how much will it cost? For what vocational areas does the study of mathematics prepare one? What are the employment trends in these vocational areas? What are the beginning and average salaries? What are the associated satisfactions? What prestige is associated with these jobs? What is the principal activity employed in these jobs? Is a preliminary job tryout available? What college courses should supplement mathematics if one of these vocational areas is chosen?

It is gratifying to learn[11] that the Board of Directors of the National Council of Teachers of Mathematics has agreed that the next job of its Commission on Post-War Plans "is to create, or cause to be written, a relatively small but effective pamphlet on mathematics to be used in the guidance of the junior and senior high school pupil." Such a project deserves the full cooperation of all who are interested in mathematics. Since this full cooperation will doubtless be forthcoming, the pamphlet will be a great aid to counselors on the secondary level and it will furnish valuable clues to counselors on the college level. Unfortunately, the completion of such a task is time consuming and it may be a year before the pamphlet is published. The problem of advising the veteran is here and now. The newspapers recently announced that the War Department expects to discharge 1,100,000 in October, 1945, alone. Perhaps as many as 100,000

[9] Part 1, Definition of Titles, U. S. Government Printing Office, 1939.

[10] For example, see Thornton C. Fry's discussion of Industrial Mathematics, reprinted from Research—A National Resource—II, Section VI, Part 4, pp. 268–288, A House Document, 77th Congress. Printed also by the Bell Telephone Laboratories.

Mathematics—Its vocational aspects, Bulletin, Oklahoma A. and M. College, vol. 42, No. 13, May, 1945.

[11] See Zant, J. H., The next step in planning for post-war mathematics, The Mathematics Teacher, October 1945, pp. 276–277.

of these men and women will desire further schooling. How can those veterans who are interested in mathematics learn about mathematics as a career?

Many colleges are establishing centers to which servicemen can come for advice. Many cities are establishing guidance centers for veterans. The Veterans Administration has a "contact man" and a "training officer" near you. All of these agencies are willing and able to place information at the disposal of the veteran. What information can we give these agencies?

If your college has an efficient central placement bureau you can quickly learn where each of your majors, for the past ten years say, found his first job and at what rate of pay. It will be simple to check his advancement and learn the contribution which his training allowed him to make to the war effort. Don't neglect the teacher training placement officer. His records will show both the frequency of calls for high school teachers of mathematics and the average starting pay in comparison with other fields. If no placement bureau records are available, an informal departmental meeting will yield an amazing amount of information about the jobs which your majors have pursued successfully.

All of the information assembled in this fashion should be concisely summarized and appended to information concerning those opportunities for college students of mathematics which are well known to mathematicians but not necessarily known to vocational counselors who are concerned with 25,000 or more occupations. I refer here to college teaching, actuarial work, statistical work, industrial mathematics, the service Academies, and Civil Service positions. This information should then be delivered in person to all agencies associated with veterans in your locality and, of course, any unusual items should be submitted to the Department of General Information in this MONTHLY.

CALENDAR OF FUTURE MEETINGS

Twenty-eighth Summer Meeting, Ithaca, New York, August 19–20, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April 13, 1946
ILLINOIS, Normal, May 10–11, 1946
INDIANA
IOWA, Grinnell, April 19–20, 1946
KANSAS, Emporia, April 6, 1946
KENTUCKY, Louisville, April 27, 1946
LOUISIANA-MISSISSIPPI, Ruston, La., March 15–16, 1946
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, N. Y., May 11, 1946
MICHIGAN, Ann Arbor, April 13, 1946
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, April 4, 1946
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN, Boulder, Colo., April 19–20, 1946
SOUTHEASTERN, Raleigh, N. C., April 19–20, 1946
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REHABILITATION OF GRADUATE WORK*

J. R. KLINE, University of Pennsylvania

Your commitee has asked me to discuss the subject of the Rehabilitation of Graduate Work. In this address I shall confine the discussion to the rehabilitation of graduate work in mathematics. Emphasis must, however, be placed upon the fact that the rebuilding of graduate work in *all fields of knowledge* is of vital concern to mathematicians and, in fact, to any group whose primary interest is in a particular science. Just as the human body is a unit and all parts must be strong and well-coordinated to insure a healthy being, so physical science and mathematics cannot flourish while the humanities and other academic fields are neglected, so far as research and graduate activity are concerned. If an overwhelming and disproportionate amount of the nation's superior talent is drawn into science, the result will be detrimental to the nation and to science. Plans for the recovery and development of scientific talent must be related to society's other needs for intellectual leadership. The problem of the rehabilitation of graduate work in the social sciences and the humanities is perhaps an even more difficult one than in mathematics, for full-scale training at the graduate level has been interrupted in those fields for a longer period than in science.

Let us review briefly the growth of graduate work in mathematics in the United States as shown by the number of Ph.D. degrees granted in various periods.

Period	Average Number of Ph.D.'s Granted Annually
1900–04	12
1905–09	18
1910–14	24
1915–19	24
1920–24	24
1925–29	45

It is interesting to note that during the period 1915–19, the same average was maintained as in the previous five years, in spite of the fact that the First World War occurred in this interval and that in the academic year 1918–19, there were only nine doctorates in mathematics. In the period 1920–24, the number did not increase because of the interruption of training of undergraduate students during the war; these men would have been the potential graduate students for the first half of the twenties. In the second half of the twenties, graduate work was increasing very rapidly; in 1929, there were sixty-three doctorates awarded in mathematics.

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* Address delivered at the Annual Meeting of the Mathematical Association of America on November 25, 1945.

For the remainder of the period which I am discussing, the figures are given for each year.

Year	No. of Ph.D.'s	Year	No. of Ph.D.'s
1930	78	1938	69*
1931	76	1939	104
1932	80	1940	89
1933	77	1941	104
1934	96	1942	76
1935	86	1943	51
1936	82	1944	39
1937	72*		

During the period 1940–44, the number of master's degrees granted were as follows:

Year	No. of Master's Degrees
1940	303
1941	175
1942	168
1943	112
1944	114

Observe the very rapid decrease in the number of doctorates and master's degrees since 1941. You will note that, in 1944, American universities granted less than forty per cent of the number of doctorates conferred in 1941. Figures are not available for 1945 but the number is probably smaller than in 1944.

Let us examine the reasons for this rapid decrease. The first of these is the fact that great numbers of graduate students and prospective graduate students, who were undergraduate majors in mathematics, entered the armed forces. Some of these were driven by social pressure to volunteer when it would have been to the best interests of the country for them to continue their scientific training. A much greater number were inducted under the operation of the Selective Service Act. Until July, 1942, mathematicians did not qualify for deferment and both graduate students and teachers of mathematics were classified I-A, with practically no opportunity for successful appeal. Finally, success crowned the efforts of those who argued that the country should conserve and increase its supply of mathematicians and in July, 1942, mathematicians were listed among those engaged in critical occupations. However, even then a graduate student was eligible for deferment only if he were also teaching at least fifteen hours per week. This left very little time for effective graduate work. Under the operation of this directive, deferment of graduate students and teachers of mathematics was not automatic. Cases were sometimes decided adversely by local boards which, while doing a magnificent piece of work in general, were not in possession of the expert knowledge necessary for the appraisal of highly specialized personnel.

* The drop in the number of Ph.D.'s in these years may be due, in part, to the decrease in undergraduate attendance in our colleges and universities during the depression.

However, there was a period, beginning in May, 1943, when Selective Service machinery was put on a satisfactory basis by the creation of the National Committee on Physicists and Mathematicians. This was again changed in November, 1943, by the abolition of this committee and the substitution of procedures involving certification by the United States Employment Service. This machinery was abolished in April, 1944, insofar as men under twenty-six were concerned and, consequently, almost all physically able men engaged in teaching and studying mathematics were inducted. As a result, graduate work in mathematics was carried on only by women students, men classified as IV-F, and men in war industry who carried on graduate work in the evening. The number of doctorates awarded since 1941 would have been still smaller, if there had not been a group of men and women who had completed the course work and had started on dissertations before being drawn into war work. Many of these persons were able to finish their dissertations while engaged in war activities.

Another reason for the decrease in the number of doctorates is that many persons entered war work in connection with various projects such as the Radiation Laboratory and Aberdeen Proving Ground, because they wished to use their mathematical training as fully as possible for the nation's war effort. They were not at all certain that their training would be used effectively by the armed forces and, consequently, they chose to enter fields where there was emphasis on mathematical and physical training. In many cases, men of this group are now being inducted because the agencies can no longer ask for their deferment. Unless remedial action can be taken as indicated later, these men will probably be in the service for a long time because they will have very low point scores and will be the last persons demobilized.

It must be borne in mind that many women who were potential graduate students also interrupted their work to enter various phases of war activity as computers and statisticians.

A discouraging feature of this situation is the fact that we cannot rely upon the undergraduate majors in science to furnish us with the new supply of graduate students. Undergraduate students, other than engineers and pre-medical students, had even less protection than graduate students and, consequently, the number of available male graduates from colleges and universities, with majors in mathematics, is very far below normal.

M. H. Trytten of the Office of Scientific Personnel has made an extensive study of the shortages in the various sciences, due to the cessation of training during the war. It is his conservative opinion that it will be 1955 before we will be producing the number of Ph.D.'s that we would have expected to produce each year, if the growth of graduate work had continued at the same rate as in the decade 1930–40. Thus we will have lost a large number of trained scientists during the period 1940 to 1955.

Let us examine the effect of this loss of scientists. In the first place, *it lowers the scientific competence of the nation.* I hold it as fundamental that one of the greatest sources of strength that any nation can have is that power which comes

to it through its *research* in fundamental science. Kill the impulse that leads scholars to devote their full efforts to the search for truth for the sake of truth alone, and you have destroyed the impetus for intellectual progress and one of the nation's richest heritages. Mathematics, that was developed merely as a purely intellectual investigation into the consequences of certain assumed hypotheses, has made tremendous contributions to all phases of our national war effort. Likewise, it has been the means by which other sciences have been able to make much more significant contributions.

In order that real progress may be made, scientific research must always continue as a great adventure into the unknown. I quote from the physicist, Rabi: "To set out a detailed program with merely practical goals for truly scientific research is like trying to make a map of a country no one has ever seen and the very existence of which is in grave doubt. Pure science cannot have any goal other than the appeasement of the human spirit of intellectual adventure." I hold that if progress in pure science is not maintained by training a future generation that will make up the deficit that now exists in our personnel, then truly we shall lose the peace, even though we have won the war.

At this point I should like to quote from a recent article by Vannevar Bush:

"Americans who sincerely want to share in securing and maintaining the peace of the world through a strong international organization have their work plainly before them. We must all judge our courses as good or bad according to how they will help or hinder the strength with which the nation goes forward, not as a possible combatant in a world of isolated, suspicious states but as one free people among other free peoples. This applies to the engineer in the fields, the scientist in the laboratory, the industrialist at his desk, the mechanic at the lathe—to all of us as individuals, and to all of us as groups. No individual, no group has any right to override the good of the nation as a whole. At this critical time, any individual who places a selfish interest above the good of his country, above its strength in a difficult world, should pause to think that by so doing he may be making it necessary for his children or grandchildren to fight in a desperate war.

"The realistic sense of moral responsibility which I stress here is basic. With it, we have the foundation on which the bulwarks of national strength can be built.

"I place high in the list of these the maintenance of vigorous research in fundamental science, with Federal financial aid for the support of research programs and for the education of future scientists, and with stress on the fact that fundamental research demands at the same time the highest degree of freedom and initiative for the individual."

In the second place, *these scientists are needed by colleges and universities* for their research staffs and for their teaching staffs. The amount of teaching of mathematics which must be done by our colleges and universities in the post-war years will be very great. Those of you who remember the period just after the First World War will recall the tremendous influx of students to our colleges in

the years 1919 and 1920. Even though graduate training had been interrupted for a much shorter period, the number of men who had completed the training up to the doctorate was inadequate to meet the demand. Many institutions of good standing were not able to maintain the requirement that an instructor hold the Ph.D. degree; indeed, many were unable to do so until about 1925 and 1926.

Let us look at the present situation. The figures to be given are based on a study made during the past summer of staffs of all colleges and universities. According to these statistics, there were 1,675 persons with the Ph.D. degree among the 4,600 teachers of mathematics who were employed by the colleges and universities of this country. The number 4,600 includes all members of mathematical teaching staffs of the rank of instructor and above. Included in the tabulation were colleges and universities, some professional and technological institutions, most junior colleges, and teachers colleges. Of the 4,600 teachers of mathematics, about 11% were on leave of absence.

It is interesting to analyze these figures still further. I have divided the institutions as follows: (1) Member institutions of the Association of American Universities; (2) Institutions approved by the A.A.U.; (3) Institutions not approved by the A.A.U. (*i.e.*, those institutions which, according to A.A.U. judgment, do not maintain curricula and standards which fit their students for graduate work).

(1) In the first group, member institutions of the Association of American Universities (33 institutions), there are 449 persons with the rank of instructor or higher. Of these, 355 (or approximately 80%) hold the Ph.D. degree. This percentage is surely lower than normal because there are a number of notable instances where institutions have been forced to appoint to instructorships persons without doctor's degrees, in order to meet their teaching demands. This is a policy which these institutions would not have countenanced in pre-war years.

(2) Institutions on the approved list of the A.A.U. In this list there are 266 colleges and universities, having 1,240 members of the mathematics departments. Of these 1,183 are of the rank of instructor or higher and 677 hold the Ph.D. degree; in other words, 57% of those who are instructors have attained the doctorate. In addition, there are 23 approved technological institutions, such as Georgia School of Technology, with 315 members of the mathematics departments, of whom 251 are of the rank of instructor or higher. Of these, 146, or 58%, hold the doctorate. Thus, in the institutions of each type in this group, there is approximately the same percentage of persons with the Ph.D. degree.

(3) Non-approved institutions. Slightly more than 2,500 of the teachers of mathematics in colleges and universities are in institutions not on the approved list of the A.A.U. About 500 persons with the Ph.D. degree are at institutions of this group. Thus in institutions of this group, about 20% of the mathematics faculties hold the Ph.D. degree.

We may expect that there will be a great demand from institutions in this third group for mathematicians with advanced training up to the doctorate, in order to improve their academic standing and thereby gain recognition and

approval of their students for graduate work. It is the general practice for most institutions to refuse admission to the graduate schools of students from non-approved institutions, unless these students qualify on the basis of the Graduate Record Examination. Prospective graduate students from this group are not, by any means, to be disregarded; from some of the small colleges we may secure excellent graduate students. In fact, this spring there came to my attention a candidate from a non-approved institution who made one of the best records on the Graduate Record Examination I have ever seen.

It should be borne in mind that the figures given above display a somewhat distorted picture with respect to the number of doctors on mathematical staffs. Many departments of mathematics in our universities and colleges have suffered losses by death and retirement during the war and have made no permanent solution of the problem of filling these vacancies but have made temporary appointments. Many professors who have reached the age of retirement have been requested to continue until the close of the war, but these men will wish to retire now.

Another factor that must be borne in mind is the fact that industry and governmental agencies are likely to employ men trained in graduate mathematics in increasing numbers, thus making still greater demands upon our supply of mathematicians.

To meet the demands of both the academic and the industrial world, we must first see that there is an adequate supply of superior men to teach and to supervise the research of those taking graduate work in mathematics. In this country, we have been fortunate that there have been no casualties among mathematicians in war service, except indirectly as the deaths of certain men may have been hastened by overwork occasioned by their devotion to the nation's war effort. In this respect, our situation is entirely different from that of Poland where probably two-thirds of the nation's mathematicians were murdered for political and racial reasons, were killed in battle, or died from causes directly attributable to the war. Hungary has also lost heavily.

However, this does not mean that all trained in mathematics are available to the universities and colleges. Many of them are still on leave from their permanent posts. Many will receive attractive offers to remain in or enter government service and industry. If university research and training is to keep pace with the growth of our economy, if able men are to remain in college research and others are to be attracted to it, we must meet the following problems.

(1) New sources of financial support must be found. Incomes in other professions—medicine, law, engineering, *etc.*—have increased steadily during the war. The cost of living has risen markedly. The personal income tax bears heavily on the middle income groups. In spite of these factors, most university salaries have been frozen at a level which was none too high even before the war. A retention of this scale will have an adverse effect on the recruitment and retention of college and university personnel.

(2) University teaching loads must not be excessive, if good men are to return and others are to be retained in academic life. In many cases, research men have been drawn into positions where all their time and energy was to be devoted to research in problems connected with the war. Many of these men will wish to return to teaching, for teaching and research are complementary functions. If there is too much teaching, especially of an elementary character, research is bound to suffer and many men will be attracted to the outside positions which offer full time for research as well as better salaries.

(3) Many of the men who have been on leave from their academic positions will need "re-conditioning," in order that they may again take up their research interrupted by the war. It is extremely encouraging to see the number of men who have indicated that they are planning to spend a year or a semester at institutions like the Institute for Advanced Study for this purpose. The Guggenheim Foundation has reserved a number of fellowships for just this purpose and is encouraging outstanding young scholars, who are in need of this opportunity, to apply. Another excellent method would be the creation of more fellowships of the same type as the National Research Council fellowships for post-doctoral research. Anyone who followed the development of mathematics during the period from 1924 to 1935 will realize the tremendous increase in mathematical research among men who had just attained the doctorate and who were given the benefit of these fellowships. This is reflected in the rapid increase in the sizes of our mathematical publications in this period. The *Annals of Mathematics* and the *Transactions of the American Mathematical Society* both increased the sizes of their volumes. The *American Journal*, aided since 1927 by an annual subsidy of $2,500 from the American Mathematical Society, expanded from 300 pages to about 1,000 just before the war. The *Duke Journal*, founded in 1935 by the generous contributions of Duke University, publishes a volume of 800 to 900 pages of high-grade research. The *Quarterly of Applied Mathematics*, founded in 1943, is the latest addition to the list of journals publishing mathematical research.

Unless new personnel is recruited for mathematics by the influx of high-grade graduate students, the whole structure will eventually break down. For the immediate future, this supply must come from those students who had started their graduate work or who had completed undergraduate majors and who were either inducted into the armed forces or entered war work in one of the large physics laboratories, the Aberdeen Proving Ground, the Oak Ridge or Los Alamos Project or others of a similar nature. For these there are available:

(a) The benefits of the GI Bill for men who were in the armed forces.

(b) The pre-doctoral fellowships of the National Research Council. For these fellowships The Rockefeller Foundation has provided a sum of $335,000. They are particularly for men who have been engaged in war work and who are not eligible for the benefits of the GI Bill. There is provision whereby these fellowships may be used to supplement the amount received under the GI Bill. However, in this case, the amount awarded is such as to bring the total amount

available to the student up to the sum of $1,200 plus tuition for an unmarried man and $1,800 plus tuition for a married man. These fellowships are administered by a board set up in the National Research Council under the chairmanship of Professor J. T. Tate, Professor of Physics at the University of Minnesota, and has as members: H. A. Barton, C. W. Bray, D. W. Bronk, L. P. Eisenhart, R. G. Harrison, ex officio, W. A. Noyes, Jr. I was told recently that this committee had already received more than 250 applications. It is evident that this fund will not be able to handle all worthy cases.

(c) Regular university fellowships. A number of universities have not awarded their full quota of fellowships during the war years, because there were not sufficient numbers of candidates available who met their normal standards. These fellowship funds have been allowed to accumulate and consequently at many institutions it will be possible to make more than the usual number of awards in the next few years.

Here there must be a word of warning. Without any doubt, we shall soon be faced with a very serious teacher shortage. Enrollments started to mount very rapidly at the beginning of the present semester. At my own university, we admitted the June high school graduates as freshmen engineers in July and therefore expected a small number in November. Instead, we were required to provide three times as many sections of engineering mathematics as had been expected. There were corresponding increases in liberal arts sections. Another department chairman told me that ten days before college opened, the most liberal estimate as to probable enrollment had been placed at 4,500. The beginning of the semester found that figure had been exceeded by more than 1,000. This state of affairs could be duplicated in many more instances. I predict that the situation with respect to procurement of adequately trained teachers of collegiate mathematics, which is difficult now, will become worse by March and that by September we shall be faced with a shortage more serious than that which faced us in July and August of 1943. The easiest manner of solving our difficulties, at least in part, is to appoint as a full-time instructor the returned service man who is now trying to finish his graduate work. This leaves little time for graduate work and, I contend, is absolutely unfair. We must not overload these men by giving them teaching schedules of 12 to 15 hours but should give them all possible assistance in finishing the work for their degrees. Before these men were inducted, they were not able to do effective graduate work because of heavy teaching loads. Many of these men were deferred because they fulfilled the Selective Service requirement of teaching 15 hours per week. Let us see that they have a chance for proper graduate work when they are released.

One very disturbing factor with respect to these men is the fact that they entered the service toward the end of the war and now have very low point scores. Thus, under normal working of the point system, they will be the last people released, even though they are vitally needed for the restoration of the scientific competence of the nation.

Dr. A. H. Compton, Nobel Prize winner and Chancellor of Washington University, said in a recent radio address:

"It takes 6 years for a capable 18-year-old to train himself for an effective research career in science. Even if we start now, it will be 6 years before we again have a normal supply of young scientists for our universities and industry. Can we afford to wait any longer?"

Admiral Furer has stated: "There is a growing belief that important as it may be to maintain after the war, ground forces and sea forces of a size commensurate with our national responsibilities, it may be even more important to keep the weapons and material which we supply these forces in step with the advances of science. Stocking our arsenals with the weapons of this war is no guarantee that we can win the next war. In fact, it is probably the surest way of losing it. It would be wiser to maintain only arsenals of modest size when we are speaking of ships, or guns or aircraft and to use the money saved thereby to replace the old things with the creations of research laboratory and American genius."

In face of these and many other pronouncements, our promising scientists have not been given preferential demobilization. This state of affairs exists in spite of the fact that neither our allies nor, so far as we know, our enemies have permitted such a condition to develop. On the contrary, they have maintained or increased their programs for the training of scientists and technologists during the war. The War Policy Committee of the Society and Association discussed the desirability of preferential demobilization of prospective graduate students and teachers of mathematics as early as November, 1944. Through the committee, the question was presented to the Office of Scientific Personnel and eventually we placed the problem squarely before the subcommittee, headed by Dr. Henry Allen Moe of the Guggenheim Foundation, which was to study the question of the discovery of scientific talent, placed before Dr. Vannevar Bush by President Roosevelt. I should like to read the recommendations of the Moe committee, made shortly after V-E Day:

"Arrangements should be made now with the Army and the Navy whereby now that it is militarily feasible, these talented students should be ordered to duty in the United States for fully independent, integrated scientific study of a grade available to civilians in peace times. This should be adopted as the considered policy of the armed services and no desire of a commanding officer to retain a potential scientist for his usefulness on the spot should be allowed to interfere with the operation of the policy.

"It is recommended that this plan be carried out, not in terms of a stated number of young scientists, but rather that, now, centers of science and technology in the United States should be combed for information concerning those students who, prior to the war, had given evidence of high talent for science and technology; and that, as soon as militarily possible, these students by name, should be ordered to duty as students. Probably no more than 100,000 of the 10,000,000 men in the Army and Navy would be involved and now, following

V-E-Day, that number could not be militarily significant. Likewise, we recommend that the armed services comb their records for men, who, during the war, have given evidence of high talent for science and technology, and they also be included in this plan."

During the last summer I sent a questionnaire to chairmen of departments of mathematics in colleges and universities on the approved list of the Association of American Universities, asking for the names of promising young mathematicians who should be considered for such a program. More than 450 names were secured and were forwarded to the Office of Scientific Personnel, directed by Dr. M. H. Trytten, who is to prosecute this with the proper authorities.

There is one hopeful sign that no further losses will be incurred in the ranks of those who are now graduate students in science. I have been informed that on November 23 there went to press, for distribution to local boards, a directive providing for the deferment of all graduate students and senior students in physical science and mathematics. This is also to apply to teachers of mathematics. The information which has come to me is that this memorandum will be released on November 27.

In addition to immediate plans, we must look at the long-range picture. In the post-war years there must be a continuous supply of young men capable of entering upon research careers in science. Here there is a very disturbing factor for there has been continually looming on the horizon the spectre of universal military training in peace time. This training period would delay the time when young men could start their scientific careers and many others who would thus be drawn into military service might be lost to higher education entirely. In 1944, the War Policy Committee appointed a subcommittee to study the problem of universal military training. This committee, while making no statement as to whether universal military training should or should not be adopted, made strong pronouncements as to the type of training which should be given to men who could benefit by scientific training and who would be subject to peace time military service. This subcommittee recommended a program with very strong emphasis on science. This report has been approved by the War Policy Committee, the Council of the Society and the Board of Governors of the Association, and will be published in the near future in the *Bulletin*, this MONTHLY, and *The Mathematics Teacher*. We hope that this factor of universal military training will not retard the growth of graduate work in science.

A hopeful sign for the future lies in the interesting recommendations contained in various science bills before Congress at the present time. One of these would create as many as 6000 four-year scholarships to enable young men who show scientific promise to work for the bachelor's degree, in addition to creating a large number of fellowships to enable the recipients to obtain advanced training leading to the doctorate in science and engineering. However, it must be pointed out that all of this is prefaced upon the fact that good training is to be given to these young prospective scientists in our secondary schools. All of us know the lamentable condition which now exists with respect to the teaching of

arithmetic and algebra in our secondary schools. I am afraid that the conditions which were so eloquently described in the famous Nimitz letter still exist and are likely to continue to exist, unless a very vigorous step is taken to see that the level of the teaching of mathematics in the secondary schools is raised. I am not pleading for an increased amount of required mathematics but I do plead that those who wish to take advanced mathematics in secondary schools should be given the opportunity to do so and that that teaching should be done by competent teachers. In this field, this Association and the National Council of Teachers of Mathematics have an opportunity to do pioneer work of a very significant nature.

Likewise, if increasing numbers of persons are to come to college and take science at the collegiate level, we must be certain that the grade of instruction which we offer in these fields is inspiring and sound, so that those with an aptitude for mathematics will be brought to a full realization of its beauty and majesty and will be prepared to enter upon scientific careers. Here again I do not think that we are entirely without fault and I believe that there can be real improvement in the teaching of mathematics at the collegiate level. There should also be a re-studying of our graduate programs to insure the best possible training for our future research men of mathematics. The United States has assumed world leadership in mathematics. That leadership brings responsibilities, both for imparting the results to coming generations and for ever widening the frontiers of mathematical truth. This responsibility faces both the Mathematical Association of America and the American Mathematical Society. We must see that we meet the challenge.

MATHEMATICS AT THE AMERICAN UNIVERSITIES IN EUROPE

THE STUDY CENTER IN FLORENCE

T. A. BANCROFT, Iowa State College

The University Study Center at Florence, Italy was organized by the United States War Department to offer courses of study on the standard university level to army personnel of the Mediterranean Theater of Operations. It was one of three such institutions in Europe, the other two being at Shrivenham, England and Biarritz, France. The Center was situated about 3 miles from downtown Florence in buildings formerly used by a Fascist School of Aeronautics. The buildings were completed in 1938 and are of brick, stone, and concrete in modern functional design. The eight principal buildings were re-named for American colleges and universities as follows: Princeton Hall, Harvard Hall, Stanford Hall, Yale Hall, Citadel Hall, Cornell Hall, Vanderbilt Hall, and Duke Hall.

The staff at the Study Center was composed of both military and civilian personnel. The military personnel was in complete charge of administration and

also did a large share of the actual classroom teaching. The military personnel was made up of both officers and enlisted men. One rule in the school catalog stated "for purposes of disciplinary control in the classroom, the instructor of whatever rank or grade is the representative of the Commanding General and will be accorded the respect and courtesy of an officer of the highest rank present. The instructor is in absolute authority during the period of his control of a classroom and in respect to the work in his course." For reasons of expediency the first man to arrive at the Study Center in each department was made the head of that department. This meant that all departments were headed by military personnel, since the civilian instructors did not arrive until the organization had been completed. All members of the teaching staff, no matter what their previous rank or grade, were given the title of instructor. Members of the Mathematics Department were as follows:

MILITARY PERSONNEL

H. M. Anderson, 1st Lt., F.A., Thomas Anderson, Cpl., F.A., Stephen Dickler, 1st Lt., S.C., Thomas Dyer, Pfc., Inf., Edmund Fontanella, 1st Lt., Inf., Head of the Department, William Lassow, T/Sgt., F.A., Theodore Love, 1st Lt., C.E., K. O. May, 2nd Lt., Inf., E. Mehr, S.K.3.C., U.S.N.R., William Prahl, 1st Sgt., Inf., Mathew C. Resick, Capt., F.A., Edwin Shotland, T/Sgt., A.C., Warman Welliver, Capt., F.A.

CIVILIAN PERSONNEL

T. A. Bancroft, E. E. Blanche, H. C. Christofferson, W. H. Fagerstrom, Chester Jaeger. Pomona College.

Courses offered in Mathematics were as follows: solid geometry, college algebra, trigonometry, analytic geometry, slide rule, differential calculus, integral calculus, statistics, and a reading course for advanced undergraduate students. One year and a half of high school algebra was made the prerequisite for college algebra. Plane geometry and college algebra were made the prerequisites for trigonometry. Trigonometry was made the prerequisite for the slide rule course. The reading course for advanced undergraduate students was handled on an individual basis. Topics studied included: statistics, probability, projective geometry, advanced calculus, and complex variable. It was found that most of the advanced undergraduute students had been studying mathematics on their own initiative, during spare time, at their home military unit.

The Study Center at Florence ran for four sessions, from July 2, 1945 to November 17, 1945. A session lasted for four weeks, with classes meeting one hour each day for five days a week. The instructor was asked to evaluate each student's work at the end of each week by means of tests, problems to be handed in, or a combination of these two. A final examination was given at the end of each session. Certificates of course completion with letter grade earned

were given the students at the end of each session. The school catalog defined the letter grades as follows: "A" representing an achievement surpassing that of 95% of a normal student body, "B" inferior in achievement to "A" but surpassing that of 70% of a normal student body, "C" inferior in achievement to "B" but surpassing that of 30% of a normal student body, "D" inferior in achievement to "C" but surpassing that of 5% of a normal student body, "E" representing the achievement of the lowest 5% of a normal student body. No student was allowed to remain for a second session whose scholastic record showed an "E" in more than one course. A maximum enrollment of 2700 students was attained during the third session.

Standard text books, prepared for the War Department by well-known writers of mathematical texts, were issued to the students free of charge. Since it was impossible to cover the usual college semester's or quarter's work in one session, students were given a chance to continue a course from one session to the next session provided as many as five enrolled for the second session's work. As a result of this, two sessions of work were offered in most of the mathematics courses, and in some courses three sessions of work were offered. By this means and in such courses, an attempt was made to give a student the necessary background for a subsequent course in college mathematics in the American college or university of his choice. The University Study Center, however, issued no transferable semester or quarter hours of credit; it being left up to the returned veteran's college or university to evaluate the certificate of course completion.

A number of G.I.'s came to the Study Center in order to review previous college courses taken one to four years in the past. Most of these students expected to enter some American college or university upon being discharged. In each course in mathematics students were sectioned according to whether they were reviewing the subject or were taking it for the first time. No attempt was made to section students according to ability or past performance.

It was found in the statistical study entitled, "Soldier Attitudes Toward the University Study Center," prepared for the Commandant, University Training Command MTOUSA, by the Research Branch, Information and Education Section, Hq., MTOUSA; that the students, for the most part, were pleased with the work done at the University Study Center at Florence. A number of the members of the Mathematics Department expressed themselves as pleased with the interest and industry of their students. For the most part they took advantage of this chance to further their intellectual and professional training. Also, as one remarked, thinking about mathematical problems left few brain cells with which to worry about the sweating out of the trip home. Everything that was possible was done by the administration and the faculty to make the University Study Center like a typical American college or university. Probably the Study Center's biggest contribution was to get the G.I.'s back into college patterns of behavior, *i.e.* attending classes, studying, thinking for himself, *etc.* At least, some success was attained in these directions.

BIARRITZ AMERICAN UNIVERSITY

R. M. WINGER, University of Washington

Among the distinctive features of the American University located at Biarritz, seaside resort of romance and glamor, may be mentioned: (1) It is a general educational institution of college grade, comprising schools of Liberal Arts, Science, Engineering, Education, Commerce, Agriculture, Fine Arts, and Journalism, operated directly by an arm of the U. S. Government. (2) It is located in a foreign country. (3) The faculty and students with a few exceptions are Americans in uniform, the secretaries are British girls in the governmental service, the waiters in the dining halls and the technical workmen are French civilians, and the janitors are German prisoners of war. (4) The school sprang full fledged into being with an enrollment of 4,000 students. (5) There is no campus as such, there are no college colors, no college songs and no college yells, but we have a band and an orchestra and publish a newspaper five days a week and operate a theater and radio station. In addition the university enjoys a number of unusual cultural advantages including some of the finest French operas, concerts and lectures, as well as unique Basque festivals and sports.

The difficulties of transforming a beach resort of fashionable hotels and villas into residence halls, class rooms and laboratories were tremendous, especially when it is recalled that the town had been under German occupation for several years and that materials and supplies of all kinds had to be brought in from the United States or Germany while the war in the Pacific was at its height. The ingenuity of branch heads and supply sergeants was taxed to the limit. Thus, painted plywood was used for black boards, while bits of carpet tacked to wooden blocks served as erasers. But through the united efforts of the Army and civilian faculty—and to the surprise of many—the "youngest university in the world" opened its doors on schedule, August 20, 1945.

The following men comprised the mathematical staff for the first term: J. I. Tracey (Branch Head), Frank Ayres, F. R. Bamforth, I. A. Barnett, Lt. Col. A. C. Cohen, Lester Hill, Capt. A. O. Lindstrum, R. B. McClenon, E. R. C. Miles, J. H. Neelley, A. L. Nelson, C. G. Phipps, J. C. Polley, B. P. Reinsch Capt. W. T. Scott, C. R. Sherer, Gabor Szego, H. S. Thurston, Lt. Col. A. S. Turner, Major V. R. Walker, R. M. Winger, Lt. Stephen Oselinsky, Lt. H. E. Temmer, Sgt. E. F. Arndt, Cpl. J. C. Gibson, Cpl. S. D. Conte. No fewer than eight of this faculty are department heads in their respective institutions in the United States.

Of the 4,000 students there were some 800 registrations (including duplicates) in mathematics. Courses were scheduled in the usual lower division subjects, but in response to demand, classes were organized in several advanced and graduate courses. The courses, with the number of sections in each were: college algebra (21), trigonometry (12), analytic geometry (8), mathematics of finance (2), spherical trigonometry and astronomy (1), differential calculus (8), integral calculus (4), differential equations (1), higher algebra (1), advanced

analytic geometry (1), higher mathematics for engineers and physicists (2). In addition a few graduate students have engaged in special study under the direction of Dr. Szego. A mathematical seminar, meeting fortnightly, has also been maintained.

After two weeks the students in college algebra were regrouped and a number of sections of intermediate algebra were organized, the better to adapt the work to the varying needs of the students. This proved so satisfactory that for the next term several sections in intermediate algebra and one in beginning algebra and one in plane and solid geometry are scheduled. Otherwise the courses given during the first term will be repeated and opportunity will be offered to qualified advanced students for individual study under supervision.

As a reward for exceptional performance, 80 students or 2% of the whole student body were selected to continue their work in the second term. These were prorated among the several departments according to the number of majors, the quota for science being 13, of which 6 were students of mathematics. For the second term the international aspect of the university has been increased by the admission of 50 British, 50 French and 25 Canadian students—dubbed the "foreign legion" by the Americans. The mathematical faculty remains intact, except for a few of the military members who are being separated, while the enrollment in mathematics exceeds 1,000.

With the close of the second term, this experiment in higher education will pass into history. How shall we appraise the results? First of all it was apparent from the beginning that any apprehension concerning the psychological problems that these battle-weary veterans might present were happily unfounded. While they were drawn largely from combat troops who had seen the toughest fighting from the beach heads of Normandy to the Elbe River, they appeared not too unlike civilian college boys—older to be sure, somewhat more serious, yet with a disconcerting tendency to prefer refresher courses.

It was the policy in mathematics to maintain college standards in the work covered. It is the concensus of opinion of the faculty that the achievement of the students was about on a par with that in civilian colleges. Classes were kept small, ranging in size from 10 to 20. Grades were assigned on the usual five letter basis with about the usual number of failures being recorded.

Aside from the academic results, there is a universal conviction that the intangible values alone have justified this utopian educational venture. It was reassuring to the soldier to find that he is not merely a cog in a military machine, but a human being who is rated, not by a stripe on a uniform, but by his own intellectual capacity and attainments. The association of men from privates to Lt. Colonels in the same classes, some of them taught by privates and sergeants, was a demonstration in practical democracy. To the mathematical faculty, representing a score of colleges and universities, it has proved an unforgettable experience, this teaching of mathematics in storied Biarritz.

QUATERNIONS AND REFLECTIONS*

H. S. M. COXETER, University of Toronto

1. Introduction. It is just a hundred years since Cayley began to use quaternions for the discussion of rotations. He was followed by Boole, Donkin, Clifford, Buchheim, Klein, Hurwitz, Hathaway, Stringham, and Study. Apparently none of these men thought of considering first the simpler operation of *reflection* and deducing a rotation as the product of two reflections. This procedure will be described in §§3 and 5, and its consequences developed in the later sections.

Every quaternion $a=a_0+a_1i+a_2j+a_3k$ determines a point $\mathbf{P}_a=(a_0, a_1, a_2, a_3)$ in euclidean 4-space, and a hyperplane $a_0x_0+a_1x_1+a_2x_2+a_3x_3=0$. The reflection in that hyperplane is found to be the transformation $x\rightarrow -a\bar{x}a/\mathrm{N}a$. This leads easily to the classical expression

$$x \rightarrow axb \qquad (\mathrm{N}a = \mathrm{N}b = 1)$$

for the general displacement preserving the origin. Cayley obtained this elegant expression by "brute force" as early as 1855. It became somewhat more natural in the hands of Klein and Hurwitz, forty years later. The treatment in §7 will possibly serve to clarify it still further.

We begin with a few algebraic lemmas, mostly due to Hamilton.

2. Elementary properties of quaternions. A *quaternion* is a hyper-complex number $a=a_0+a_1i+a_2j+a_3k$, where a_0, a_1, a_2, a_3 are real numbers ("scalars"), and multiplication is defined by the rules

$$i^2 = j^2 = k^2 = ijk = -1,$$

which imply $jk=i=-kj$, $ki=j=-ik$, $ij=k=-ji$. Thus quaternions form an associative but non-commutative algebra.

It is often convenient to split a quaternion into its "scalar" and "vector" parts:†

$$a = \mathrm{S}a + \mathrm{V}a, \qquad \mathrm{S}a = a_0, \qquad \mathrm{V}a = a_1i + a_2j + a_3k.$$

We define also the *conjugate* quaternion‡

$$\bar{a} = \mathrm{S}a - \mathrm{V}a = a_0 - a_1i - a_2j - a_3k$$

and the *norm*

$$\mathrm{N}a = \bar{a}a = a\bar{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

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* This paper is an amplification of an invited address delivered at the annual meeting of the Mathematical Association of America in Chicago on November 24, 1945.

† W. R. Hamilton, Elements of Quaternions, vol. 1, London, 1899, pp. 177, 193. By his special convention, p. 186, Sxy means S(xy), not (Sx)y; similarly for Vxy and Nxy.

‡ Klein's symbol $\bar{a}$ seems preferable to Hamilton's Ka.

In terms of a and $\bar{a}$, we have $Sa=\frac{1}{2}(a+\bar{a})$, $Va=\frac{1}{2}(a-\bar{a})$.

Since $\overline{ii}=-1=i\bar{i}$ and $\overline{ij}=-k=j\bar{i}$, *etc.*, we easily verify that

$$\overline{ab} = \bar{b}\bar{a},$$

whence $Nab=\overline{a}\overline{b}ab=\bar{b}\bar{a}ab=\bar{b}(Na)b=NaNb$. If $Na=1$, we call a a *unit* quaternion. To every non-vanishing quaternion a there corresponds a unit quaternion*

$$Ua = a/\sqrt{Na}.$$

A quaternion p is said to be *pure* if $Sp=0$. Then $\bar{p}=-p$ and $p^2=-Np$. Thus a pure unit quaternion is a square root of -1.

The point $\mathbf{P}_x=(x_1, x_2, x_3)$ in ordinary space may be represented by the pure quaternion $x=x_1i+x_2j+x_3k$. This representation resembles the Argand diagram in the plane, in that the distance $\mathbf{P}_x\mathbf{P}_y$ is given by $\mathbf{P}_x\mathbf{P}_y^2=N(y-x)$. Since

$$xy = -(x_1y_1+x_2y_2+x_3y_3)+\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} i+\begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix} j+\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} k,$$

we see that $-Sxy$ and Vxy are the ordinary "scalar product" and "vector product" of the two vectors $\mathbf{P}_0\mathbf{P}_x$ and $\mathbf{P}_0\mathbf{P}_y$. If x and y are pure *unit* quaternions, then $\mathbf{P}_x$ and $\mathbf{P}_y$ lie on the unit sphere around the origin $\mathbf{P}_0$, and we have $\angle\mathbf{P}_x\mathbf{P}_0\mathbf{P}_y=\theta$, where

$$\cos\theta = -Sxy = -\tfrac{1}{2}(xy+yx).$$

Thus the condition for $\mathbf{P}_x$ and $\mathbf{P}_y$ to lie in perpendicular directions from $\mathbf{P}_0$ is

$$xy + yx = 0.$$

LEMMA 2.1. *For any quaternion a we can find a unit quaternion y such that $ay=y\bar{a}$.*

Proof. Take any pure quaternion p for which $\mathbf{P}_0\mathbf{P}_p$ is perpendicular to $\mathbf{P}_0\mathbf{P}_{Va}$.† Then

$$(Va)p + p(Va) = 0.$$

But since the scalar Sa commutes with p,

$$(Sa)p - p(Sa) = 0.$$

By addition,

$$ap - p\bar{a} = 0.$$

The desired *unit* quaternion is $y=Up$.

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* Hamilton, *op. cit.*, p. 137.

† If an explicit formula is desired, we may write $p=V(Va)q$, where q is any pure quaternion (other than the scalar multiples of Va). In fact, if $p=V\alpha q$, where $\alpha=Va$, then $2p=\alpha q-q\alpha$, and

$$2(\alpha p + p\alpha) = \alpha(\alpha q - q\alpha) + (\alpha q - q\alpha)\alpha = \alpha^2 q - q\alpha^2 = 0,$$

α^2 being scalar. Thus our appeal to geometry could have been avoided.

LEMMA 2.2. *Let a and b be two quaternions having the same norm and the same scalar part.* Then we can find a unit quaternion y such that $ay=yb$.*

Proof. If $b=\bar{a}$, this is covered by Lemma 2.1; so let us assume $b\neq\bar{a}$. Since $Sa=Sb$, we have

$$a+\bar{a}=b+\bar{b}, \qquad a-\bar{b}=b-\bar{a},$$
$$a(a-\bar{b})b=a(b-\bar{a})b,$$
$$a(ab-Nb)=(ab-Na)b.$$

Thus $ac=cb$, where $c=ab-Nb=ab-Na=a(b-\bar{a})$. Since $b\neq\bar{a}$, $c\neq 0$; so we can take $y=Uc$.

LEMMA 2.3. *Any quaternion is expressible as a power of a pure quaternion.*†

Proof. Since $Np^t=(Np)^t$, it will be sufficient to prove this for a *unit* quaternion, a. Since $(Sa)^2-(Va)^2=Na=1$, such a quaternion may be expressed as

$$a=\cos\alpha+p\sin\alpha,$$

where $\cos\alpha=Sa$ and p is a *pure* unit quaternion. Since $Va=p\sin\alpha$, we have $p=UVa$. Since $p^2=-1$, de Moivre's Theorem shows that

$$a^n=\cos n\alpha+p\sin n\alpha$$

for any real number n. In particular, $a^{\pi/2\alpha}=p$, so $a=p^{2\alpha/\pi}$. Thus

$$a=p^t,$$

where $p=UVa$ and $\cos\frac{1}{2}t\pi=Sa$ (so that we may suppose $0\leqq t\leqq 2$).

3. Reflections and rotations in three dimensions. We have seen that $\mathbf{P}_x$ and $\mathbf{P}_y$ lie in perpendicular directions from the origin $\mathbf{P}_0$ if the pure quaternions x and y satisfy the relation

$$xy+yx=0.$$

If $Ny=1$, so that $y^2=-1$, this condition may be expressed as

$$x=yxy.$$

Since $\overline{yxy}=\bar{y}\bar{x}\bar{y}=-yxy$, yxy is pure for *any* position of $\mathbf{P}_x$. Thus the linear transformation $x\rightarrow yxy$ (where $Ny=1$) represents a collineation which leaves invariant every point $\mathbf{P}_x$ in the plane through $\mathbf{P}_0$ perpendicular to $\mathbf{P}_0\mathbf{P}_y$, *i.e.*, the plane

$$y_1x_1+y_2x_2+y_3x_3=0.$$

Moreover, it reverses the vector $\mathbf{P}_0\mathbf{P}_y$:

$$y\rightarrow y^3=-y.$$

* Two such quaternions satisfy the same "rank equation" $x^2-2mx+n=0$, where $m=Sa=Sb$ and $n=Na=Nb$. For a general discussion of the equation $ay=yb$, see Arthur Cayley, On the quaternion equation $qQ-Qq'=0$, Messenger of Mathematics, vol. 14, 1885, pp. 108–112.

† Hamilton, p. 399.

But the only collineation having these properties is the *reflection* in the plane of invariant points. Hence

THEOREM 3.1. *The reflection in the plane* $\sum y_r x_r = 0$ *is represented by the transformation*

$$x \rightarrow yxy \qquad (\mathrm{N}y = 1).$$

The product of two such reflections, $x \rightarrow yxy$ and $x \rightarrow zxz$ (with $\mathrm{N}y = \mathrm{N}z = 1$), is the transformation

$$x \rightarrow zyxyz.$$

Accordingly, this is a *rotation*, in the plane $\mathbf{P}_0\mathbf{P}_y\mathbf{P}_z$, through twice the angle $\mathbf{P}_y\mathbf{P}_0\mathbf{P}_z$. In other words, it is a rotation through ϕ about the line $\mathbf{P}_0\mathbf{P}_{\mathrm{V}yz}$, where

$$\cos \tfrac{1}{2}\phi = -\mathrm{S}yz.$$

By Lemma 2.1, any unit quaternion a may be expressed as $-zy$, where $ay = y\bar{a} = z$. (Here z, like y, is a pure unit quaternion; for,

$$z + \bar{z} = ay - y\bar{a} = 0, \quad \text{and} \quad \mathrm{N}z = \mathrm{N}a\mathrm{N}y = 1.)$$

Since $\bar{a} = -yz$, this shows that the transformation

$$x \rightarrow axa \qquad (\mathrm{N}a = 1)$$

is a rotation through ϕ about $\mathbf{P}_0\mathbf{P}_{\mathrm{V}a}$, where $\cos \frac{1}{2}\phi = \mathrm{S}a$.

In the notation of the proof of Lemma 2.3, we have $a = \cos\alpha + p \sin\alpha$, where $\mathbf{P}_0\mathbf{P}_p$ is the unit vector along the axis of rotation, and $\alpha = \pm\frac{1}{2}\phi$. The sign is a matter of convention. Taking the plus, we find that the rotation through $\frac{1}{2}\pi$ about the x_3-axis $\mathbf{P}_0\mathbf{P}_k$ transforms $\mathbf{P}_i$ into $\mathbf{P}_j$:

$$i \rightarrow ai\bar{a} = \frac{1+k}{\sqrt{2}}\, i\, \frac{1-k}{\sqrt{2}} = \frac{(i+j)(1-k)}{2} = j.$$

(The opposite convention would have given $i \rightarrow -j$.) We sum up in

THEOREM 3.2.* *The rotation through* ϕ *about the line with direction cosines* (p_1, p_2, p_3) *is represented by the transformation* $x \rightarrow axa$, *where*

$$a = \cos \tfrac{1}{2}\phi + (p_1 i + p_2 j + p_3 k) \sin \tfrac{1}{2}\phi.$$

Since the product of two rotations, $x \rightarrow axa$ and $x \rightarrow bxb$, is another rotation, *viz.*, $x \rightarrow baxba$, we can immediately deduce

THEOREM 3.3. *All the rotations about lines through the origin in ordinary space form a group, homomorphic to the group of all unit quaternions.*

Since the rotation through ϕ is indistinguishable from the rotation through $\phi + 2\pi$ about the same axis, there are *two* quaternions, $\pm a$, for each rotation.

* Arthur Cayley, On certain results relating to quaternions, Philosophical Magazine (3), vol. 26, 1845, p. 142. George Boole, Notes on quaternions, *ibid.*, vol. 33, 1848, p. 279. W. F. Donkin, On the geometrical theory of rotation, *ibid.* (4), vol. 1, 1851, p. 189.

Alternatively, we may say that the group of rotations is *isomorphic* to the group of all "homogeneous quaternions," in accordance with the formula

$$x \to axa/\mathrm{N}a, \quad \text{or} \quad x \to axa^{-1}.$$

(A homogeneous quaternion is the class of all scalar multiples of an ordinary quaternion.) To make this an isomorphism rather than an anti-isomorphism, we must agree to multiply group elements from right to left.

Combining $x \to axa$ with the *inversion* $x \to -x$, we obtain the transformation

$$x \to -axa \qquad (\mathrm{N}a = 1)$$

which may be regarded either as a *rotatory-inversion* of angle ϕ or as a *rotatory-reflection* of angle $\phi+\pi$.

4. The general displacement. Two orthogonal trihedra at the same origin can be brought into coincidence by the successive application of the following reflections: one to interchange the two x_1-axes, another to interchange the two x_2-axes (without disturbing the x_1-axis), and a third (if necessary) to reverse the x_3-axis. Thus the general orthogonal transformation in three dimensions is the product of at most three reflections—an even or odd number according as the transformation preserves or reverses sense. In particular, any *displacement* (or "movement") leaving the origin fixed must be the product of only two reflections *i.e.*, a *rotation*. (This is a famous result, due to Euler.) Thus the group considered in Theorem 3.3 is the group of all such displacements, and is a subgroup of index 2 in the group of all orthogonal transformations. The latter contains also the rotatory-reflections $x \to -axa$, which are products of three reflections.

Similarly in four dimensions, the general orthogonal transformation (*i.e.*, congruent transformation with a fixed origin) is a product of at most four reflections (in hyperplanes). Thus the general displacement (leaving the origin fixed) is the product of two or four reflections. But the product of two reflections is a rotation (about the common plane of the two hyperplanes, through twice the angle between them). Hence a displacement is either a single rotation or the product of two rotations about distinct planes. If any point besides the origin is invariant, the displacement can only be a single rotation; for it operates essentially in the 3-space perpendicular to the line of invariant points. In the general case of a double rotation, where only the origin is fixed, it is well known (though far from obvious) that the axial planes of the two rotations may be chosen to be completely orthogonal. This was first proved by Goursat in 1889. We shall obtain a new proof in §9.

5. Reflections and rotations in four dimensions. A point $\mathbf{P}_x$ in euclidean 4-space has four Cartesian coordinates (x_0, x_1, x_2, x_3) which may be interpreted* as the constituents of a quaternion

$$x = x_0 + x_1 i + x_2 j + x_3 k.$$

* A. S. Hathaway, Quaternions as numbers of four-dimensional space, Bulletin of the American Mathematical Society, vol. 4, 1897, pp. 54–57.

The distance $\mathbf{P}_x\mathbf{P}_y$ is given by

$$\mathbf{P}_x\mathbf{P}_y^2 = (y_0 - x_0)^2 + (y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2 = \mathrm{N}(y - x).$$

If x and y are *unit* quaternions, then $\mathbf{P}_x$ and $\mathbf{P}_y$ lie on the unit hypersphere around the origin $\mathbf{P}_0$, and we have $\angle \mathbf{P}_x\mathbf{P}_0\mathbf{P}_y = \theta$, where

$$\cos\theta = x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = \mathrm{S}x\bar{y} = \tfrac{1}{2}(x\bar{y} + y\bar{x}).$$

Thus the condition for $\mathbf{P}_x$ and $\mathbf{P}_y$ to lie in perpendicular directions from $\mathbf{P}_0$ is

$$x\bar{y} + y\bar{x} = 0.$$

If $\mathrm{N}y = 1$, this condition may be expressed as

$$x = -y\bar{x}y.$$

Thus the linear transformation $x \rightarrow -y\bar{x}y$ (where $\mathrm{N}y = 1$) represents a collineation which leaves invariant every point $\mathbf{P}_x$ in the hyperplane through $\mathbf{P}_0$ perpendicular to $\mathbf{P}_0\mathbf{P}_y$, *i.e.*, the hyperplane

$$y_0x_0 + y_1x_1 + y_2x_2 + y_3x_3 = 0.$$

But it reverses the vector $\mathbf{P}_0\mathbf{P}_y$:

$$y \rightarrow -y\bar{y}y = -y.$$

Hence

THEOREM 5.1. *The reflection in the hyperplane* $\sum y_\nu x_\nu = 0$ *is represented by the transformation*

$$x \rightarrow -y\bar{x}y \qquad (\mathrm{N}y = 1).$$

The product of two such reflections, $x \rightarrow -y\bar{x}y$ and $x \rightarrow -z\bar{x}z$ (with $\mathrm{N}y = \mathrm{N}z = 1$), is the transformation

$$x \rightarrow z\,\overline{y\bar{x}y}\,z = z\bar{y}x\bar{y}z.$$

Accordingly, this is a rotation, *in* the plane $\mathbf{P}_0\mathbf{P}_y\mathbf{P}_z$, through twice the angle $\mathbf{P}_y\mathbf{P}_0\mathbf{P}_z$. In other words, it is a rotation *about* the completely orthogonal plane

$$\sum y_\nu x_\nu = \sum z_\nu x_\nu = 0$$

through angle ϕ, where $\cos\frac{1}{2}\phi = \mathrm{S}y\bar{z} = \mathrm{S}z\bar{y} = \mathrm{S}\bar{y}z$. This proves

THEOREM 5.2. *The general rotation through angle* ϕ *(about a plane) is*

$$x \rightarrow axb,$$

where $\mathrm{N}a = \mathrm{N}b = 1$ and $\mathrm{S}a = \mathrm{S}b = \cos\frac{1}{2}\phi$.

Conversely, the transformation $x \rightarrow axb$ (where $\mathrm{N}a = \mathrm{N}b = 1$) is a rotation whenever $\mathrm{S}a = \mathrm{S}b$; for then, by Lemma 2.2, we can find unit quaternions y and z such that $ay = yb = z$, enabling us to write

$$a = z\bar{y}, \qquad b = \bar{y}z.$$

6. Clifford translations. The product of the two rotations $x \to axa$ and $x \to ax\bar{a}$ is the so-called *left translation**

$$x \to a^2x,$$

while the product of $x \to axa$ and $x \to \bar{a}xa$ is the *right translation*

$$x \to xa^2.$$

Clearly, left translations $x \to ax$ form a group, so do right translations $x \to xb$, and any left translation commutes with any right translation. Every left translation has a *unique* expression $x \to ax$ ($Na = 1$); for, the equation $ax = bx$ would imply $a = b$. Similarly every right translation has a unique expression $x \to xb$. Hence

THEOREM 6.1. *The group of left (or right) translations is isomorphic to the group of unit quaternions.*

(In the case of right translations, we can make this a true isomorphism, and not merely an anti-isomorphism, by letting the quaternion b correspond to the translation $x \to x\bar{b}$, so that the product ab corresponds to $x \to x\overline{ab} = xb\bar{a}$.)

Comparing this with Theorem 3.3, we see that the group of left (or right) translations is homomorphic to the group of rotations about a fixed origin in ordinary space, with two Clifford translations for each rotation.

A Clifford translation (*i.e.*, a left or right translation) has the remarkable property of turning every vector through the same angle. For, if $Nx = 1$, so that also $Nax = 1$, the left translation $x \to ax$ transforms $\mathbf{P}_x$ into $\mathbf{P}_{ax}$, and

$$\cos \angle \mathbf{P}_x\mathbf{P}_0\mathbf{P}_{ax} = Sax\bar{x} = Sa,$$

which is the same for all vectors $\mathbf{P}_0\mathbf{P}_x$. Similarly, $x \to xb$ transforms $\mathbf{P}_x$ into $\mathbf{P}_{xb}$, and $\cos \angle \mathbf{P}_x\mathbf{P}_0\mathbf{P}_{xb} = Sx\bar{x}b = Sb$.

Can a left translation be also a right translation? This would make $ax = xb$ for every x. The case $x = 1$ gives $a = b$. Now take x to be the y of Lemma 2.1. Then

$$ya = ay = y\bar{a}, \qquad y(a - \bar{a}) = 0, \qquad Va = 0,$$

and so, since $Na = 1$, $a = \pm 1$. Thus the only left translations that are also right translations† are the *identity* $x \to x$ and the *inversion* $x \to -x$.

Instead of deriving Theorem 5.1 from the condition for $\angle \mathbf{P}_x\mathbf{P}_0\mathbf{P}_y$ to be a right angle, we might have observed that $x \to ax$ (where $Na = 1$) must be some kind of congruent transformation (since $Nax = Nx$), and that this transforms the special reflection $x \to -x$ into the general reflection

$$x \to -a\bar{a}\bar{x} = -a\bar{x}a.$$

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* Felix Klein, Vorlesungen über nicht-Euklidische Geometrie, Berlin, 1928, p. 240.

† William Threlfall and Herbert Seifert, Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes, Mathematische Annalen, vol. 104, 1931, p. 10.

7. The group of displacements. The product of a left translation and a right translation is, of course, $x \to axb$ (where $Na = Nb = 1$). The product of two displacements of this form is another of the same form. In particular, as we saw in Theorem 5.2, a *rotation* is of this form (with the special relation $Sa = Sb$). But the general displacement with a fixed origin is the product of two rotations. Hence

THEOREM 7.1.* *The general displacement preserving the origin is*

$$x \to axb \qquad (Na = Nb = 1).$$

Of course, axb is the same as $(-a)x(-b)$. With this exception, each displacement has a unique expression $x \to axb$. For, the equation $axb = a'xb'$ would imply $a'^{-1}ax = xb'b^{-1}$ for every x, whence $a'^{-1}a = b'b^{-1} = \pm 1$. In other words, every displacement (with a fixed origin) is the product of a left translation and a right translation in just two ways. Thus the direct product of the groups of all left translations and of all right translations is homomorphic to the group of all four-dimensional displacements preserving the origin (with two elements of the direct product for each displacement), and this in turn is homomorphic to the direct square of the group of all three-dimensional displacements preserving the origin (with two displacements for each element of the direct square).

8. The general orthogonal transformation in four dimensions. The general "opposite" or "sense-reversing" transformation leaving the origin invariant is the product of an odd number of reflections. Hence, in four dimensions, it is either a single reflection or a product of three. But in the latter case the three reflecting hyperplanes intersect in a line of invariant points, and every hyperplane perpendicular to this line is invariant; so this scarcely differs from a rotatory-reflection in ordinary space. As such, it has an axis and a special reflecting plane. Its product with the special reflection $x \to -\bar{x}$ is a displacement $x \to axb$; so the rotatory-reflection itself must be $x \to -a\bar{x}b$, or, after changing the sign of a,

$$x \to a\bar{x}b.$$

Since $a\,\overline{a+b}\,b = a+b$, the line of invariant points is $\mathbf{P}_0\mathbf{P}_{a+b}$. Since $a\,\overline{a-b}\,b = -(a-b)$, the axis is $\mathbf{P}_0\mathbf{P}_{a-b}$.

We sum up our conclusion in

THEOREM 8.1. *Every orthogonal transformation in four dimensions is either*

$$x \to axb \quad \text{or} \quad x \to a\bar{x}b.$$

9. The general displacement expressed as a double rotation. By Theorem 5.2, the general *half-turn* is

$$x \to pxq,$$

* Arthur Cayley, Recherches ultérieures sur les déterminants gauches, Journal für die reine und angewandte Mathematik, vol. 50, 1855, p. 312.

where p and q are pure unit quaternions. This is the half-turn about the plane containing all points $\mathbf{P}_x$ for which $x=pxq$, or

$$px + xq = 0.*$$

Since $p-q$ and $1+pq$ are particular solutions of this equation for x, we may describe the plane as $\mathbf{P}_0\mathbf{P}_{p-q}\mathbf{P}_{1+pq}$. A rotation through $t\pi$ about the same plane is, of course, $x\rightarrow p^t x q^t$. Replacing q by $-q$ $(=\bar{q}=q^{-1})$, we deduce that $x\rightarrow p^t x q^{-t}$ is a rotation through $t\pi$ about the completely orthogonal plane $\mathbf{P}_0\mathbf{P}_{p+q}\mathbf{P}_{1-pq}$. (The fact that these two planes *are* completely orthogonal is most easily verified by observing that the product of the half-turns $x\rightarrow pxq$ and $x\rightarrow pxq^{-1}$ is the inversion $x\rightarrow p^2x=-x$.)

Thus the product of rotations through $t\pi$ and $u\pi$ about the respective planes $\mathbf{P}_0\mathbf{P}_{p\mp q}\mathbf{P}_{1\pm pq}$ is $x\rightarrow p^{t+u}xq^{t-u}$. Setting $t\pi=\alpha+\beta$, $u\pi=\alpha-\beta$, and observing that

$$\cos\alpha + p\sin\alpha = p^{2\alpha/\pi}$$

(see the proof of Lemma 2.3), we deduce that the product of rotations through angles $\alpha\pm\beta$ about planes $\mathbf{P}_0\mathbf{P}_{p\mp q}\mathbf{P}_{1\pm pq}$ is

$$x \rightarrow (\cos\alpha + p\sin\alpha)x(\cos\beta + q\sin\beta).$$

In other words,

THEOREM 9.1.† *The general displacement $x\rightarrow axb$ is the double rotation through angles $\alpha\pm\beta$ about planes $\mathbf{P}_0\mathbf{P}_{p\mp q}\mathbf{P}_{1\pm pq}$, where*

$$\cos\alpha = \mathrm{S}a, \qquad \cos\beta = \mathrm{S}b, \qquad p = \mathrm{UV}a, \qquad q = \mathrm{UV}b.$$

10. Lines in elliptic space. The above considerations can be translated into terms of *elliptic* geometry by identifying pairs of antipodal points on the hypersphere. Now all scalar multiples of a quaternion x represent the same point $\mathbf{P}_x$, whose coordinates (x_0, x_1, x_2, x_3) are homogeneous. The transformation

$$x \rightarrow -y\bar{x}y$$

is the reflection in the polar plane of $\mathbf{P}_y$, and this is the same as the inversion in $\mathbf{P}_y$ itself. We now say that the group of all displacements is precisely the direct product of the groups of left and right displacements; accordingly, it is isomorphic to the direct square of the group of displacements preserving the origin. Instead of Theorem 9.1, we say that the general displacement $x\rightarrow axb$ is the product of rotations through angles $\alpha\pm\beta$ about the respective *lines* $\mathbf{P}_{p\mp q}\mathbf{P}_{1\pm pq}$, where

(1) $$\cos\alpha = \mathrm{S}a, \qquad \cos\beta = \mathrm{S}b, \qquad p = \mathrm{UV}a, \qquad q = \mathrm{UV}b.$$

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* Irving Stringham, On the geometry of planes in parabolic space of four dimensions, Transactions of the American Mathematical Society, vol. 2, 1901, p. 194.

† Édouard Goursat, Sur les substitutions orthogonales et les divisions régulières de l'espace, Annales Scientifiques de l'École Normale Supérieure (3), vol. 6, 1889, p. 36.

The line $\mathbf{P}_{p-q}\mathbf{P}_{1+pq}$, which is the axis of the half-turn $x \rightarrow pxq$, is conveniently denoted by $\{p, q\}$,* or equally well by $\{-p, -q\}$. Thus *any two pure unit quaternions determine a line* $\{p, q\}$. The absolute polar line is $\{-p, q\}$ or $\{p, -q\}$.

Two lines $\{p, q\}$ and $\{p', q'\}$ have, in general, two common perpendicular lines, which are the transversals of the four lines $\{\pm p, q\}$ and $\{\pm p', q'\}$. These are preserved by either of the half-turns $x \rightarrow pxq$, $x \rightarrow p'xq'$, and so also by their product

$$x \rightarrow p'pxqq'. \tag{2}$$

Thus they are the two axes of this double rotation. Any point on either axis will be reflected first in $\{p, q\}$ and then in $\{p', q'\}$; altogether it will be translated along that axis through twice the distance between the lines. Hence *the two distances between the lines* $\{p, q\}$ *and* $\{p', q'\}$, *measured along their common perpendiculars, are* $\frac{1}{2}|\alpha \pm \beta|$, *where*

$$\cos \alpha = -\mathrm{S}p'p = -\mathrm{S}pp', \qquad \cos \beta = -\mathrm{S}qq'; \tag{3}$$†

and the common perpendiculars themselves are the lines

$$\{\pm \mathrm{UV}pp', \mathrm{UV}qq'\}. \tag{4}$$

It follows from (3) that the condition for $\{p, q\}$ and $\{p', q'\}$ to *intersect* is

$$\mathrm{S}pp' = \mathrm{S}qq', \tag{5}$$

and then the angle between them, being half the angle of the rotation (2), is arc cos $(\pm \mathrm{S}pp')$. Similarly, the condition for $\{p, q\}$ and $\{p', q'\}$ to be *perpendicular* (*i.e.*, for one to intersect the polar of the other) is

$$\mathrm{S}pp' + \mathrm{S}qq' = 0. \tag{6}$$

Thus the condition for them to intersect at right angles is

$$\mathrm{S}pp' = \mathrm{S}qq' = 0. \tag{7}$$

The common perpendicular lines (4) cease to be determinate if $\mathrm{V}pp'$ or $\mathrm{V}qq'$ vanishes, *i.e.*, if either $p' = \pm p$ or $q' = \pm q$. Lines $\{p, q\}$ and $\{p, q'\}$ are said to be *left parallel*. They have an infinity of common perpendiculars

$$\{\mathrm{UV}pp', \mathrm{UV}qq'\}, \tag{8}$$

where p' may range over *all* unit pure quaternions (except $\pm p$). To verify

* This notation was used by A. S. Hathaway, Quaternion space, Transactions of the American Mathematical Society, vol. 3, 1902, p. 53. It is closely associated with the representation of a line in elliptic space by an ordered pair of points on a sphere; see Eduard Study, Beiträge zur nichteuklidische Geometrie, American Journal of Mathematics, vol. 29, 1907, pp. 121–124.

† Here we are using (1) with $a = -p'p$, $b = -qq'$. Plus signs would have given the supplementary distances, which are equally valid; but it seems preferable to use the sign that makes α and β small when p' and q' are nearly equal to p and q.

this we merely have to observe that the line (8), which intersects $\{p, q\}$ at right angles, also intersects $\{p, q'\}$ at right angles. The distance between these left parallel lines, measured along any of the common perpendiculars, is $\frac{1}{2}$ arc cos $(-Sqq')$.

Similarly, *right parallel* lines $\{p, q\}$ and $\{p', q\}$ are distant $\frac{1}{2}$ arc cos $(-Spp')$ along any of an infinity of common perpendiculars (8), only now it is q' that can vary. Thus the common perpendicular lines of right parallels are left parallel, and vice versa.

By the remark at the beginning of §9, the condition for the line $\{p, q\}$ to contain the point $\mathbf{P}_x$ is

$$px + xq = 0. \tag{9}$$

Regarding this as an equation to be solved for q or p, we see that *we can draw through a given point* $\mathbf{P}_x$ *just one left and one right parallel to a given line* $\{p, q\}$, namely

$$\{p, -x^{-1}px\} \quad \text{and} \quad \{-xqx^{-1}, q\}.$$

Thus the set of all left (or right) parallels to a given line is an elliptic congruence: there is just one member of the set through every point of space.

If $\{p, q\}$ contains $\mathbf{P}_y$, its polar line lies in the polar plane of $\mathbf{P}_y$. Replacing q by $-q$, we deduce that the condition for the line $\{p, q\}$ to lie in the plane $\sum y_\nu x_\nu = 0$ is

$$py = yq. \tag{10}*$$

Instead of insisting that the p and q of the symbol $\{p, q\}$ shall be *unit* pure quaternions, we could just as well allow them to be any two pure quaternions of *equal* norm. Then $\{p, q\}$ is the same line as $\{\lambda p, \lambda q\}$ for any non-zero scalar λ; *i.e.*, the two pure quaternions are "homogeneous coordinates" for the line. Instead of (3) we must now write

$$\cos\alpha = -Spp'/\sqrt{Npp'}, \qquad \cos\beta = -Sqq'/\sqrt{Nqq'};$$

but instead of (4) we find that the common perpendiculars to $\{p, q\}$ and $\{p', q'\}$ are simply $\{\pm Vpp', Vqq'\}$, or

$$\{Vpp', Vqq'\} \quad \text{and} \quad \{Vp'p, Vqq'\}.$$

Formulas (5), (6), (7), (9), (10) remain valid, but (8) takes the simpler form $\{Vpp', Vqq'\}$.

To express the line $\mathbf{P}_a\mathbf{P}_b$ in the form $\{p, q\}$ we have to find pure quaternions p and q satisfying

$$pa + aq = pb + bq = 0.$$

We may take $p = a\bar{b} - b\bar{a}$ and $q = \bar{a}b - \bar{b}a$, or, halving these, $p = Va\bar{b}$ and $q = V\bar{a}b$. Thus *the line* $\mathbf{P}_a\mathbf{P}_b$ *is* $\{Va\bar{b}, V\bar{a}b\}$.

Similarly, the line of intersection of planes $\sum a_\nu x_\nu = 0$ and $\sum b_\nu x_\nu = 0$ is $\{Va\bar{b}, V\bar{b}a\}$.

* Hathaway, Quaternion Space, p. 52.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE EQUATION OF JOUKOWSKI'S AEROFOILS

J. D. MANCILL and BETTY THOMAS, University of Alabama

The conformal transformation

$$W = z + a^2/z,$$

where $z = x + iy$ and $W = u + iv$, and a is any positive real constant, maps suitably chosen circles in the z-plane into curves in the W-plane which resemble the cross section of a modern aerofoil. Cylinders having these images as sections are known as Joukowski's aerofoils. We shall consider in this note the problem of transforming any circle K in the z-plane into a curve in the W-plane by means of this transformation. The equation of this resulting curve L in the W-plane will be shown to be a polynomial of degree four or less. In this way we shall show that the equation of any Joukowski aerofoil section is a polynomial of degree four. There are known methods of constructing the aerofoil point by point for any given circle K as well as approximations to its equation* but so far as the authors know the exact equation is new.

We shall make use of isotropic coordinates in carrying out the transformation. The results show the facility that can be attained in practice by the use of isotropic coordinates in problems of conformal mapping.

Let the circle K in the z-plane be represented by the equation

$$z\bar{z} - \overline{P}z - P\bar{z} + C = 0,$$

where

$$\bar{z} = x - iy, \qquad P = h + ik, \qquad \overline{P} = h - ik, \qquad C = h^2 + k^2 - r^2,$$

and

$$(x - h)^2 + (y - k)^2 = r^2.$$

Therefore, we have the three equations

$$z^2 - zW + a^2 = 0, \tag{1}$$

$$z\bar{z} - \overline{P}z - P\bar{z} + C = 0, \tag{2}$$

$$\bar{z}^2 - \bar{z}\overline{W} + a^2 = 0, \tag{3}$$

from which to eliminate z and $\bar{z}$ and obtain the equation of the curve L in isotropic coordinates.

* See, *e.g.*, Green, Hydro- and Aerodynamics, Pitman, 1937, pp. 62–67.

Substituting the values of $\bar{z}$ from (2) into (3) and simplifying, we obtain

$$rz^2 + sz + t = 0, \tag{4}$$

where

$$r = \overline{P}^2 - \overline{PW} + a^2, \quad s = P\overline{PW} + C\overline{W} - 2\overline{P}C - 2a^2P, \quad t = C^2 + a^2P^2 - PC\overline{W}.$$

The elimination of z between equations (1) and (4) gives the equation

$$\begin{vmatrix} r & s & t & 0 \\ 0 & r & s & t \\ 1 & -W & a^2 & 0 \\ 0 & 1 & -W & a^2 \end{vmatrix} = 0.$$

Expanding this determinant and simplifying the results after substituting the values of P and $\overline{P}$ in terms of h and k, we obtain the equation

$$\begin{aligned} &a_0(W^2\overline{W}^2) + a_1(W^2\overline{W} + W\overline{W}^2) + a_2(W^2\overline{W} - W\overline{W}^2) + a_3(W^2 + \overline{W}^2) \\ &\quad + a_4(W^2 - \overline{W}^2) + a_5(W\overline{W}) + a_6(W + \overline{W}) + a_7(W - \overline{W}) + a_8 = 0, \end{aligned} \tag{5}$$

where a_i, $i=0, 1, \cdots, 8$, are known functions of a, h, k, and C. This is the equation of the curve L in the W-plane.

In order to represent the curve L by means of an equation in the variables u and v, we replace the expressions in the parentheses in equation (5) by their respective values in terms of u and v. This gives the equation we have been seeking,

$$\begin{aligned} &A_0(u^2 + v^2)^2 + A_1(u^3 + uv^2) + A_2(u^2v + v^3) + A_3uv + A_4u^2 \\ &\quad + A_5v^2 + A_6u + A_7v + A_8 = 0, \end{aligned} \tag{L}$$

where

$$\begin{aligned} A_0 &= C(h^2 + k^2), \\ A_1 &= -2(C^2h + a^2h^3 + a^2hk^2 + Ch^3 + Chk^2 + a^2Ch), \\ A_2 &= -2(C^2k - a^2h^2k - a^2k^3 + Ch^2k + Ck^3 - a^2Ck), \\ A_3 &= 8(C^2hk - a^4hk), \\ A_4 &= 5C^2h^2 + C^2k^2 + 2a^2C^2 + 4a^2h^4 + 4a^2h^2k^2 + 5a^4h^2 + a^4k^2 + 6a^2Ch^2 \\ &\quad - 6a^2Ck^2 + a^4C + C^3, \\ A_5 &= 5C^2k^2 + C^2h^2 - 2a^2C^2 - 4a^2h^2k^2 - 4a^2k^4 + a^4h^2 + 5a^4k^2 + 6a^2Ch^2 \\ &\quad - 6a^2Ck^2 + a^4C + C^3, \\ A_6 &= -4(2a^2Ch^3 - 2a^2Chk^2 + 2a^4h^3 - 2a^4hk^2 + a^4Ch + a^6h + C^3h + a^2C^2h), \\ A_7 &= -4(2a^2Ch^2k - 2a^2Ck^3 - 2a^4h^2k + 2a^4k^3 + a^4Ck - a^6k + C^3k - a^2C^2k), \\ A_8 &= a^8 - 16a^4h^2k^2 + 4a^6h^2 - 4a^6k^2 + 4a^2C^2h^2 - 4a^2C^2k^2 + 8a^4Ch^2 \\ &\quad + 8a^4Ck^2 + C^4 - 2a^4C^2. \end{aligned}$$

This equation is of fourth degree except when $C=0$ or when $h=k=0$; that is except when the circle K passes through the origin or when its center is at the origin. When $C=0$, h or k not zero, the equation (L) is of third degree. When $h=k=0$, and $C\neq 0$, it is of second degree and in general represents an ellipse. For the circle $|z|=a$ the ellipse is degenerate, that is the segment of the u-axis from $u=-2a$ to $u=+2a$.

According to Green,* when the circle K passes through $A(z=a)$ and just encloses $B(z=-a)$ the form of the curve L is that of an aerofoil. The shape and dimensions of the aerofoil are dependent upon the position of the circle K. When the circle is fixed the elements h, k, and r are known and therefore the A_i, $i=0, 1, \cdots, 8$, are known functions of these elements. Therefore, the equation of any Joukowski aerofoil section can be determined and it is an equation of the form (L) wih $A_0\neq 0$.

NOTE

MICHAEL WILENSKY, Cincinnati, Ohio

The line of reasoning in my paper, *A Representation of Integers*,† can be simplified and generalized. We had there a formula (3)

$$m = k + k' + 2\sqrt{kk' + r}, \qquad m > 6,$$

where $k'>k>0$, $kk'+r>0$, $m>r$, and in the case $r<0$, $m>|2r|$, and therefore $k<m/4$. It is evident that this formula is in general a *recurrence formula*, because k' has also r as quadratic residue and k is in general the smaller coefficient of k' in the equations

$$q^2 = kk' + r \quad \text{and} \quad q'^2 = k''k' + r.$$

The radicand itself is evidently the q of k'. One can, therefore, consider k', k'' as numbers m of different orders, or as functions of the place "t" occupied by them: $m(t)$, $m(t-1)$, $m(t-2)$, $\cdots$. Thus (3) can be written

$$m(t) = k + m(t-1) + 2q(t-1).$$

Proceeding thus further, we shall finally arrive, since the numbers m are decreasing with diminishing t whereas k and r remain constant, at a number m, $m(\lambda)$, which will not satisfy the required conditions. The smallest m which satisfies these conditions we denote by $m(0)$, but the series has no greatest number; it is an infinite one. We have accordingly

$$\begin{aligned} m(0) &= m(0) \\ m(1) &= k + m(0) + 2q(0) \\ m(2) &= k + m(1) + 2q(1) \\ &\cdots\cdots\cdots\cdots \\ m(t) &= k + m(t-1) + 2q(t-1). \end{aligned} \tag{1}$$

* Loc. cit., p. 66.

† This MONTHLY, vol. 51, 1944, pp. 466–468.

Formula (4) of my previous paper

$$q = k + \sqrt{kk' + r}$$

can be written (see above)

$$q(t) = k + q(t - 1). \tag{2}$$

By the way, it follows from equations (1) and (2) that the constant second differences of the terms of the series, $[m(t) - m(t-1)] - [m(t-1) - m(t-2)]$, equal $2k$.

By induction we find from (1)

$$m(t) = kt^2 + 2q(0)t + m(0),$$

and from (2)

$$q(t) = kt + q(0) = m'(t)/2,$$

and hence

$$m(0) = kt^2 - 2q(t)t + m(t).$$

Of course, there can be, and, in case of composite k, must be other initial numbers with the same k and r which constitute different series.

Example. Let $m(0) = 9$, $q(0) = 4$, and thus $k = 1$, $r = 7$; $m(1{,}000) = 1{,}008{,}009$, $q(1{,}000) = 1{,}004$, and

$$1{,}008{,}009 = 1 + 1{,}006{,}002 + 2\sqrt{1{,}006{,}002 + 7}.$$

It is, however, not necessary to take $m(0)$ initially; one can take $m(\lambda)$ and compute in the same manner $m(\lambda \pm t)$, where $\lambda - t \geqq 0$.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1944-45

Pi Mu Epsilon, Michigan State College

The new officers who took office in April 1944 sent the club members on a scavenger hunt at the first meeting to bring back ten specified objects defined cryptically or in mathematical terms, including a cone and arcsin (1/2). These two were combined as refreshments after they had been produced for the judges. Other programs were as follows:

Magic squares, by Janey Humphreys, Jean Bullard, and Wendel Blanding

Tricks with the slide rule, by Richard Graves and Sterling Berberian

How smart are the professors? with boners illustrated by members of the Mathematics Department and detected by the students

Lives of great mathematicians, a series of five minute talks given by eight new student initiates

Probabilities in dice, poker, and bridge, by Mrs. Mary Jean Thomas, Rosa Sorenson, and Beva Dirksen

Machines for solving algebraic equations, by Professor J. S. Frame

Numbers are peculiar animals, a series of five minute talks by members of the Department

Some mathematical curves, a series of five minute talks by each of the newly elected student initiates

The date of Easter, by Thera Chiaverini, and

Continued fractions, by Mary Mortimer

The calculus method of comparing means, by Jean Bullard, and

The effect of outside work on grades, by Eleanor Treat.

Two Louis C. Plant scholarship awards were presented at the March banquet to outstanding Juniors majoring in mathematics and deemed most worthy on the basis of scholarship, interest in mathematics, and helpfulness to the Department of Mathematics. Jane Walker received a first prize of fifty dollars, and Eunice Anderson a second prize of twenty-five dollars. A quiz program followed the presentation of awards. Picnics were held in the East Lansing State Park in the spring and in the Forestry Cabin in the fall.

Officers for the year 1944–45 were: President, Mary Jean Armstrong (later Mrs. Thomas); Vice-President, Rosa Sorenson; Secretary-Treasurer, Beva Dirksen. Officers for 1945–46 are: President, Rosa Sorenson; Vice-President, Eunice Anderson; Secretary-Treasurer, Jane Walker.

Archimedeans, Winthrop College

The following talks were presented during the past year:

Men who discovered conic sections, by Katherine Lawrence

Conic sections, by Dr. Ruth Stokes

Famous women in mathematics, by Gea Dealia Wolfe and Theora Bernstein

Outstanding women in the field of mathematics, by Emily Boone, Frances Dantzler, and Althea Floyd

The origin of arithmetic, by Hester Carpenter, Ruth Cottingham, and Polly Burgess.

The initiation banquet and party were held at the college shack on October 12, 1945. Twenty-five new members were initiated during the 1945–46 session. Officers for the 1945–1946 session are: President, Althea Floyd; Vice-President, Laura Jean Newell; Secretary, Emily Boone; Treasurer, Frances Dantzler; Social Chairman, Joyce Hutto; Reporter, Frances Crosby; Faculty Sponsors, Mrs. Blanche Badger and Dr. Josephine Mitchell.

Zeno Club, Alfred University

Early in the year the former *Math Club* adopted a constitution which provided for a new name, the *Zeno Club*. Several student and faculty papers were presented as follows at the semi-monthly meetings:

Trick methods of squaring numbers, followed by a discussion of trick methods of multiplying numbers, by Henry Beerman.

Are imaginary numbers real? by Dr. L. L. Lowenstein

The paradoxes of Zeno, by Jewel Karpel

Testing our knowledge, a talk on residues, by Dean A. E. Whitford

Fermat's last theorem, by Virginia Larson

Boolean algebra, a humorous lecture, by Professor W. V. Nevins

Trisection problem, by Betty Lou Fontaine

Nature and mathematics, including a mention of early counting, by Registrar W. A. Titsworth

Radical axis, by Marion Miller

Why I changed from mathematics to education, by Dr. Joseph Seidlin.

For the first meeting prospective club members were the guests of Dr. and Mrs. Lowenstein, and other social functions followed during the year. The club purchased for the library a collection of *University Mathematical Texts*, edited by Alexander Aitken and D. E. Rutherford. Virginia Larson was tapped for membership in Alfred's mathematics honorary, *Pi Delta Mu*.

Officers for the year 1944–45 were: President, Virginia Larson; Vice-President, Henry Beerman; Secretary-Treasurer, Betty Lou Fontaine; Press Agent, Olive Cohen. Officers elected for 1945–46 are: President, Julianne Sanford; Vice-President, Marion Miller; Secretary-Treasurer, Olive Cohen.

Mathematics Club, University of Nevada

Probability and its uses was the subject around which most of the talks to the Mathematics Club of the University of Nevada were given during the fall of 1944 and the spring of 1945. The students of the Mathematics Seminar Class under the guidance of the Faculty Adviser, Dr. E. M. Beesley, were responsible for the organization program.

The following lectures were given:

Pari-mutuel betting, by Charlotte Ferris

Combinatorial analysis, by Katherine O'Leary

Mathematics in army engineering, by Major E. C. Grafton

Some Diophantine problems, by Dr. E. M. Beesley.

At the meeting devoted to entertaining the freshmen mathematics students, a written quiz contest was held, and prizes were given to students obtaining highest scores. Major E. C. Grafton of the Civil Engineering Department and veteran of the Normandy beachhead was the guest speaker. Two highly successful social events were held: the annual Christmas Party at the home of Dean Fredrick Wood, and the Spring Picnic held jointly with the Chemistry Club and Alpha Epsilon Delta. The 1944–45 officers were: President, Mary Watts; Vice-President, Katherine O'Leary; Secretary-Treasurer, Margie McQuerry.

Harvard Mathematical Club, Harvard University

Meetings of the club were frequently attended during 1944–45 by visitors from Boston University, M.I.T., Regis, and Tufts. Papers presented were:

How to skate; or, the Laplace integral, by Professor H. Whitney

The four-color problem, by Arthur Greenwood

Transfinite arithmetic, by Roger Lyndon

The mathematical theory of international politics, by Dr. R. P. Boas, Jr.

Rotation and projection, by John Breakwell of Tufts College

Language and mathematics, by Carl Cohen

Non-euclidean geometry, by Richard Arens, first prize winner

Elementary algebraic geometry, by Dr. I. S. Cohen

The Banach-Tarski paradox; or, it's a small world, by E. J. Akutowicz

Map projections, by Lindley Burton, second prize winner

e^{e^x}; *or, analysis with e's*, by G. T. Williams, subsequently published in this MONTHLY, vol. 52, 1945, pp. 323–327, under the title *Numbers generated by the functions* e^{e^x-1}

Some freak functions, by Professor A. Zygmund of Mount Holyoke College

Trivial summability, by R. C. Buck

Some borderline cases in astronomy and mathematics, illustrated by lantern slides, by Dr. Herbert Jehle of the Physics department

The axiom of choice, by Henry Helson

Dimensional analysis, by R. W. Landauer

Elementary Diophantine approximation, by Professor R. Salem

The probability of occupancy; or, how many people have the same birthday, by Professor R. von Mises.

Recipients of the Robert Fletcher Rogers Prizes for the best student papers were Mr. Arens ($35) and Mr. Burton ($15). At the last meeting the club voted to present to the Department of Mathematics a portrait of the late Professor G. D. Birkhoff. Officers for 1944–45 were: President, Mrs. Ellen Buck; Vice-President, R. F. Arens; Secretary, J. A. Zilber; Treasurer, R. C. Buck; Faculty Adviser, Dr. R. P. Boas, Jr.

Mathematics Club, Wellesley College

The Wellesley College Mathematics Club held two meetings during the year 1944–45. The number of meetings had been reduced to allow students more time for participation in war activities. The first meeting was devoted to student reports on the topic

Mathematical fallacies.

A guest speaker was invited for the second meeting, at which the Club heard an address entitled

The Jewish Passover and the Christian Easter, by Professor Bancroft Brown of Dartmouth College.

The officers of the club were: President, Patricia Boland; Vice-President, Mrs. Joy Levy; Treasurer, Jeanne Garcelon; Junior Executive, Ida Harrison; Secretary, Miriam Paul.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Mathematics of Finance. By T. E. Raiford. Boston, Ginn and Co., 1945. Without tables: 3+176 pages. $2.50. With tables: 3+176+305 pages. $3.60.

This text is intended for a first course in the subject and it consists of two parts. Part I covers the subjects of interest and annuities certain, and Part II is a brief discussion of the simpler forms of life insurance.

The technical mathematical background required from the reader of this text does not go beyond the elementary part of a course in College Algebra. However, a successful student of the text must have the maturity to learn and appreciate the meaning of the highly technical and detailed symbolism which is used; for example, the double superscript notation which is used in the annuity symbols.

Part I begins with the development of the elementary formulas involved in the computation of simple interest. The usual subjects of compound interest, annuities, amortization and sinking funds, building and loan associations, valuation of bonds, and depreciation, follow. There is also a chapter on approximations and limits in financial problems. The approach is logical and the method of presentation is clear. In most cases the general formulas are presented first and then the simpler ones are pointed out as special cases. Numerous illustrative examples of typical problems are worked out in the text to demonstrate how the various formulas are used in their solutions. The lists of exercises seem quite adequate. Indeed the content of Part I appears to be well tested class room material.

Among the noteworthy features are an excellent chapter on annuities, the treatment of loans by the direct reduction plan, and the illustrative tables worked out to show such things as amortization schedules, investment schedules for bonds, *etc.* Also the set of Glover's tables which the book contains is a most valuable and useful asset.

In Part II the author does not presuppose even an elementary knowledge of probability. This necessarily makes the work on life insurance very elementary and somewhat unsatisfactory. However, this will not detract much in using the book as a text for a course in mathematics of finance since, as the author states, only a few lessons can be given to the discussion of the simpler forms of life insurance.

The appearance and typography of this text is in keeping with the usual good standard of Ginn and Company's books.

W. W. ELLIOTT

Introduction to Non-Euclidean Geometry. By H. E. Wolfe. New York, The Dryden Press, 1945. 13+247 pages. $4.50.

In writing this book with the avowed purpose of introducing students whose mathematical training has not been extensive to non-Euclidean geometry, the author is breaking new ground. In many respects he has done an excellent job for he possesses an easy and readable style, and has devoted much time and thought to the subject. The table of contents is as follows: I, The Foundations of Euclidean Geometry; II, The Fifth Postulate; III, The Discovery of Non-Euclidean Geometry; IV, Hyperbolic Plane Geometry; V, Hyperbolic Plane Trigonometry; VI, Applications of the Calculus to Hyperbolic Geometry; VII, Elliptic Plane Geometry and Trigonometry; VIII, The Consistency of the Non-Euclidean Geometries.

In the first three chapters the reader will find an excellent account of the historical background which is particularly important here. The gradual development of ideas throughout the centuries is dramatically told, and a student cannot but be impressed by the significance of the story for the whole of mathematics. It might have been desirable to give a somewhat more detailed account of developments since the time of Riemann. For example, by considering the totality of lines and planes through a point in Euclidean space and their intersections with an arbitrary plane, the fundamental significance of the ideas of projective geometry can be explained in an elementary fashion. From this point of view the whole question of parallelism takes on a new and simpler character.

The development of Hyperbolic geometry in chapter IV is wholly synthetic and is based on the assumption that through a given point more than one line can be drawn coplanar with a given line but not meeting it. This reviewer approves of the lack of emphasis on axiomatic foundations in a book of this character. Ideal points are introduced in §39 and ultra-ideal points in §46. It is here, however, that the approach encounters difficulties. Some reference to projective ideas, possibly along the lines suggested above, would enable the student to see the need for such *inaccessible* points.

The introduction of a metric in chapter V and again in chapter VIII is a ticklish process, especially since the author does not assume a knowledge of analytical methods on the part of the reader. (Cf. Klein's beautiful treatment of the subject in his *Nicht-Euklidische Geometrie*, Springer, 1928.)

The question, then, resolves itself into a pedagogical one. To what extent can the ideas of non-Euclidean geometry be successfully transmitted to students whose geometrical background is limited? Undoubtedly, something can be done and Professor Wolfe deserves a vote of thanks for writing an interesting and readable book. Whether one agrees with his choice of the stage at which the attempt should be made or not, it is clear that this book will stimulate interest in modern geometry which far surpasses that of Euclid in its power to satisfy the aesthetic senses and provide food for philosophical thought.

In an Appendix the author gives the propositions of Euclid Book I; a summary of the properties of the circular and hyperbolic functions and their geo-

metrical interpretations; the theory of orthogonal circles, and an account of the inversion transformation,—all of which are pertinent to the subject in hand.

The book is well printed and bound and the diagrams are particularly well drawn. The proof has been carefully read, for there are no noticeable misprints.

G. dE B. ROBINSON

NEW BOOKS RECEIVED

Industrial Algebra and Trigonometry with Geometrical Applications. By J. H. Wolfe, W. F. Mueller, and S. D. Mullikin. New York and London, McGraw-Hill Book Co., Inc., 1945. 13+389 pages. $2.20.

Sequential Analysis of Statistical Data: Applications. Prepared by Statistical Research Group, Columbia University. New York, Columbia University Press, 1945. 315 pages. $6.25.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 711. *Proposed by H. S. M. Coxeter, University of Toronto*

Suppose that the vertices of a polyhedron represent places that we wish to visit, while the edges represent the only possible routes. Hamilton considered the problem of visiting all the places, without repetition, on a single journey. (See, *e.g.*, W. W. R. Ball, *Mathematical Recreations and Essays*, London, 1939, p. 262.) This is easily solved for the pentagonal dodecahedron. Prove that it cannot be done for the rhombic dodecahedron.

E 712. *Proposed by Donald Eves, Paterson, N. J.*

A man has twelve coins, all of which appear exactly alike, but one of which is counterfeit and does not weigh the same as a genuine coin. He has at his disposal a delicate set of balances, but no weights. How can he detect the false coin, and whether light or heavy, in not more than three weighings? (*Cf.* E 651 [1945, 397].)

E 713. *Proposed by Joseph Rosenbaum, Bloomfield, Conn.*

Find a euclidean construction for a non-regular pentagon which has both a circumcircle and an incircle.

E 714. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a system of numeration of base B, find the sets of cyclic numbers of three different digits such that the numbers of each set are successive terms of a geometric progression.

If $B=3p+1$ show that the three digits of at least two of these sets are consecutive terms of an arithmetic progression. Apply to the case $B=10$.

E 715. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

Suppose $ABCD$ is a proper plane convex quadrilateral and P a point exterior to this plane. Consider the four tetrahedra $PABC$, $PABD$, $PACD$, $PBCD$. If PH is the shortest of all the altitudes of these four tetrahedra, show that H must be interior to $ABCD$. (By a proper quadrilateral is meant one with no three vertices linear.)

SOLUTIONS

Mrs. Miniver's Intersecting Circles

E 681 [1945, 395]. *Proposed by W. B. Campbell, Philadelphia*

From *Mrs. Miniver*: "She saw every relationship as a pair of intersecting circles. The more they overlapped, it would seem at first glance, the better the relationship; but this is not so. Beyond a certain point, the law of diminishing returns sets in, and there are not enough private resources left on either side to enrich the life that is shared. Probably perfection is reached when the area of the two outer crescents, added together, is exactly equal to that of the leaf shaped piece in the middle. On paper there must be some neat mathematical formula for arriving at this; in life, none."

Discuss the possibility of a unique solution for circles of given radii.

Solution by W. W. Johnson, John Huntington Polytechnic Institute. Let the areas of the circles of radii a and b be A_1 and A_2 $(a \leqq b)$, the area of the leaf be L, the area of the crescent of the smaller circle be C_1 and the area of the crescent of the larger circle be C_2. Then $L+C_1=A_1$ and $L+C_2=A_2$, whence $2L+(C_1+C_2)=A_1+A_2$. For the perfection Mrs. Miniver envisions $L=C_1+C_2$, whence $3L=A_1+A_2$ and $L=(A_1+A_2)/3$.

Clearly, the limit of L is A_1, whence $A_1 \geqq (A_1+A_2)/3$, or $A_2 \leqq 2A_1$. Thus it is seen that if the capacity of the more capable of the two individuals concerned is more than twice that of the less capable no ideal relationship can be realized and in case it is exactly twice, the less capable will be completely immersed in the other and no private resources will be left to that member of the pair.

In the general case set $r=b/a$; then $1 \leqq r \leqq \sqrt{2}$ for the ideal condition to exist. Let the circles of radii a and b have centers O_1 and O_2 respectively and intersect in points A and B. Designate angle AO_1B by 2ϕ and angle AO_2B by 2θ.

Then $L=$sector O_1AB+sector O_2AB-quadrilateral AO_2BO_1, so that for ideal conditions when $3L=A_1+A_2$ we must have

$$L = a^2\phi + b^2\theta - b \sin\theta(a \cos\phi + b \cos\theta) = \pi(a^2 + b^2)/3.$$

Eliminating ϕ by means of the relation

$$a \sin\phi = b \sin\theta,$$

we readily find

$$r^2\theta + \arcsin(r \sin\theta) - r \sin\theta(\sqrt{1 - r^2 \sin^2\theta} + r \cos\theta) = \pi(1 + r^2)/3.$$

This last is the equation that must be solved for θ when the ratio r is given. In a given case this transcendental equation could only be solved by trial involving much time and patience. So that instead of there being a neat mathematical formula for arriving at the solution the mathematics is true to the social problem in presenting a situation of transcendent difficulty for an ideal solution.

In the special case of a perfect mating $r=1$ and our equation becomes

$$2\theta - \sin 2\theta = 2\pi/3.$$

Even this equation is transcendental but not of too great difficulty. Solving this equation by the method of successive approximations, we find $2\theta=149°16.3'$ as the angle at the center of either of the equal circles subtended by one arc side of the common symmetrical leaf.

From the above we can deduce that even when two people are exactly mated the attainment of an ideal relationship is not a mundane affair, but a transcendental problem whose solution, however, can be realized with much less effort and patience than in the case of two people unequally yoked together.

Also solved by Murray Barbour, W. B. Clarke, Constance Davis, W. Unterberg, and the proposer.

Editorial Note: After the proposal of E 681 appeared in print it was brought to our attention that the same problem had been proposed by Otto Gruenberger in the February 1945 issue of the *Graham Dial*, publication of Graham Transmissions, Inc., Milwaukee, Wisconsin. The prize-winning solution was given by W. W. Johnson and appeared in the April 1945 issue of the *Dial*. The above solution is, with a few alterations, the same as that which appeared in the *Dial*.

A Variant of the St. Petersburg Paradox

E 683 [1945, 395]. *Proposed by Irving Kaplansky, Columbia University*

A person tosses a coin n times. For every unbroken run of i heads he will receive 2^i dollars. What is his expectation?

Solution by the Proposer. If the final run of heads is of length r ($r=0, \cdots, n$) the player gains $2^r-2^{r-1}=2^{r-1}$ from the last toss. The probability of this eventuality is $1/2^n$ for $r=n$, and $1/2^{r+1}$ for $r<n$ (for in the latter case the preceding toss must be a tail). Hence, if u_n is his expectation

$$u_n = u_{n-1} + \sum_{r=0}^{n-1} \frac{2^{r-1}}{2^{r+1}} + \frac{2^{n-1}}{2^n}$$
$$= u_{n-1} + n/4 + 1/2.$$

From this and $u_1 = 1$ we find $u_n = (n^2+5n+2)/8$.

Also solved by D. W. Alling, Murray Barbour, F. W. Herlihy, J. B. Kelly, and E. S. Keeping.

An Extension of E 663

E 685 [1945, 395]. *Proposed by Peter Scherk, University of Saskatchewan*

Prove that the equation $x^r - 1 = p^n$, where p is a prime while r and n are integers greater than 1, has only one solution. (*Cf.* E 663 [1945, 159].)

Solution by Paul Erdös, Stanford University. We have

$$(x - 1)(1 + x + \cdots + x^{r-1}) = p^n,$$

whence $x = 1 + p^t$, $t > 0$ (see editorial note). Clearly, then,

$$1 + x + \cdots + x^{r-1} \equiv r \pmod{p}, \quad \text{or} \quad r \equiv 0 \pmod{p}.$$

Therefore

$$1 + x + \cdots + x^{r-1} = (1 + x + \cdots + x^{p-1})(1 + x^p + \cdots + x^{n-p}).$$

Hence

$$1 + x + \cdots + x^{p-1} = p^u, \quad (u > 1, \text{ since } x = 1 + p^t).$$

Now

$$1 + x + \cdots + x^{p-1} \equiv p \pmod{p^t}.$$

Thus $t = 1$, or $x = 1 + p$. But then

$$1 + x + \cdots + x^{p-1} = (x^p - 1)/(x - 1) = [(p + 1)^p - 1]/p = p^u,$$

or

$$(p + 1)^p - 1 = p^{u+1}.$$

Clearly $(p+1)^p - 1 > p^p$ and $(p+1)^p - 1 < p^{p+1}$ except for $p = 2$, which proves that $(p+1)^p - 1 = p^{u+1}$ is impossible unless $p = 2$. We then have the unique solution $3^2 - 1 = 2^3$.

Also solved by D. W. Alling, R. C. Buck, and the proposer. Erdös remarks that the above solution is essentially that given by Gérona in *Nouvelles Annales de Math.*, 9, 1870, p. 469, and 10, 1870, p. 204.

Editorial Note: We may show that $t > 0$ in a manner similar to that which the proposer used. Suppose $t = 0$, then $x = 2$, and $p \neq 2$. Hence $p^2 \equiv 1 \pmod 4$ and

$$p^{2k} + 1 \equiv 2 \not\equiv 0 \pmod 4, \qquad k > 0.$$

Hence n cannot be even. Let n be odd, and let $p \exp f(2^r)$. Then $f \mid 2^{r-1}$. On the other hand $f \mid 2n$, for

$$p^{2n} - 1 \equiv (p^n + 1)(p^n - 1) \equiv 0 \pmod{2^r}.$$

Therefore, n being odd, we have $f=1$ or $f=2$. From $p^f \equiv 1 \pmod{2^r}$ it follows that

$$p^f = b \cdot 2^r + 1 > 2^r - 1 = p^n,$$

hence $f > n$. Thus we must have $n=1$, a contradiction of the hypothesis of the original problem.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4193. *Proposed by Hüseyin Demir, Columbia University*

If on the sides of an arbitrary pentagon $A_1A_2A_3A_4A_5$ the triangles $B_iA_{i+2}A_{i+3}$ (with indices reduced mod 5) are constructed such that $B_iA_{i+2} \| A_iA_{i+1}$, and $B_iA_{i+3} \| A_iA_{i+4}$, then the lines A_iB_i concur in a point C.

4194. *Proposed by R. Goormaghtigh, Bruges, Belgium*

In each of the triangles formed by three of the vertices of a cyclic quadrilateral, we consider the projection of the orthocenter on the circumdiameter parallel to the Simson line of the fourth vertex of the quadrilateral with respect to the triangle. The four projections form a quadrilateral inversely similar to the one given and are on a circle concentric to the circumcircle of that quadrilateral.

4195. *Proposed by R. Goormaghtigh, Bruges, Belgium*

There are ten ways to divide six points on a circle into two groups of three so as to form pairs of triangles having no common vertex. The midpoints of the segments joining the orthopoles of a given straight line with respect to each pair of triangles are ten collinear points.

4196. *Proposed by V. Thébault, Tennie, Sarthe, France*

Let A_1, B_1, C_1 be vertices of equilateral triangles constructed exteriorly, or interiorly, on the sides BC, CA, AB of a triangle, and let A_2, B_2, C_2 be the intersections (BC_1, CB_1), (CA_1, AC_1), (AB_1, BA_1). Then, V and W being the first (or second) centers isogonal and isodynamic of ABC, show that $1/VW = 1/A_1A_2 + 1/B_1B_2 + 1/C_1C_2$.

4197. *Proposed by V. Thébault, Tennie, Sarthe, France*

In a tetrahedron $T \equiv ABCD$ the perpendiculars from an arbitrary point P to the planes of the faces meet again the pedal sphere of P in A', B', C', D'. (1) The intersections of corresponding faces of T and $A'B'C'D'$ belong to a hyperboloid. Generalize. (2) If P is on a sphere with center O and if the perpendiculars to the faces meet it in A'', B'', C'', D'', the orthologic center of T and $A''B''C''D''$, other than P, is the focus of the inscribed paraboloid with the axis parallel to OP.

SOLUTIONS

Tangents to a Twisted Cubic

4141 [1944, 593]. *Proposed by H. S. M. Coxeter, University of Toronto*

Prove synthetically that the four lines of 4018 [1943, 125] are (in general) four tangents to a twised cubic.

Solution by James Jenkins, Student, University of Toronto.

LEMMA 1. *Given a triangle and a point O in the same plane, but not on a side, the locus of trilinear poles of the lines through O is a conic through the vertices of the triangle.*

Proof. If p varies in a pencil, we have, in the notation of the Editorial Note to E 481 [1942, 259], $A_p \wedge B_p$, whence $P_a' \wedge P_b'$ and $AP' \wedge BP'$. Thus P', the trilinear pole of p, traces a conic through A and B (and similarly through C).

LEMMA 2. *If a plane π meets the face BCD of a tetrahedron ABCD in a line a, then the tetrahedral pole of π projects from A into the trilinear pole of a with respect to BCD.*

Proof. Let π meet the four faces $\alpha, \beta, \gamma, \delta$ of the tetrahedron in lines a, b, c, d; let $\alpha', \beta', \gamma', \delta'$ denote the planes Aa, Bb, Cc, Dd; and let A' denote the point $(\beta', \gamma', \delta')$, and so on. Since the tetrahedra $ABCD$ and $A'B'C'D'$ have their four corresponding pairs of faces meeting in coplanar lines, the four lines AA', BB', CC', DD' are concurrent in a point P, which is the tetrahedral pole of π. Let a meet the lines CD, DB, BC in L, M, N (on b, c, d). Then the coaxial planes γ', δ', and APB meet the plane α or BCD in concurrent lines, two of which are CM, DN. So the line (α, APB) joins B to the point (CM, DN). Similarly (α, APC) is $C(DN, BL)$, and (α, APD) is $D(BL, CM)$. These lines thus meet in the trilinear pole of LMN (or a) with respect to BCD. But the planes APB, APC, APD are coaxial in the line AP. Hence the tetrahedral pole P projects into the trilinear pole of a.

LEMMA 3. *Given a tetrahedron and a line o not in a face, the locus of tetrahedral poles of the planes through o is a twisted cubic through the vertices of the tetrahedron.*

Proof. Such an axial pencil of planes meets each face in a flat pencil of lines whose center is not on a side of that face. Hence, by Lemmas 1 and 2, the points of the required locus lie on four quadric cones which join the vertices of the

tetrahedron to conics circumscribing the respectively opposite triangles. These cones evidently have common generators in pairs, so their residual intersection is a twisted cubic through the four vertices.

THEOREM. *A line of general position meets the face-planes of a tetrahedron in four points whose trilinear polars, with respect to those faces, are four tangents of a twisted cubic.*

Proof. The dual of Lemma 3 states that the tetrahedral polar planes of all the points of a line not through a vertex osculate a twisted cubic which further osculates the face-planes of the tetrahedron. As we approach one of the points where the line meets a face-plane, the tetrahedral polar plane tends to coincide with that face-plane, and its limiting intersection is the trilinear polar line of the point in question with respect to that face. From the general theory of envelopes it follows at once that this trilinear polar line is a tangent to the twisted cubic. Hence the four such trilinear polars all touch one twisted cubic.

COROLLARY. *The four trilinear polars touch an infinity of twisted cubics.* See A. C. Dixon, *Quarterly Journal of Mathematics*, vol. 23 (1889), p. 351.

Game of the Red Cross

4148 [1945, 102]. *Proposed by B. M. Stewart, Michigan State College*

A variation of the game of Solitaire is this game of the Red Cross, which is played with checkers on a field of squares arranged in six rows and five columns. During the play there can be at most one piece on a square, and at the outset one square is free so that the play, which consists of a series of jumps, may begin. A jump may be made if in three consecutive squares A, B, C of any row or column there are pieces on A and B while C is free. The jump is moving the piece on A over B to C and removing the piece on B from the play. Jumps along the diagonals of the field are forbidden. The object of the play is to remove all the pieces from the field except five, which are to be left in the form of the Red Cross, as in figure 1.

By the known theory for Solitaire (for example, see this MONTHLY, April, 1941, p. 228) a necessary condition is that the initially empty square be either as in figure 2 or 3. Show that in each of these cases a solution is possible.

x x x x x	0 0 0 0 0	0 0 0 0 0
x x 0 x x	0 0 0 0 0	0 0 0 0 0
x 0 0 0 x	0 0 x 0 0	0 0 0 0 0
x x 0 x x	0 0 0 0 0	0 0 0 0 0
x x x x x	0 0 0 0 0	0 0 0 0 0
x x x x x	0 0 0 0 0	0 0 x 0 0
Figure 1	Figure 2	Figure 3

Solution by E. D. Schell, Arlington, Va. The 30 squares of the board are numbered consecutively along the rows from left to right. For figure 2 where 13 is the only square without a piece, the moves are as follows:

3 — 13, 6 — 8, 16 — 6, 1 — 11, 26 — 16, 16 — 6, 5 — 3, 3 — 1,
1 — 11, 9 — 7, 18 — 8, 7 — 9, 10 — 8, 11 — 13, 20 — 18, 30 — 20,
15 — 25, 29 — 19, 18 — 20, 20 — 30, 27 — 29, 30 — 28, 28 — 18, 22 — 12.

For figure 3 where 28 is the only square without a piece, the moves are

18 — 28, 25 — 23, 20 — 18, 22 — 24, 12 — 22, 29 — 19, 19 — 17, 27 — 29,
21 — 23, 30 — 28, 28 — 18, 11 — 21, 1 — 11, 26 — 16, 16 — 6, 17 — 19,
10 — 20, 20 — 18, 14 — 12, 3 — 13, 6 — 8, 5 — 3, 2 — 4, 4 — 14.

Solution by the Proposer. For figure 2 the moves are

15 — 13, 24 — 14, 17 — 19, 20 — 18, 30 — 20, 27 — 17, 29 — 27, 26 — 28,
17 — 19, 8 — 18, 19 — 17, 11 — 13, 28 — 18, 18 — 8, 3 — 13, 21 — 11,
1 — 3, 5 — 15, 6 — 16, 16 — 18, 4 — 2, 2 — 12, 20 — 10, 10 — 8.

For figure 3 replace the first seven moves for figure 2 by

30 — 28, 19 — 29, 22 — 24, 25 — 23, 28 — 30, 15 — 25, 30 — 20,

and continue with the remaining moves for figure 2.

Tetrahedron and Powers

4149 [1945, 102]. *Proposed by N. A. Court, University of Oklahoma*

The powers of the vertices of a tetrahedron (T) with respect to the spheres determined by the centroid of (T) and the circles of intersection of the respectively opposite faces of (T) with a sphere (L), of arbitrary radius, concentric with the circumsphere of (T), are equal.

Solution by Howard Eves, College of Puget Sound. Let us take the circumcenter of (T) as origin and let $\mathbf{s}$, $\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}$ be the position vectors of the vertices and $\mathbf{g}$ the position vector of the centroid of (T). Let R be the radius of the circumsphere and S the radius of (L). Then

$$\mathbf{s}\cdot\mathbf{s} = \mathbf{t}\cdot\mathbf{t} = \mathbf{u}\cdot\mathbf{u} = \mathbf{v}\cdot\mathbf{v} = R^2,$$

and the equation of (L) is $\mathbf{x}\cdot\mathbf{x} = S^2$. Now the equation of a general sphere coaxal with (L) and having the plane of the vertices $\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}$ as radical plane is

$$\mathbf{x}\cdot\mathbf{x} + k\,|\,\mathbf{x}-\mathbf{t}, \mathbf{x}-\mathbf{u}, \mathbf{x}-\mathbf{v}\,| - S^2 = 0,$$

where k is a scalar parameter. If we insist that this sphere pass through the centroid $\mathbf{g}$ we find

$$k = (S^2 - \mathbf{g}\cdot\mathbf{g})/|\,\mathbf{g}-\mathbf{t}, \mathbf{g}-\mathbf{u}, \mathbf{g}-\mathbf{v}\,|,$$

giving the sphere

$$\mathbf{x}\cdot\mathbf{x} - S^2 + (S^2 - \mathbf{g}\cdot\mathbf{g})\,|\,\mathbf{x}-\mathbf{t}, \mathbf{x}-\mathbf{u}, \mathbf{x}-\mathbf{v}\,|\,/\,|\,\mathbf{g}-\mathbf{t}, \mathbf{g}-\mathbf{u}, \mathbf{g}-\mathbf{v}\,| = 0.$$

The power of vertex s with respect to this sphere is then

$$R^2 - S^2 + (S^2 - \mathbf{g}\cdot\mathbf{g})\,|\,\mathbf{s}-\mathbf{t},\mathbf{s}-\mathbf{u},\mathbf{s}-\mathbf{v}\,|\,/\,|\,\mathbf{g}-\mathbf{t},\mathbf{g}-\mathbf{u},\mathbf{g}-\mathbf{v}\,| = R^2 + 3S^2 - 4\mathbf{g}\cdot\mathbf{g},$$

since

$$|\,\mathbf{s}-\mathbf{t},\mathbf{s}-\mathbf{u},\mathbf{s}-\mathbf{v}\,|\,/\,|\,\mathbf{g}-\mathbf{t},\mathbf{g}-\mathbf{u},\mathbf{g}-\mathbf{v}\,| = \text{volume (stuv)/volume (gtuv)} = 4.$$

In this manner we see that the four required powers are all equal. The analogous theorem for the plane is also true.

Solved also by the proposer.

Editorial Note. The proposer made the following remark:

An indirect proof of this proposition is contained in my paper, *A porism on eleven spheres*, art. 3b, Mathematical Student, vol. 10, no. 3, Sept. 1942. The proposition is worthy of a direct proof; perhaps one of your contributors will supply one.

The theorem of the problem is true for a simplex σ in euclidean space of n dimensions, the vertices being denoted by B_i, $1 \leqq i \leqq n+1$. Select arbitrarily one vertex as origin O of vectors, then the vectors of the other n vertices A_i are denoted by $\mathbf{a}_i$, $1 \leqq i \leqq n$, and this system of n vectors is linearly independent. There exists an associated system of vectors $\mathbf{a}_i'$ such that $\mathbf{a}_i' \cdot \mathbf{a}_j = 0$, $j \neq i$, $\mathbf{a}_i' \cdot \mathbf{a}_i = 1$; see 3752 [1937, 404]. Denote by $\mathbf{g}$, $\mathbf{c}$ the vectors of the centroid G and the circumcenter C of σ. The equation of the face π_i opposite A_i, of π_0 opposite O, and of the circumsphere (C) are

$$(1) \qquad \pi_i\colon\ \mathbf{a}_i'\cdot\mathbf{x} = 0; \qquad \pi_0\colon\ \mathbf{x}\cdot\sum_{i=1}^{n}\mathbf{a}_i' = 1; \qquad (C)\colon\ \mathbf{x}^2 - 2\mathbf{c}\cdot\mathbf{x} = 0;$$

where $\mathbf{c}^2 = R^2$, the square of the radius of (C). The sphere (L) with center C and radius S has the equation $(\mathbf{x}-\mathbf{c})^2 - S^2 = 0$. The system of spheres

$$(2) \qquad (\mathbf{x} - \mathbf{c})^2 - S^2 + k\mathbf{a}_i'\cdot\mathbf{x} = 0,$$

has the radical plane π_i where k is an arbitrary constant, we determine the one passing through G; and we find that $k = (n+1)\{S^2 - (\mathbf{g}-\mathbf{c})^2\}$. Thus the equation of the sphere through G and the intersection of (L) with face π_i is

$$(3) \qquad (\mathbf{x} - \mathbf{c})^2 - S^2 + (n+1)\{S^2 - (\mathbf{g}-\mathbf{c})^2\}\mathbf{a}_i'\cdot\mathbf{x} = 0,$$

and the power of A_i with respect to this sphere is

$$(4) \qquad P^{(i)}(A_i) = R^2 + nS^2 - (n+1)(CG)^2, \qquad 1 \leqq i \leqq n,$$

after setting $\mathbf{a}_i^2 - 2\mathbf{c}\cdot\mathbf{a}_i = 0$, and $(\mathbf{g}-\mathbf{c})^2 = (CG)^2$. In a somewhat similar manner using the equation for π_0 in (1) we obtain for $P^{(0)}(O)$ the right member of (4); but there is no need for this task, since if we take for the origin any other vertex of σ, we must get again the right member of (4) for the other n vertices. This completes the proof.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Assistant Professor Foster Brooks has returned to Kent State University, Ohio, and has been promoted to an associate professorship.

Assistant Professor G. F. Cramer of Tulane University has been promoted to an associate professorship.

Assistant Professor Douglas Derry of the University of Saskatchewan has been promoted to an associate professorship.

C. B. Helms has been appointed to an assistant professorship at Pennsylvania Military College, Chester, Pennsylvania.

Dr. S. B. Littauer has been appointed chairman of the mathematics department of Newark College of Engineering, Newark, New Jersey.

Assistant Professor A. K. Mitchell of Trinity College, Hartford, has accepted a position in the engineering department of the Pratt and Whitney Aircraft division of the United Aircraft Corporation.

Dr. A. L. Putnam of Yale University has been appointed to an assistant professorship at the College of the University of Chicago.

Dr. Moses Richardson of Brooklyn College has been promoted to an assistant professorship.

C. F. Stephens has returned to Prairie View University, Prairie View, Texas, and has been promoted to a professorship.

Dr. S. L. Thorndike has been appointed to an associate professorship at Winthrop College, Rock Hill, South Carolina.

Professor Oscar Zariski of Johns Hopkins University has been appointed Research Professor of Mathematics at the University of Illinois.

The following appointments to instructorships have been announced:
Brown University: Dr. W. M. Chen
Drake University, Des Moines, Iowa: Dr. A. A. Grau
Johns Hopkins University: Dr. Norman Davids
Ohio State University: Dr. Joshua Barlaz
Oregon State College: Dr. Emund Pinney
University of Michigan: Dr. R. D. Schafer
University of Saskatchewan: W. Buscombe, Dr. W. Fraser, Alma Hiebert, Dr. W. Kozakiewicz, J. S. Vigder
University of Wisconsin: Dr. B. H. Colvin

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE A.A.A.S. COOPERATIVE COMMITTEE ON SCIENCE TEACHING

RALEIGH SCHORLING, University of Michigan

The Cooperative Committee on the Teaching of Science and Mathematics has been reconstituted and is now a standing committee of the A.A.A.S. and operates under the title *The A.A.A.S. Cooperative Committee on Science Teaching.* The new committee met on Saturday and Sunday, October 20–21. The members of the committee, with the organization which each man represents, are as follows: K. Lark-Horovitz, Purdue University, *chairman*, American Association of Physics Teachers; G. W. Warner, Woodrow Wilson Junior College, American Association of Physics Teachers; O. J. Lee, Northwestern University, American Astronomical Society; B. S. Hopkins, University of Illinois, American Chemical Society; L. W. Taylor, Oberlin College, American Institute of Physics; G. W. Blaydes, Ohio State University, Botanical Society of America; L. L. Quill, Michigan State College, Division of Chemical Education, American Chemical Society; E. C. Stakman, University of Minnesota, Executive Committee of the A.A.A.S.; G. A. Thiel, University of Minnesota, Geological Society of America; Raleigh Schorling, University of Michigan, Mathematical Association of America; E. H. C. Hildebrandt, Northwestern University, National Council of Teachers of Mathematics. It is probable that two or three persons, to be selected by several associations not now represented, will presented be added.

The Committee will, in the near future, publish in *School Science and Mathematics* the fourth of a series of reports dealing with the broad problems of science teaching which require collaboration of representatives of the various sciences, entitled "The Preparation of High School Science and Mathematics Teachers." There are five recommendations as follows:

1. *A policy of certification in closely related subjects within the broad area of the sciences and mathematics should be established and put into practice.*

Specifically, any combination of three of the following four subjects is recommended: biological science (including both botany and zoology), chemistry, mathematics, and physics. A combination of general science with any two or three subjects is also recommended.

More and more the colleges have been adopting standards of concentration in a major department which make it difficult and often impossible for an undergraduate to secure a good preparation for teaching in as many as three science fields. While this may be appropriate for the future research scientist, it is not

good for the future high school teacher. The colleges should provide opportunity for a "concentration" which shall lie not in one department but shall spread (not too thinly) over at least three subjects among the sciences and mathematics. The program for the liberal arts degree should allow for such a program of study.

The Committee proposes this policy as a minimum for beginning teachers. As the teacher gains experience and perhaps looks toward a position in a larger school, he should be required to take further work in the subjects in which he plans to specialize.

2. *Approximately one-half of the prospective teacher's four year college program should be devoted to courses in the sciences.*

The Committee recommends that approximately one-half of the four-year college program should be given to courses in science and mathematics. Sixty semester hours, divided among three subjects, will allow for a 24-hour major in one science subject and 18 hours in each of the two others.*

Such a program allows for a reasonable degree of specialization and insures enough breadth to prepare the student reasonably well for the teaching assignments he may expect during his first years of teaching. This program leaves ample room for the usual "general education" requirements of the college and for the required professional courses in education. For the student who plans carefully and is given advice early enough, it even leaves room for 18 or 20 semester hours of a subject such as history or foreign language or music which a student may wish to take as an extra subject to be added to his teaching repertoire.

The Committee is convinced that progress will be made in the preparation of science teachers only if the science departments consider that this preparation is one of their major functions and treat it as such. There should be a section in the college catalog describing the program for the preparation of science teachers. This program should be worked out cooperatively by the science departments and the department of education. Such collaboration will result in a better solution of the problems of student teaching and the courses in methods of teaching, which the Committee recognizes as an essential part of the preparation of a teacher.

The prospective science teacher should make a study of the social consequences and historical backgrounds of science. This may be done through a thorough-going course in this field, offered for junior and seniors, through reading and honors work, and through assisting the instructor of college courses of the "cultural" type. Also the prospective teacher could gain much from observing experimental teaching at the secondary level which aims to correlate and integrate the sciences.

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* When biology is one of the three subjects, the Committee recommends a minimum of 24 hours, rather than 18 hours, because preparation in botany, zoology, and physiology is necessary.

3. *Certificates to teach general science at the 7th, 8th, or 9th grade level should be granted on the basis of not less than 42 semester hours of college courses in the subjects covered in general science.*

Because it provides activities and learning relating science to everyday living in our modern technological world, because it cuts across the boundaries of specialized subject-matter lines, and because it provides an opportunity for all, or nearly all, pupils to explore science phenomena and applications, the Committee recognizes general science as extremely important. It is therefore particularly desirable that a teacher be well equipped to teach such a subject.

The Committee proposes that certificates for teaching general science be granted only to candidates who have college courses in all of the basic sciences. Exception might be made only in the case of people who show proficiency in the broad field of general science through some type of examination such as is suggested later in this report.

The Committee's best guess is that approximately 42 semester hours is an appropriate minimum requirement, with a minimum of 15 hours in biological science, 15 hours in physical science (including both physics and chemistry), and 12 hours in the earth sciences (geology, physiography, meteorology) and astronomy.

The earth sciences should receive more attention in the general science courses. In recent years they have been largely displaced as separate subjects in high-school curricula. The Committee believes that if science teachers had some formal training in that area, there would be a re-introduction of some subject-matter dealing with man's natural environment. Never before has the average person had so wide an opportunity of coming into contact with the diverse features of the earth's surface as is afforded him today through our modern means of transportation, and never before has intelligent citizenship been so much dependent upon understanding of the earth and its natural resources.

4. *Colleges and certification authorities should work toward a five-year program for the preparation of high-school teachers.*

All of the goals of science education would be better achieved if the prospective teacher had a five-year period of preparation. There would then be more time for obtaining the broad liberal education which promotes growth of the student as a person and makes him a better teacher. There would also be time for both the broad training in science which the Committee recommends and the specialization in one subject which is desirable for the teacher in the larger schools.

The only obstacle to the rapid spread of the five-year program is economic. Most of the large cities and several of the wealthier states now require it. Probably the states and areas with less wealth will not be able to pay salaries which justify five years of preparation until legislation is passed to furnish federal aid toward the raising of teachers' salaries.

5. *Curriculum improvements in the small high school should go hand in hand with improvement in teacher preparation.*

If the teachers of science and mathematics were more broadly prepared, they would still have difficulty in the small high school because the number of students is too few to permit the offering of a variety of courses. No school with less than two hundred students, for example, could offer one course in algebra for pupils going to college and a separate course in general mathematics for those not going to college. Yet variety is desirable—at any rate enough variety to provide a sequence of courses for those planning to specialize in the service area and a different sequence for those who will not specialize and in most cases will not go to college. When teachers are broadly enough prepared to teach the variety of courses that are needed, ways must be found in the small school to offer these courses.

One way to solve this problem is to teach two courses simultaneously in the manner employed by every teacher of a one-room rural school. Thus a teacher might, in a single period for ninth and tenth grade pupils, teach algebra to one small group and general mathematics to the others. A teacher of physical science might teach a general course for his entire class and add a couple of periods a week for work on problems and for laboratory work in physics with two or three students who plan to specialize in science in college.

A second suggestion is that the college preparatory courses in mathematics and science be offered in a cycle. Trigonometry and advanced algebra might be offered in alternate years, and similarly with chemistry and physics.

A third suggestion is that the local board of education pay students' tuition for correspondence courses or for Saturday courses taught in larger schools in neighboring cities. Correspondence courses are offered by the extension departments of many state universities, as well as by private schools. Local teachers could easily supervise the work of an occasional student in a correspondence course. The high school in the county seat might offer Saturday courses for pupils from the smaller schools nearby.

The Committee considered several very interesting proposals. Perhaps most significant is a project relating to audio-visual aids. A number of large industrial corporations, on their own initiative, suggested that they were willing to finance a series of movies (both sound and silent) and film strips to aid in the teaching of science and mathematics, and, furthermore, that they were willing to make these available to schools and teachers at low cost. It was further proposed that the Committee contact resourceful teachers and encourage them to write the script for movies and film strips, and that the Committee criticize the scripts, aid the corporations in securing audience responses to be used in the further revision of the films, and, finally, to sponsor the films. The Committee appointed a sub-committee to consider the various proposals.

THE REPORT OF THE HARVARD COMMITTEE

The report of the Harvard Committee on "The Objectives of a General Education in a Free Society," published by the Harvard University Press under the title *General Education in a Free Society*, is a document which every educator should read. Different from many recent studies on general education in the sense that it "presents a view of the total American educational scene," the Harvard Report was developed only after three years of intensive effort on the part of a committee of twelve members. Conferences were held with numerous other persons of wide experience and high standing in the fields of education and business, and a sum of approximately $60,000 was spent upon the undertaking. As President Conant has warned, any judgment of the Report based on incomplete or fragmentary reading is almost certain to be unfair; yet some sections are of sufficient interest to mathematicians that they are reviewed here.

It appears to be the opinion of many critics that the Report's treatment of the problems of secondary education is especially realistic and constructive. Attention is called to the fact that in 1870 three-fourths of those who attended high school went on to college. So the high school's function was clear; it was simply to prepare for college. Now, three-fourths of the high school students do not expect to go to college. This condition raises the fundamental question, "Can the interests of the three-fourths who go on to active life be reconciled with the equally just interests of the one-fourth who go on to further education?" To complicate the problem, "in many states nearly the whole population of high-school age is now in high school, and the same may presently be true of most states. Thus within a generation the problem of how best to meet this immense range of talent and need has grown up, like the fabled beanstalk, to overshadow virtually every other educational problem. It is in truth at the heart of any attempt to achieve education for democracy." Mathematicians are challenged by the assertion that abstract concepts like demonstrative proof in geometry, generalized number in algebra, or hypothesis in science ask for an effort of mind of which many adolescents now in high school seem incapable; a similar statement is made of such general concepts as justice and even democracy. As a result of this condition pictured by the authors of the Report, a plea is made for "even a greater diversity than exists at present in the still largely bookish curiculum" and at the same time for a central course of general education which will give unity to the program of study.

It is proposed that about half, *i.e.*, eight units, of the secondary curriculum of each student be organized as the core of the program of general education. The common and desirable division within the eight units would be three in English, three in science and mathematics, and two in the social studies. The Report is emphatic, however, in insisting that such a program of general education is only a minimum for all students. For those students who will be ending their formal education with high school, another course in each of the three areas seems essential, and for those who are going to college, a deeper knowledge of one or more of the areas is also imperative.

The Report calls attention to the fact that the need of elementary mathematics involves a constantly growing section of the general population. "Beyond this, however, mathematics has an important intrinsic role in general education. It helps build some of the skills and comprehensions that make the effective individual. . . . Mathematics may be defined as the science of abstract form. It is concerned with the universal pattern within the concrete situation." Consequently, mathematics has lost none of its traditional relevance in a general education, though it now must include considerations of its enormous utility in modern life.

It is recommended in the Report that by the end of the seventh or the middle of the eighth grade every student should have acquired a reasonable facility in the language of arithmetic and should possess some competence in the solution of arithmetical problems. Also, by this time, every student should have learned by induction or by intuitive reasoning some of the common facts of geometry. The next stage in the mathematics program, "and the last for those students who are least apt in the subject," should develop some appreciation of and skill in the use of simple algebra, graphs, and trigonometry. "Probably little more than half the pupils enrolled in the ninth grade can derive genuine profit from substantial instruction in algebra or can be expected to master demonstrative geometry. Those who have the requisite abilities should certainly receive such instruction." As a general principle, the Report insists that when a student "has reached his limit of tolerance in handling abstractions, his *general* education in mathematics must also come to an end." For those who show themselves to be inept in mathematics, any further work in the subject might be in the form of shop mathematics, business arithmetic, mathematics of the farm, and so on.

Students who have the capacity for and interest in science and mathematics are encouraged by the Report to continue their work in mathematics by studying a substantial course in algebra and a comparatively rigorous course in demonstrative geometry. Beyond demonstrative geometry, it is recognized that such students would normally study advanced algebra, solid geometry, and trigonometry, but it is urged that consideration be given to the creation of a senior course which "might include—besides elementary trigonometry and some solid geometry—analytic geometry and an elementary approach to the principles of the calculus." It is strongly recommended that provision be made for competent students going into science or mathematics to receive adequate preparation in high school so that the first mathematics studied in college will be the calculus.

In the case of students going to college who have decided that their interest in science and mathematics will not extend beyond a general education in these areas, only two years of high school mathematics is prescribed. However, further mathematics is advised so that the student will have full freedom of choice if he later pursues scientific or technological training. In particular, it is recommended that a new senior high-school course might be created for such students which would include an introductory survey of elementary trigonom-

etry, statistics, precision of measurement, and the use of graphs.

The Report's treatment of the college program is specialized to a consideration of the particular problems of Harvard College. It is concluded that the present Harvard system affords rich opportunities for specialization but general education has been neglected. Consequently, it is proposed that of the sixteen courses required for the bachelor's degree, students should be required to take six courses in general education. Moreover, in any individual program no one of these general courses may be counted for both concentration and general education. Of the six courses, it is specified that at least one shall be in the humanities, one in the social sciences, and one in the sciences. The nature of the courses is discussed in some detail; in general, they are not to be survey courses of the traditional type. In science, for example, it is expected that the general courses should represent reasonably broad syntheses; they should be taught so as "to convey some integrative viewpoint, scientific method, or the development of scientific concepts, or the scientific world-view."

Most of the students now entering Harvard College have studied at least one year of school mathematics beyond the minimal two-year program previously discussed. Consequently, *from the point of view of general education*, the authors of the Report state that "one might be permitted the optimistic view that our students have already completed a minimal program in mathematics before entering college." Moreover, the position is taken that for the student whose work in college makes no specific mathematical demands there is little point, whatever his level of performance, in prescribing remedial mathematics. Since the Report is concerned essentially with the problem of general education, there is no discussion of college courses in mathematics.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE TWENTY-NINTH ANNUAL MEETING OF THE ASSOCIATION

The twenty-ninth annual meeting of the Mathematical Association of America was held at Chicago, Illinois, on Saturday and Sunday, November 24 and 25, 1945, in conjunction with meetings of the American Mathematical Society. About three hundred and nineteen persons attended the meetings, including the following two hundred and five members of the Association:

L. K. ADKINS, State Teachers College, La-Crosse, Wisconsin
A. A. ALBERT, University of Chicago
G. E. ALBERT, Ohio State University
N. R. AMUNDSON, University of Minnesota
K. J. ARNOLD, University of Wisconsin
MAX ASTRACHAN, Antioch College
W. L. AYRES, Purdue University

R. W. BABCOCK, Kansas State College
RUTH M. BALLARD, Illinois Institute of Technology
R. H. BARDELL, University of Wisconsin in Milwaukee
R. W. BARNARD, University of Chicago
C. F. BARR, University of Wyoming
WALTER BARTKY, University of Chicago
W. D. BATEN, Michigan State College
GRACE E. BATES, University of Illinois
S. LOUISE BEASLEY, Drury College
S. F. BIBB, Illinois Institute of Technology
G. A. BLISS, University of Chicago
L. M. BLUMENTHAL, University of Missouri
C. R. BLYTH, University of Toronto
A. O. BOATMAN, Carthage College
H. F. BOHNENBLUST, Indiana University
W. M. BORGMAN, JR., Wayne University
C. C. BRAMBLE, U. S. Naval Academy
RICHARD BRAUER, University of Toronto
R. H. BRUCK, University of Wisconsin
E. L. BUELL, Northwestern University
R. S. BURINGTON, Case School of Applied Science
H. E. BURNS, Indiana University Extension
I. W. BURR, Purdue University
L. E. BUSH, College of St. Thomas

S. S. CAIRNS, Queens College
W. D. CAIRNS, Oberlin College
R. H. CAMERON, University of Minnesota
H. H. CAMPAIGNE, University of Minnesota
C. S. CARLSON, St. Olaf College
W. B. CARVER, Cornell University
W. F. CHENEY, JR., University of Connecticut
P. L. CHESSIN, University of Wisconsin
E. W. CHITTENDEN, University of Iowa
LAURA E. CHRISTMAN, Senn High School, Chicago
R. V. CHURCHILL, University of Michigan
PAUL CIVIN, University of Buffalo
E. H. CLARKE, Hiram College
C. J. COE, University of Michigan
NANCY COLE, Connecticut College
J. J. CORLISS, DePaul University
RICHARD COURANT, New York University
N. A. COURT, University of Oklahoma
H. S. M. COXETER, University of Toronto
A. E. CURRIER, U. S. Naval Academy

FLORA DINKINES, University of South Carolina
J. E. DOTTERER, Indiana Central College
W. L. DUREN, JR., Tulane University
W. H. DURFEE, Northwestern University

D. H. ERKILETIAN, Jr., University of Missouri, School of Mines and Metallurgy
H. J. ETTLINGER, University of Texas
H. P. EVANS, University of Wisconsin
H. S. EVERETT, University of Chicago

EDNA M. FELTGES, Wilson Junior College
L. R. FORD, Illinois Institute of Technology
TOMLINSON FORT, University of Georgia
J. S. FRAME, Michigan State College

M. P. GAFFNEY, JR., University of Chicago
B. H. GERE, U. S. Naval Academy
J. S. GEORGES, Wright Junior College
J. W. GIVENS, Northwestern University
A. M. GLEASON, U. S. Navy
L. M. GRAVES, University of Chicago
EDISON GREER, University of Kansas

V. H. HAAG, Hershey Junior College
D. W. HALL, University of Maryland
P. R. HALMOS, Syracuse University
O. H. HAMILTON, Oklahoma A. and M. College
M. L. HARTUNG, University of Chicago
E. D. HELLINGER, Northwestern University
R. G. HELSEL, Ohio State University
FRITZ HERZOG, Michigan State College
M. R. HESTENES, University of Chicago
E. H. C. HILDEBRANDT, Northwestern University
T. H. HILDEBRANDT, University of Michigan
J. D. HILL, Michigan State College
D. L. HOLL, Iowa State College
SISTER MARY CHARLOTTE HOLLAND, St. Xavier College
G. B. HUFF, University of Texas
R. C. HUFFER, Beloit College
H. K. HUGHES, Purdue University
RALPH HULL, University of Nebraska
J. A. HYDEN, Vanderbilt University

M. H. INGRAHAM, University of Wisconsin

R. L. JEFFERY, Queen's University
B. W. JONES, Cornell University

IRVING KAPLANSKY, University of Chicago
DORA E. KEARNEY, Iowa State Teachers College

M. W. KELLER, Purdue University
A. J. KEMPNER, University of Colorado
E. C. KIEFER, James Millikin University
J. R. KLINE, University of Pennsylvania
W. C. KRATHWOHL, Illinois Institute of Technology
JOHN KRONSBEIN, Evansville, Indiana
W. H. KURZIN, Chicago City College, Herzl Branch

JOSEPH LANDIN, University of Notre Dame
E. P. LANE, University of Chicago
LUISE LANGE, Chicago City College, Wilson Branch
R. E. LANGER, University of Wisconsin
H. D. LARSEN, University of New Mexico
GILLIE A. LAREW, Randolph-Macon Woman's College
J. W. LASLEY, JR., University of North Carolina
J. H. LEWIS, New York, N. Y.
LEO LIOLIOS, St. Louis University
Z. L. LOFLIN, Southwestern Louisiana Institute
A. T. LONSETH, Northwestern University
R. C. LUIPPOLD, University of Buffalo

C. C. MACDUFFEE, University of Wisconsin
MORRIS MARDEN, University of Wisconsin in Milwaukee
ELLA MARTH, Harris Teachers College
W. T. MARTIN, Syracuse University
SISTER MARY FELICE, Mt. Mary College
J. R. MAYOR, Southern Illinois Normal University
W. C. MCDANIEL, Southern Illinois Normal University
E. J. MCSHANE, University of Virginia
KARL MENGER, University of Notre Dame
E. J. MICKLE, Ohio State University
A. N. MILGRAM, University of Notre Dame
D. D. MILLER, U. S. Navy
H. J. MISER, Ohio State University
W. L. MISER, Vanderbilt University
E. J. MOULTON, Northwestern University
J. R. MUSSELMAN, Western Reserve University

J. L. NAGLE, Chicago
C. V. NEWSOM, Oberlin College
MARY W. NEWSON, Eureka College
IVAN NIVEN, Purdue University
E. P. NORTHROP, University of Chicago
F. S. NOWLAN, University of British Columbia

SISTER JEANETTE OBRIST, Mount St. Scholastica College
J. M. H. OLMSTED, University of Minnesota
ISAAC OPATOWSKI, University of Chicago

P. M. PEPPER, University of Notre Dame
L. C. PLANT, Michigan State College
J. E. POWELL, Michigan State College
G. B. PRICE, University of Kansas
A. L. PUTNAM, Chicago

TIBOR RADÓ, Ohio State University
P. H. RAKER, General Motors Institute
J. F. RANDOLPH, Oberlin College
W. R. RANSOM, Tufts College
RUTH B. RASMUSEN, Chicago City College, Wilson Branch
C. H. RAWLINS, JR., U. S. Naval Academy
C. B. READ, University of Wichita
MINA S. REES, Hunter College
F. A. REIBER, Chicago
W. T. REID, Northwestern University
HAIM REINGOLD, Illinois Institute of Technology
G. DE B. ROBINSON, University of Toronto
W. H. ROEVER, Washington University
R. E. ROOT, U. S. Naval Academy
G. F. ROSE, University of Wisconsin
A. E. ROSS, St. Louis University
J. B. ROSSER, Cornell University

HANS SAMELSON, Syracuse University
R. G. SANGER, University of Chicago
J. B. SCARBOROUGH, U. S. Naval Academy
SAMUEL SCHECTER, Indiana University
E. W. SCHREIBER, Western Illinois State Teachers College
M. E. SHANKS, University of Missouri
M. F. SMILEY, Lehigh University
C. V. L. SMITH, U. S. Naval Academy
G. W. SMITH, University of Kansas
H. L. SMITH, Louisiana State University
ABRAHAM SPITZBART, University of Wisconsin
C. E. SPRINGER, University of Oklahoma
R. C. STEPHENS, Knox College
B. M. STEWART, Michigan State College
M. H. STONE, Harvard University
E. B. STOUFFER, University of Kansas
W. T. STRATTON, Kansas State College
J. S. STUBBE, University of Illinois
ARTHUR SVOBODA, DePaul University
A. G. SWANSON, General Motors Institute
J. L. SYNGE, Ohio State University

H. P. THIELMAN, Iowa State College
J. M. THOMAS, Duke University
H. C. TRIMBLE, Iowa State Teachers College
A. W. TUCKER, Princeton University
H. L. TURRITTIN, University of Minnesota

E. P. VANCE, Oberlin College
G. B. VAN SCHAACK, U. S. Navy
E. C. VARNUM, Rockford, Illinois
J. I. VASS, University of Wisconsin in Milwaukee
H. E. VAUGHAN, University of Illinois

K. W. WEGNER, Carleton College
E. T. WELMERS, Michigan State College
MARY K. WERKMAN, Parker High School, Chicago
M. E. WESCOTT, Northwestern University
P. M. WHITMAN, Los Alamos, New Mexico
W. M. WHYBURN, Texas Technological College
L. R. WILCOX, Illinois Institute of Technology
R. L. WILDER, University of Michigan
F. B. WILEY, Denison University
S. S. WILKS, Princeton University
R. S. WOLFE, Northwestern University
C. L. WOODS, Emmanuel Missionary College
F. L. WREN, Peabody College

J. H. ZANT, Oklahoma A. and M. College

The meetings were held in the auditorium of the Museum of Science and Industry in Jackson Park at 57th Street and South Shore Drive, and lunches and dinners were available at the cafeteria in the Museum.

The sessions of the American Mathematical Society were held on Friday and in the morning and afternoon on Saturday. On Friday at 7:30 P.M. the Josiah Willard Gibbs Lecture on "Physics and the wave equation" was given by Professor J. C. Slater; and on Saturday at 11:00 A.M. Professor S. M. Ulam gave an address "On the stability of solutions of functional equations." The twenty-seventh Colloquium of the Society consisted of four lectures on Friday and Saturday by Professor Tibor Radó on "Length and area."

The Mathematical Association held its sessions on Saturday at 7:30 P.M. and on Sunday at 9:30 A.M. The program was prepared by a committee consisting of R. L. Jeffery, chairman; S. S. Cairns, and H. P. Thielman.

FIRST SESSION OF THE ASSOCIATION

"Introduction of vector algebra and vector analysis in elementary mathematics courses," by Professor M. R. Hestenes, University of Chicago.

"Quaternions and rotations," by Professor H. S. M. Coxeter, University of Toronto.

SECOND SESSION OF THE ASSOCIATION

"Rehabilitation of graduate work," by Professor J. R. Kline, University of Pennsylvania.

"Graduate training in statistics," by Professor W. G. Cochran, Iowa State College.

"The formulation of statistical tests in terms of admissible decisions and acceptable risks," by Professor Churchill Eisenhart, University of Wisconsin.

MEETING OF THE BOARD OF GOVERNORS

The Board met Saturday at 10:00 A.M. and again at 2:15 P.M. in the Windermere Hotel. Nineteen members of the Board were present, including nine Regional Governors.

The following twenty-nine persons were elected to membership on applications duly certified:

R. V. ANDREE, Ph.M. (Wisconsin) Grad. Teaching Asst., Univ. of Wisconsin, Madison, Wis.

W. T. ARGUE, A.B. (Tulsa) Chief Computer, Seismograph Service Corp., Tulsa 1, Okla.

SISTER M. RITA CECILE BOYLE, M.S. (Michigan) Asst. Prof., Barry Coll., Miami 38, Fla.

W. H. CARNAHAN, A.M. (Indiana) Asst. Prof., Purdue Univ., Lafayette, Ind.

J. F. DALY, Ph.D. (Princeton) Lt. Comdr., U.S.N.R. Bureau of Ships, Navy Dept., Washington, D. C.

T. C. DOYLE, Ph.D. (Princeton) Asst. Prof., Dartmouth Coll., Hanover, N. H.

SAMUEL EILENBERG, Ph.D. (Warsaw) Asso. Prof., Univ. of Michigan, Ann Arbor, Mich.

DANIEL FINKEL, B.S. (C.C.N.Y.) Instr., Amherst Coll., Amherst, Mass.

SISTER MARY SERAPHIM GIBBONS, A.M. (Minnesota) Coll. of St. Catherine, St. Paul 1, Minn.

A. M. GLEASON, B.S. (Yale) Lt., U.S.N.R.

C. H. HOLTON, A.M. (Harvard) Asst. Prof., Georgia School of Tech., Atlanta, Ga.

JOHN KRONSBEIN, Ph.D. (Leipzig) Research Engr., Hoosier-Cardinal Corp., Evansville, Ind.; Consultant, Illinois Tool Works, Chicago, Ill.

H. M. LASHIER, A.M. (Southern California) Asst. Prof., Math. and Physics, Pacific Union Coll., Angwin, Calif.

NEIL LOCKWOOD, A.B. (Baker Univ.) Operator, Red River Broadcast Co., Duluth, Minn.

M. L. MADISON, M.S. (Colorado) Asst. Prof., Colorado State Coll. of A. and M.A., Fort Collins, Colo.

N. S. MENDELSOHN, Ph.D. (Toronto) Lecturer, Queen's Univ., Kingston, Ont., Canada

ERIC MICHALUP. Actuary, Compania Nac. Anonima de Seguros, La Previsora, Apartado 848, Caracas, Venezuela

M. H. MILLER, B.S. (Anderson Coll., Ind.) Instr., Anderson Coll. and Theolog. Seminary, Anderson, Ind.

REV. P. M. MINO, A.B. (St. Francis Coll., Pa.) Teacher, St. Francis Prep. School, Loretto, Pa. On leave, Grad. student, Columbia Univ.

E. F. MOORE, A.M. (Texas) Dean, Head of Dept., Hannibal-LaGrange Coll., Hannibal, Mo.

ARTHUR PORGES, M.S. (Ill. Inst. of Tech.) 2743 W. San Marino St., Los Angeles 6, Calif.

J. R. PURDY, Ph.D. (Illinois) Asso. Prof., Southern Illinois Normal Univ., Carbondale, Ill.

B. S. M. RAO. Prof., Head of Dept., Central Coll., Bangalore, India

ELSJE SPRUITENBURG, Dr.Math. (Amsterdam) Teacher, High School, The Hague, Netherlands

H. L. STEINBERG, A.B. (U.C.L.A.) Grad. Student, Univ. of California at Los Angeles, Los Angeles, Calif.

H. W. STRAND, A.M. (Michigan) Instr., Purdue Univ., Lafayette, Ind.

A. F. SVOBODA, M.S. (DePaul) Instr., DePaul Univ., 64 E. Lake St., Chicago, Ill.

JEANETTE A. VAN OS, A.B. (Harris T. C., Mo.) Computer, General American Life Ins. Co., St. Louis, Mo.

S. E. WALKLEY, B. S. in E.E. (Tufts Coll.) RT 2/C, U.S.N.R.

The Secretary reported the deaths of the following members of the Association:

W. A. Bratton, Emeritus President, Whitman College. (June 1945)

P. J. da Cunha, Professor of mathematics, University of Lisbon. (February 4, 1945)

H. A. DoBell, Professor of mathematics, New York State College for Teachers. (December 8, 1944)

N. Durairajan, Executive Engineer, Mylapore, Madras. (July 15, 1945)

R. R. Fleet, Professor of mathematics and astronomy, Central College, Fayette, Missouri. (December 1, 1944)

J. L. Gibson, Dean and Professor Emeritus of mathematics, University of Utah. (February 10, 1945)

T. F. Holgate, Professor Emeritus of mathematics, Northwestern University. (April 10, 1945)

K. S. K. Iyengar, Professor of mathematics, Central College, Bangalore, India. (June 23, 1944)

M. M. S. Moriarty, Emeritus, High School, Holyoke, Massachusetts. (April 30, 1944)

W. P. Ott, Head of department of mathematics, University of Alabama. (December 25, 1944)

F. E. Riley, Jr., Teacher, High School, Phoenix, Arizona. (Killed in action October 22, 1944)

C. H. Rowe, Professor of mathematics, University of Dublin. (November 1944)

S. T. Sanders, Jr., Professor of mathematics, Southwestern Louisiana Institute. (April 10, 1945)

G. T. Sellew, Professor Emeritus of mathematics, Knox College. (December 27, 1944)

J. A. Swenson, Associate in mathematics, Teachers College, Columbia University. (May 2, 1945)

J. D. Tamarkin, Professor of mathematics, Brown University. (November 1945)

A. L. Underhill, Associate professor of mathematics, University of Minnesota. (January 18, 1945)

P. H. Underwood, Teacher, retired, High School, Houston, Texas. (December 14, 1944)

Willis Whited, Retired, Harrisburg, Pennsylvania. (April 28, 1945)

N. R. Wilson, Professor of mathematics, University of Manitoba. (December 27, 1944)

Among the items of business transacted by the Board were the following:

The By-laws of the new Pacific Northwest Section were presented and approved by vote of the Board.

Professor Sophia Levy McDonald of the University of California was appointed by the Board as Regional Governor for Region 14, to fill the vacancy caused by the change of residence of the recently elected Governor from that region.

The Board recommended two changes in the By-laws of the Association to be voted upon by the Association at a business meeting in 1946:

1. Section 8g to be amended by replacing the words "by the Board" at the end of the first sentence by the words "by the President subject to the approval of the Board"; and to replace the word "Board" in the sixth and eighth lines of the section by the words "Nominating Committee."

2. Amendments to various sections of the By-laws to do away with our present regions and provide for the election of a Governor from each Section of the Association.

Since there are no Sections of the Association in New England or in eastern Canada, it was suggested that if the second proposed change in the By-laws is adopted, special arrangements might be made to have Governors elected from these areas.

The Board voted that the Association become a "constituent member" of the American Council on Education.

The President was authorized to appoint a committee to study the mathematical curriculum at the Junior College level, this committee to work in cooperation with the Committee for Coordination of Studies in Mathematical Education.

The President was also authorized to appoint a committee to study the use of motion picture films, models, and other similar aids in the teaching of mathematics, and to report back to the Board on the matter.

On the recommendation of a special committee that had been appointed to consider the question, it was voted that the Association should not take over the publication of the National Mathematics Magazine.

The Board voted approval of the final report of the War Policy Committee to the Rockefeller Foundation. Also, in accordance with similar action already taken by the Council of the American Mathematical Society, it was voted to discharge this committee, which had been a joint-committee of the Society and the Association.

A proposal from the American Mathematical Society for the formation of a Mathematical Policy Committee was discussed but no action was taken on the matter at this time.

The report of the Committee on Universal Military Service in Peace Time, a sub-committee of the War Policy Committee, was approved.

The Committee on Slaught Memorial Papers reported that the first of these papers would soon be ready for publication. The papers will appear as supplements to the MONTHLY and be sent free of charge to all members of the Association and subscribers to the MONTHLY. The Board voted that no honorarium would be paid to the authors of the papers, but that each author should receive fifty copies of his paper.

The resignation of Professor Saunders Mac Lane as chairman of the Committee on Carus Monographs was reluctantly accepted by the Board, with thanks to him for the service he had given to the work of this committee. The President was authorized to appoint a new chairman for the committee. In line with a suggestion from Professor Mac Lane the Board voted that members of the Committee on Carus Monographs should be appointed for periods of six years, one new member each year, and the Secretary was empowered to institute this rotation among the present members of the committee.

A communication from the George Banta Publishing Company indicated that it would be necessary to draw up a new contract for the publication of the MONTHLY involving an increase of something like fifteen percent in cost. The Executive Committee was authorized to make such a new contract after conference with representatives of the Banta Company.

On nomination of the Editor-in-Chief the Board elected the following associate editors of the MONTHLY for the year 1946:

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The Board elected Professor J. F. Randolph of Oberlin College as a member of the Finance Committee for a term of four years to succeed Professor D. R. Curtiss.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Sunday at 9:30 A.M., President MacDuffee presiding.

The results of the election of officers were announced as follows:

First Vice-President for the term 1946–47: W. L. Ayres, Purdue University.

Governors at Large for the term 1946–48: Deane Montgomery, Smith College; C. V. Newsom, Oberlin College.

Resolutions were adopted expressing the thanks of the Association to Major L. R. Lohr, the Director of the Museum of Science and Industry, and his staff for their hospitality and courtesy in extending to the visiting members the facilities of the Museum; and to Professor R. G. Sanger and the other members of his Committee on Arrangements for their thoughtfulness and efficiency in planning for the comfort and convenience of those in attendance.

W. B. CARVER, *Secretary-Treasurer*

DECEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pa., on Saturday December 1, 1945. Professor C. A. Nelson, Chairman of the Section, presided at the morning and afternoon sessions.

There were forty-eight present, including the following twenty-eight members of the Association: C. B. Allendoerfer, Laura M. Ashbaugh, L. E. Boyer, W. B. Campbell, P. A. Caris, J. E. Davis, F. L. Dennis, Arnold Dresden, Michael Golomb, W. H. Gottschalk, C. E. Heilman, Aida Kalish, J. R. Kline, D. H. Lehmer, F. L. Manning, Clifford Marburger, D. L. McDonough, Lillian Moore, W. R. Murray, C. A. Nelson, J. C. Oxtoby, C. J. Rees, I. J. Schoenberg, G. L. Walker, A. D. Wallace, R. M. Walter, Jean B. Walton, Anna Pell Wheeler.

At the business meeting the following officers were elected for the coming year: Chairman, P. A. Caris, University of Pennsylvania; Secretary, W. H. Gottschalk, University of Pennsylvania. The program committee for the next meeting will be J. A. Clarkson, University of Pennsylvania (Chairman), G. C. Webber, University of Delaware, and W. R. Murray, Franklin and Marshall College. The next meeting will be held at the University of Pennsylvania, Philadelphia, Pa., on Saturday, November 30, 1946.

The program consisted of the following papers:

1. *Homotopy groups*, by Professor R. H. Fox, Princeton University, introduced by Professor J. C. Oxtoby.

The homotopy groups $\pi_n(Y)$ were defined, and some of the difficulties involved in the unsolved problem of finding an algorithm for their calculation were discussed.

2. *Some graphical methods in the theory of numbers*, by Professor D. H. Lehmer, University of California and Aberdeen Proving Ground.

The speaker discussed elementary graphical processes for establishing arithmetical theorems, and their corresponding analytical identities, concerning compositions and partitions of numbers into parts which are unrestricted or which are restricted in various ways.

3. *Some unsolved problems in the theory of trigonometric series*, by Professor Antoni Zygmund, University of Pennsylvania, introduced by Professor J. C. Oxtoby.

Professor Zygmund remarked that the history of mathematics shows how beneficial the development of the theory of trigonometric series was for the clarification of various general notions of mathematical analysis. The notion of function, that of the definite integral, the theory of sets of points, to mention a few examples only, were developed in close connection with the theory of trigonometric series. The hope was expressed that the still unsolved problems of that theory will considerably influence other branches of analysis. The most important, and still open, problems of trigonometric series are: (1) Is there a continuous function whose Fourier series diverges everywhere? (2) What is the structure of the "sets of uniqueness"?

W. H. GOTTSCHALK, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-eighth Summer Meeting, Ithaca, New York, August 19–20, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., October, 1946
ILLINOIS, Normal, May 10–11, 1946
INDIANA
IOWA, Grinnell, April 19–20, 1946
KANSAS, Emporia, April 13, 1946
KENTUCKY, Louisville, April 27, 1946
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, N. Y., May 4, 1946
MICHIGAN, Ann Arbor, April 13, 1946
MINNESOTA
MISSOURI
NEBRASKA, Omaha, May 4, 1946
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, April 4, 1946
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN, Boulder, Colo., April 19–20, 1946
SOUTHEASTERN, Raleigh, N. C., April 19–20, 1946
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APRIL 1946

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ON THE CHANGE OF FORM

HENRY BLUMBERG, Ohio State University

1. Strategy and mathematical exploration. Military science has its principles of strategy. So has chess,—as indeed, every game not of pure chance. The arts, too, have their basic principles of workmanship,—literature and oratory their "rhetoric," sculpture and painting their "techniques," *etc.* (We are not thinking of these principles in the derogatory sense of *mere* rhetoric or *mere* techniques, but in the broad sense comprehending their potential, intimate relationships with expression.)

How about mathematical exploration? Has it likewise an underlying strategy? As is inevitably the case for every developed field of human endeavor, the answer cannot but be in the affirmative.

And yet, while the strategy of war has from time immemorial engrossed the attention of military men, and has an immense literature devoted to it,—a somewhat similar assertion applying to literature and the arts—the "strategy of mathematical exploration" is likely to have a strange sound to mathematical ears. Is this not curious?

Every active mathematician is, of course, possessed of some system of motivating principles, which operate more or less consciously, clearly and dynamically. But mathematical literature has as yet not undertaken a broad and serious discussion of the "rhetoric" of mathematical exploration.

This phenomenon cannot be attributed to the deeper imaginative mysteries of mathematical invention as compared, say, with those of literature and the arts. Partly, perhaps, we may seek to explain it as due to the comparative recency of the grand development of mathematics, with its flood of new ideas in widely separated fields,—the call for strategic coordination being consequently delayed. In part, it may be due to the unique character of mathematical invention and discovery, which traditionally tends to regard every significant result as holding a peculiar secret of its own. Then there is the consideration that mathematicians, unlike artists and writers, say, do not much go in for sharp, open debate on the proper *modus operandi* for their vocation. (Scattered phases of how mathematicians work are treated in the autobiographical sketches of [6].)

2. What is "change of form"? But leaving it to the reader to assay the plausibility of these or other suggested explanations, we return to our main theme, which is to call attention to the need and potential utility of a more conscious and systematic strategy in mathematical work.

In the present article, we treat that phase of this strategy which we denote by "*change of form.*"

And what do we understand by "change of form"?

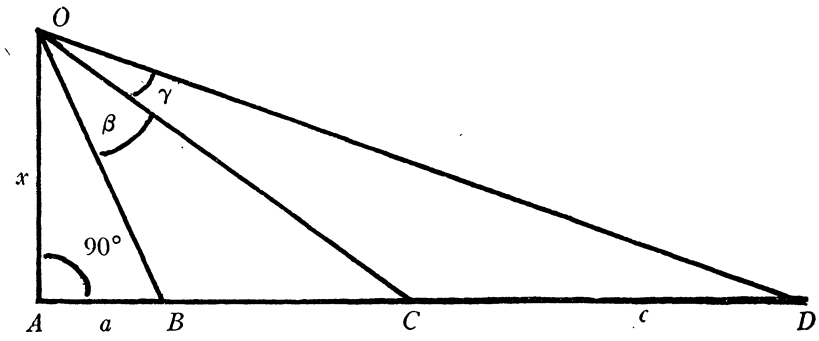
Military strategy involves the base of operations, the objective, and the line of operations. We might say—speaking in rough analogy—that change of form addresses itself to the line of operations. We could, without inexactness, use the

term *substitution* instead of *change of form*, but prefer the latter for its more appropriate abstractness and more dynamic connotation. For change of form combines both change and non-change—in a sense made clearer in the sequel. Or to put it otherwise, change of form is the "differential stroke" of mathematical exploration—in analogy to the similar role of the brush stroke in painting (*cf.* [1, 462–463]). And in accordance with this analogy; we may anticipate that the art of change of form, if duly developed, would be of basic service in mathematical exploration.

3. A preliminary remark. One more observation before proceeding with specificities. Principles of strategy are necessarily simple, and mostly obvious when pointed out. For example, the principle of a strong center in chess is anything but far-fetched. So is the principle which Rodin declared to be of prime importance in his work, namely, that of viewing every curve as the boundary of a surface, every surface as the boundary of a solid. But though all may nod quick assent to an announced principle, people differ very much in their grasp of its meaning and its range of application, and their ready use of it in situations where it can function. This consideration is of vital importance in the development of strategic power.

4. Types of change of form. We now turn to the classification and elucidation of the important types of change of form. Some types will appear as basic or primary, others as sub-types under these. Alternative classification may be devised with allowable transfer of our primary types to a secondary status, and vice versa. The virtue of a proposed classification lies not in its logical structure, but in its serviceability for exposition and exploration.

I. *Partial to comprehensive relationships.* An elementary illustration is furnished by the so-called "indirect problem" of freshman trigonometry. By this is meant a problem on triangles not directly reducible to the successive solution of triangles by the standard cases of trigonometry. For example:



Given $AB=a$, $CD=c$, $\angle BOC=\beta$, $\angle COD=\gamma$, $\angle A=90°$; find $AO=x$.

To solve this problem, we convert the slanted question of expressing x in terms of the given parts, into the more comprehensive one of determining a relationship among the five numbers a, c, β, γ and x. We are thus enabled to take

a, β, γ and x as known and c as unknown, rendering the problem direct (since we can compute c in terms of the other four letters by means of the standard cases of triangles, and thus secure the desired relationship connecting x and the given numbers).

By this method of treating knowns and unknowns on a par, all such indirect problems can be "solved"—*i.e.*, reduced to the solution of equations—whether the number of unknowns is one or more. It is frequently convenient to introduce additional variables.

The textbooks in trigonometry, as it seems, do not take up this general method, which removes an ostensible mystery.

Similarly, the so-called "reading problems" of elementary algebra, may be essentially reduced to computation. (Some recent algebra texts still speak as if a systematic plan for dealing with such problems were something not to be thought of.)

Another illustration is related to E. H. Moore's notion of "complete independence." For n postulates $P_1, P_2, \cdots, P_n$, we have 2^n possibilities with respect to the validity or non-validity of each. If there is an instance realizing each of these 2^n possibilities, the set of postulates is said to be "completely independent." If no instance exists for a particular one of these possibilities, we have a theorem—statable in one form or another—which asserts this non-realizability.

The situation is similar for n given properties (set properties, function properties, *etc.*), postulates, too, being interpretable as properties.

Curiously, a writer will sometimes take up just one or several of these possibilities, when there is no apparent reason why some of the other possibilities are of inferior interest or promise. This observation is not to be construed as an intended encouragement of problems of merely technical conception. When a certain type of work has lost its mystery for an artist—possessed of a high degree of faith in himself—he will turn to deeper mysteries. One versed in mathematical strategy will—*caeteris paribus*—more readily discern new problems, and consequently be more likely to bypass those of minor significance.

Under our present heading we may subsume the case of "putting the cart before the horse" or "reverse action." The applicability of such a maneuver may sometimes be overlooked by the student.

One instance of "reverse action" is the substitution of a variable for a number, as in the series

$$\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + \cdots$$

for the purpose of evaluating it. Rewriting it in the form

$$\frac{1^2}{1.2} - \frac{1^2}{2.3} + \frac{1^4}{3.4} - \frac{1^c}{4.5} + \cdots$$

and substituting x for 1 in the powers 1^n, we have a power series which, upon two differentiations, yields a geometric series.

Similar substitutions of a variable for a number in the series $1-1+1-1+\cdots$ leads to its "evaluation," and thus suggests one way for introducing the definition of the sum of a divergent series.

Less familiar is the substitution of the general set for a point (interpreted as a set consisting of a single point), the general interval function (or property) for a particular interval function (or property), *etc.* (An interval function is an association of a number with every interval. An interval property α is a property pertaining to intervals such that every interval has or has not property α.) The role of such substitutions is illustrated under *Generalization* (*infra*).

Among other types that may be brought under the present heading, we mention the following:

a) *From expression to process.* An instance is the following generation of the polynomial $P(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$: Take a_0 and multiply by x, getting a_0x; add a_2 and multiply by x, getting $a_0x^2+a_1x$; add a_2 and multiply by x; *etc.* This reduction to process yields immediately a simple, graphical method for computing $P(x)$ for a given value of x, hence a graphical method for solving algebraic equations. (*Cf.* [5].)

b) *From symbol to meaning.* This type is illustrated in the generalization from positive integral, to fractional and negative exponents; from ordinary to fractional differentiation; from power series to "operational series"; and in general, from formal to actual entities (numbers, functions, *etc.*).

c) *Equivalences*, especially those revealing significant discriminations, as in the instances: axiom, postulate, assumption, definitional property; set, property, variable; abstraction, generalization, essence (*cf.* below under *Generalization*).

II. *Simplification.* We shall discuss this type under *Reduction*, *Decomposition*, and *Representation*, introducing subheadings under these.

1. *Reduction.*

a) *From the infinite to the finite.* This type is basic to the "arithmetization of analysis." Many graduate students, for example, may not quite grasp the salient motive for converting the equation $\lim a_n=a$ to a statement not involving the limit process—"for every positive ϵ there exists an n such that for every $\nu>n$ we have $|a_\nu-a|<\epsilon$. It would be good, in this regard, to bolster the student's sense of strategy by means of varied exercises exhibiting the gain through this change of form.

Similar remarks apply to alternative definitions of other concepts, such as continuity and *lim sup*, to arguments pertaining to double sequences, and to diverse other cases (*cf.*, for example, the discussion below under the change *from the exact to the approximate*).

In this connection, we may recall the retention by Kowalewski, in his *Grundzüge der Differential und Integralrechnung* [3], of the limit definition of continuity. This yields novel proofs, which, however, are not as simple and natural as when the ϵ, δ-definition is adopted, Moreover, the ready geometric interpretation of the latter speaks for its strategic superiority (see below under *Representation*).

b) *From the non-denumerable to the denumerable.* This type, analogous to a), ties in with the property of the continuum of containing a dense, denumerable set—a crucial property for proof and discovery. One consequence is the "replaceability" of all neighborhoods by a denumerable number of them, namely, the "rational" ones; another, the replaceability of a non-denumerable covering T (*i.e.*, a set consisting of a non-denumerable number of linear open intervals) by a denumerable subset of T; *etc.* No wonder, too, that this type of change of form comes naturally into evidence in the postulations for various abstract spaces.

c) To mention some other illustrations of reduction: In certain essential respects, a measurable set partakes of the character of (can be "reduced" to) the sum of a finite number of intervals; and a non-measurable set to a measurable one (*cf.* IV below). Likewise, a continuous function may be "reduced" to a step-function (*i.e.*, one composed of a finite number of intervals of constancy of the function); a measurable, to a continuous function; a non-measurable, to a measurable one (*cf.* [2, 276]). We should, however, underscore that such reduction is just "*in a certain sense.*" And it would be an interesting problem to secure a substantial clarification of this phrase, thereby rendering such "reduction" a more precise tool.

d) We add an observation in regard to the organization of high school geometry in the light of strategic principles—using the process of "reduction" as an illustration. Due elaboration of the principles underlying the art of reduction would greatly facilitate the discovery of proofs and of new theorems. But more interesting is the complementary, reverse process of "composition," which consists in building "compounds" from "elements." In elementary euclidean geometry, we would, for example, begin with a simple figure (like a circle and a point, or two circles), and "compose" a more involved one, thus gaining more opportunities for finding relationships, by adding "promising" construction lines, with whatever assistance strategic principles might afford. The variety of types of such constructions, even for the more complex "originals" that appear in the texts, is very restricted, as will be found upon examination. And simple principles for "promising" construction lines can be formulated without difficulty. (One such, for example, is that of the "*double-entendre*"—as we might call it—which enjoins the drawing of a new part that "fits," *i.e.*, has a simple relationship with, two parts already in the figure, and thus mediates a relationship that may be of some interest.) Thus, with a modicum of devoted interest and practice, every high school teacher could learn to compose "originals." And this is *a fortiori* so if the strategic principles are developed not merely for *reduction* and *composition*, but more comprehensively. Learning to do this is easier than, say, for an intelligent chess player to learn how to compose two-movers or three-movers.

Of course, many attempts have been made through the ages to organize, after one manner or another, a systematic approach to the solution of "originals"—not to speak of the discovery of new theorems. But, as it seems, these attempts were not actuated by a keen strategic sense, and a felt need of working

out the principles of mathematical strategy in its broad relations. *Compartmental* strategy, without the larger view, suffers from limitations that hinder success. Herein, as it appears to the writer, is to be found an explanation for the indifferent success hitherto attained in organizing the operative principles for proofs and "discoveries" in elementary geometry.

Similar remarks apply to the strengthening of the strategic power of students in projective geometry and other fields of geometry.

2. *Decomposition*, or breaking up into parts. This itself is a major type. Theoretically, it might be supposed, that consummate skill in decompositon could render exceptionally difficult problems accessible to solution by dint of a suitable gradation of steps.

To mention some illustrations: The fact that $\aleph_0 \cdot \aleph_0 = \aleph_0$ suggests decomposition in proving denumerability. Indeed, by means of this fact, and the one discussed above that the continuum contains a dense denumerable subset, we may without difficulty, prove virtually all the important theorems on denumerability appearing in treatises on point sets or real functions. And it would be advisable, for example, in graduate teaching, once the sense and art of strategy is developed, or to the end of promoting such development, to give as exercises the proofs of the denumerability of a variety of sets on the basis of the two facts just mentioned; such as that of the algebraic numbers, of the proper maxima of a real function, of the set of points which are limit points, from one side only, of a given linear set, *etc.*

I recall the incident of one of my beginning graduate students who seemed to be impressed, as barely any other student, with my remarks on the strategy of decomposition. Working on a theorem on denumerability (related to the fundamental problem of the calculus), he evolved a novel proof consisting of three successive decompositions (or was it four? I do not exactly remember). Most everybody would have lost heart before coming through this way, and have turned to other devices.

We mention additionally:

a) The trite example of breaking up the proof that $a=b$ into the cases $a \leqq b$, $a \geqq b$.

b) The breaking up of postulates or properties into component parts for purposes of generalization—as when the inequality $|f(x)-f(\xi)| < \epsilon$, in the definition of continuity, is split into two, $-\epsilon < f(x)-f(\xi)$, $\epsilon > f(x)-f(\xi)$, and one of them used in introducing the generalized notion of semi-continuity.

3. *Representation.* This type includes numerous cases that come immediately to mind—such as the various representations of real numbers (by means of Dedekind cuts, sequences, continued fractions, products, *etc.*) and functions (by means of Taylor, Fourier, Dirichlet series, *etc.*), maps of all sorts, concrete representations of abstract groups, fields, spaces and other abstract systems.

We add the following remarks:

a) Dedekind's definition of infinite set—as one that may be biuniquely mated with a proper subset of itself—is a classic example of the strategic ad-

vantage, wherever possible, of replacing negative by positive definitions. The former permit only abstract logical inference; the latter characterize structure, furnishing ready stimulus for intuitive surmises and deductions.

It was Liouville's discovery of a positive property of a real number satisfying no algebraic equation of degree $\leqq n$ which enabled him at once to define (for the first time) a class of real, transcendental numbers.

An analogous idea, stressed by G. Hessenberg, is that the *reductio ad absurdum* should be used most sparingly, inasmuch as it rests only on deductive logic, and proves only one proposition, while a direct proof deals with actualities and representations, and yields factual results all along the way.

A curious example of representation as tool was furnished by I. Schur in deriving his theorem for the commutativity of two linear homogeneous differential expressions [4]. The proof is based on the introduction of "ideal," *infinite* differential expressions.

b) That something as abstract as an unconditioned set admits a strategic representation is truly remarkable, but this is the substance of Zermelo's illustrious theorem that every set may be normally ordered. The late chess master Emanuel Lasker (who was also a fine mathematician), when he was first informed of this theorem incisively remarked, "It can't be true—it would be too powerful." Indeed, this theorem offers the only available means for proving various basic theorems in algebra and geometry, not to speak of set theory.

c) A notable instance is furnished by Hilbert in his grand project of guaranteeing against mathematical paradox, the fundamental means being the representation of every proof by means of certain elemental entities and processes.

d) We finally mention the "algebraization" of euclidean and projective geometry, among other geometries. It should, however, be observed that the representation of a geometry by means of an algebra cannot become a valuable strategic tool unless the algebra is, or is made into, a flexible instrument for making up and solving problems. The literature, at times, flatly ignores this strategic consideration.

III. *Generalization.* This topic is discussed at some length in [1]. We shall confine ourselves here to summary statements and some observations.

Generalization is applicable to concepts, theorems and processes. For concepts, we have the strategic principle that (1) *generalization*, (2) *abstraction, and* (3) *extraction of essence are equivalent procedures.* To explain: *Generalization* is used in the sense of extension of domain, as when we pass from integers to rationals or to reals; *abstraction*, in the sense of *substituting "partial" for "total" conditioning*, as in the passage from function in the pre-modern sense (involving expressibility), to the modern conception, which discards this property; and essence is understood, of course, not in an absolute, but in a relative sense. The determination of essence, however, is not a mere subjective matter, but based on mathematical experience and cultivated insight, as when the essence of functionality comes eventually to be recognized universally as mere correspondence.

The equivalence of generalization, abstraction and extraction of essence does

not, of course, signify identity—as may be said of all proper changes of form. When one of these procedures is to be applied, the fact of its equivalence to the other two is of potential assistance in suggesting cues for its realization.

Another procedure equivalent to these three is *extension of meaning* as when fractional exponent, or fractional differentiation is invested with meaning for the first time.

The appreciation of these equivalences will fortify the strategic skill of the student with reference to the important process of generalizing concepts.

Following is an instructive example of the "strategic" generalization of a theorem:

One proof of the non-denumerability of the continuum runs as follows: If $\{x_n\}$ is a given sequence of real numbers, there is a closed interval I, of length <1, not containing x_1; a closed subinterval I_2 of I, of length $<1/2$ not containing x_2; a closed subinterval I_3 of I_2, of length $<1/3$, not containing x_3; and so on. There is just one point x common to all the I_n, $n=1, 2, \cdots$, and this point is different from each x_n, $n=1, 2, \cdots$. We may secure a generalized theorem by retaining the framework of this argument and substituting for x_n (interpreted as a set consisting of a single point) the (linear) set S_n and then, if possible, determining the condition to be imposed upon each S_n so as to permit the definition of the intervals I_n, with their requisite character for effecting the proof. We can see immediately that the desired condition is that S_n be non-dense. We thus obtain the following generalized theorem: For every given sequence of non-dense, linear sets, there exists a point contained in none of them. (The generalization to euclidean n-space is immediate.)

Strategically speaking, this generalized theorem might well have been the original incentive for introducing the notion of "exhaustible" set (=set of first category, as it is known in the literature), namely, of a set which is the sum of $\aleph_0$ non-dense sets.

We observe that to be a good generalizer one must be a good changer of form, every change of form—whether in the definition of a concept, formulation of a process, or proof of a theorem—potentially offering new cues for generalization. Thus other proofs of the theorem we have just generalized lead to other interesting generalizations.

We remark further that the question of what constitutes "essence" deserves circumspect consideration. To promote a clearer and more comprehensive understanding of "essence" in its various bearings, it would be desirable, in graduate work, to take up more frequently and thoroughly than is the present vogue, questions like the following: What—from the point of view of proposing a definition—are the "essential" properties of a measurable set? Of a topological space? Of the sum of a divergent series? *etc.*

IV. *From the exact to the approximate.* We cite the following interesting illustration: Suppose we had the problem of developing in a graduate class a plausible generalization of the *length of an interval.* We might then take the following line of reasoning: Let us try to apply the change of form "from the exact to the

approximate." Knowing the meaning of length for an interval—hence for a finite interval sum (*i.e.*, the sum of a finite number of intervals)—we are to conceive of an appropriate approximation of a finite interval sum. A promising approximation would be a *refined* one—one differing from a finite interval sum by "as little as you please." A first thought, then, is to take as the desired approximation a (linear) set which, for every positive ϵ is representable in the form $T-E_1+E_2$, where T is a finite interval sum, and E_1 and E_2 are each coverable by intervals (finite or infinite in number) of length sum $<\epsilon$. We call such a set an "*approximate interval sum.*"

It turns out that this definition is equivalent to the definition of a measurable set, as first given by Lebesgue. Thus an *L-measurable set is nothing other than an approximate interval sum*, in the sense just defined.

The latter conception furnishes a representation, a structural picture, of a measurable set, in contrast to the descriptive definition of Lebesgue, and accordingly lends itself more readily to intuitive surmises of relationships and proofs. It offers, indeed, the simplest and most direct way of developing the basic theory of L-measurable sets.

Jordan content may be treated similarly; so may measurable function.

If we wish, the change from the exact to the approximate may be subsumed under *generalization.*

We finally mention the following types:

V. *Relativization*—the change from the absolute to the relative. This is exemplified not only in the Theory of Relativity, but in many other fields. As is patent, the potentialities of relativization are far from having been exploited. In projective geometry, there is the well-known relativization of euclidean theorems—based on the relativization of parallel, midpoint, circle, perpendicular, distance, *etc.*—which can well be made a generating principle for the main substance of the subject, leading readily to the cues and the proofs of most of the principal theorems and results in the treatises on projective geometry. In the field of real functions, we have relative closure, relative measure, relative convergence, *etc.*

VI. *Logical changes of form.* As, for instance, from a proposition to its contrapositive. It may be mentioned that with the use of the contrapositive, the Borel theorem might be presented at the beginning of real function theory as an exercise immediately implied by the fundamental theorem of the Dedekind theory of the irrational (that for every Dedekind cut, the lower section has a maximum, or the upper a minimum).

VII. *Integral to differential, and conversely.* This signifies a passage from "local" to "universal" (*im Kleinen* to *im Grossen*) properties, and conversely. This type is a basic motivating idea of Relativity, as when action at a distance is replaced by field theory (differential equations). The Borel and Vitali theorems may be interpreted as furnishing a passage from "local" to "universal" properties. And in the light of this change, students will more readily recognize the strategic significance of covering theorems.

5. Some additional observations. In the interest of further elucidation we subjoin the following observations.

a) We wish to reinforce the point, made in §3, that there is nothing mandatory in our adopted classification of changes of form, the guiding objective in our choice having been expository convenience and strategic serviceability. There is no absolute criterion for determining which changes of form are basic, no more than there is a compelling reason for taking a particular one of the categories Truth, Beauty, Virtue, Will, Love, *etc.*, as more basic than the others in the formulation of a philosophy of life. Thus, for example, generalization may be made to appear as a type of change of form, or conversely, change of form may be subsumed under generalization. Likewise more or fewer categories may be listed. There is no reason, too, why personal predilections are to be excluded; and more detailed classifications may commend themselves in some fields. The crux is the availability and sharpening of basic thought-tools.

b) To amplify somewhat another observation of the Introduction: No matter how elementary an item of strategy may appear in its formulation, it is not to be assumed that, even for the expert, it is a facile instrument in its manifold interpretations and applications. What counts is not mere logical endorsement of a principle, but making it a reliable constituent of one's thought and work. The comprehensive mastery of a strategic principle, it would seem—whether in mathematics or other fields—requires that it be brought into consciousness, and then—but this is rare, perhaps only a distant ideal, for abstract principles—made into the substance of our second nature, as a ready thought-tool in its various bearings. Only thenceforward could we without loss afford to permit its "submergence" beneath workaday consciousness, and draw on its full power in that organically comfortable state.

And,—as is apparent in the case of the arts and the competitive skills, like tennis, golf, *etc.*,—it is not to be assumed that a mathematician, even if truly great, has necessarily developed a particular elementary strategic principle to a superlative degree.

c) To urge the importance of consciously appreciated, readily applicable, strategic principles, does not, of course, imply a diminished esteem for the deeper mysteries of creative work, suggested by such figures of speech as "flights of the imagination," "converse with the Muses," *etc.* The quintessence of Beauty has not been captured by man, yet those who devotedly seek it gladly use all available means of advancing their comprehension of it, including sustained cultivation of technical knowledge and skill—as the life histories of artists, writers and musicians amply attest.

No doubt technique and strategy may be permitted to become the operative enemies of the creative spirit, just as science has been utilized for destructive ends. But properly employed, strategy is a true ally of the intuitive and imaginative faculties.

Indeed, as in the controversy over "free enterprise" *vs.* government planning, the question here is not one of just *pro* or *con*, but of difference in conception as

to the kind and degree of strategy which may be "legitimately espoused." No one would knowingly favor mystification which analysis can dissipate—as in the "reading problems" of algebra, the "originals" of geometry, the "indirect" problems of trigonometry, and many analogous instances on the so-called "higher levels." And in general, it is well to refrain from a dogmatism which would pre-determine—inevitably, with a cloud of vagueness—the type or degree of technique one may "legitimately" cultivate.

One who does not pursue art may marvel interminably at the mystery of a drawing that calls for only a modicum of technique. But when such technique is acquired, the search for beauty and mystery mounts to higher planes. So in mathematics. The development of strategic skill should mean the withdrawal of interest from the minor mysteries and the growth of interest in the deeper ones.

6. Relation to teaching. Lastly, we make some remarks on the bearing of our subject upon teaching.

Strategic power may be developed by all who are dependably devoted and assiduous in its pursuit. Nobody is marked at birth as permanently incapacitated from attaining it. What part or degree of mathematical talent depends on strategic attainments has never been reliably determined. In the matter of developing the capacity for "creative" and "effective" writing, distinguished publishers, editors and writers differ widely on the "relative efficacy" of training *vs.* "natural endowment." Likewise, in the case of mathematics, it is well to avoid dogmatism in reference to gauging such "relative efficacy." Teachers, accordingly, have no scientific or moral justification for passing drastic, definitive judgments upon some of their students in the matter of their mathematical potentialities, that "so and so is clearly not cut out for mathematics," and the like. This does not mean that individual differences are not great, nor that the requisite combination of devotion and discipline can be easily found. But every conscientious teacher will want to be safeguarded against destroying faith on misunderstood or false premises.

Strategy cuts across different fields, discerning essential sameness under marked diversity. It thus signifies the acquisition of *power* as distinguished from *passive, factual knowledge.* The possession of such power must be a primary goal of all true education. Paraphrasing E. Mach, we may say that education consists in "advancement in economy of thought."

We have already indicated earlier—in particular, in connection with the organization of euclidean geometry—the feasibility of projecting a significant strategy for "discovering" theorems and devising proofs. Nothing contributes so much to the understanding of the publications of others as self-initiation as a writer. By the same token, right teaching cannot afford to neglect the eliciting of the powers of invention and discovery. Interest and devotion being assumed, means are available for developing inventive capacity. To this end, strategy must be reckoned a substantial aid.

One of the outstanding shortcomings in our graduate mathematics is that the

capacity of the students for *judging significance* is inadequately developed. Mathematical literature, too, is not free enough in give-and-take critical evaluations of mathematical publications. And there is a paucity of serious consideration and spirited discussion of higher ideals in research—*practical* ideals, realizable in the near or the more distant future.

To be sure, for progress toward higher ideals, diverse means are to be called into play. But here is not the place to enter into their discussion. We might mention, however, that strategy is one of these means, giving faith and clearing the decks for progressively more arduous undertakings. One versed in strategy—other things being equal—is a better asker of questions, is better prepared to discern the more important ones. (Though A knows subject S better than B, the latter may more readily discern significant problems in S if he is an expert question-asker.)

No mathematician should be satisfied with being merely an improver or subjoiner of the basic discoveries of others. Something like the Picard-Landau-Schottky-Caratheodory-Bernays . . . "Ideenkreis" typifies a phase of mathematical activity that could felicitously be much abated in favor of more independent exploration. (And this instance is cited, of course, with all due respect to the eminence of these mathematicians, and without derogation of the professional interest and quality of the follow-up contributions in question.) In this regard, literature and the arts offer a hint and higher directive, inasmuch as all artists and men of letters worthy of the name endeavor to understand and interpret life, not mediately, but in an original way, proper to their own essential individuality and experience.

As mathematical strategy comes to be more widely discussed, and its potentialities developed and more prevalently utilized, it will begin to play its proper role as a basic aid in mathematical education and research.

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GRADUATE TRAINING IN STATISTICS*

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1. Introduction. The emergence of statistics as a possible field of specialization for mathematicians is rather recent. There has, of course, been some kind of a profession of statistics for many years—the professional statistical societies are over 100 years old both in this country and in England. But until about 25 years ago statistics was regarded mainly as a useful tool in economics and politics and the deliberations in the professional societies reflected this point of view.

The beginnings of research and teaching in mathematical statistics were the results either of the insight and labors of individual enthusiasts or of demands from the users of statistical methods for better guidance. In London Karl Pearson was active and prolific in research, teaching and applications from 1893 until his retirement in 1933. An inspiring leader, he attracted into the field a number of able young men. In 1911, after some years as professor of applied mathematics, he was appointed head of a department of applied statistics, a post which he held jointly with a new professorship in eugenics. Upon his retirement the department of statistics was headed by his son, E. S. Pearson, while the chair of eugenics was also occupied by a noted mathematical statistician, R. A. Fisher.

In India Mahalanobis, professor of Physics at Calcutta, started a statistical laboratory about 1927 and carried this on at his own expense. The laboratory gradually brought together a group of young Indian mathematicians and has now become one of the outstanding centers of research in mathematical statistics in the world. These developments, it may be noted, were outside of departments of mathematics. A similar case in this country occurred at Columbia, where Hotelling, appointed professor of economics in 1931, is also a distinguished mathematical statistician. In consequence, the graduate lectures and research in mathematical statistics were at first conducted almost entirely in the department of economics.

About the same time, however, some mathematics departments began to offer work in mathematical statistics. At Michigan, for example, Carver became associate professor of mathematics and statistics in 1925 and has done much to promote the welfare of mathematical statistics in this country. Presumably as a result of his presence, Michigan now has three professors of mathematics who specialize in mathematical statistics. The late Dr. Rietz exercised a similar influence at the University of Iowa. To mention a few other dates, mathematical statisticians were appointed at Cambridge in 1931, Princeton in 1933, Wisconsin in 1937, California in 1938 and Iowa State in 1939. In several of these and similar cases, at least part of the impetus came from agricultural experiment stations which appreciated the services of a consulting statistician. At Cambridge, for instance, the Readership in statistics was established in the School of Agriculture, though some lectures are given by the Reader in the Mathematical Tripos.

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Thus, while statistics is at present represented in mathematics departments only to a minor extent, there are a number of institutions which offer a more or less complete graduate program in mathematical statistics.

Statistics has benefited tremendously from the influx of mathematicians of good caliber. Frequently the same statistical problem occurs in different fields of application. The applied statistician is likely to solve the problem purely in terms of the specific situation in which it arose, without concerning himself with other possible applications of his result. The mathematician goes to the essence of the problem and tackles this independently of the field of application. Consequently there has been built up a central fund of statistical theory, now fairly substantial, to which all applied fields may turn for new or improved techniques. Further benefit comes simply from the greater mathematical power available. Many statistical problems, whose solution was beyond the resources of most people in the field, yield readily to treatment by advanced mathematical methods. Consequently, it is important for the future progress of statistics that able mathematicians be attracted to specialize in that field.

2. Sources of employment. Before considering what constitutes suitable graduate training in statistics, it is relevant to examine briefly the job opportunities for statisticians and the types of work that these jobs involve. The "Description of the profession of statistics," issued recently by the National Roster of Scientific and Specialized Personnel, contains the following account of sources of employment.

> "Statisticians are employed in the various agencies of Federal, state and local government; in educational and research institutions, both public and private; in commercial, utility and manufacturing firms; in business, labor and other types of associations; in social agencies; in insurance companies; and in advertising and market research firms."

It is evident that there is a wide range of possible sources of employment. At the moment educational institutions probably rank as the chief single source for young mathematical statisticians. A tabulation of the present employment of 17 persons known to me who recently received advanced degrees in statistics shows that 11 have gone to colleges and universities, almost all of these to posts which combine teaching and research in mathematical statistics with the provision of consulting help to research groups on the campus. Of the remainder, 4 have gone into the federal service and 2 into industry. Opportunities in these three fields seem relatively plentiful for the near future.

3. Type of work. A concise definition of the functions of the statistician is extremely difficult to give. The definition given in the National Roster pamphlet is as successful as any and carries some authority, having been compiled in collaboration with the two leading statistical societies in this country.

> "The statistician uses inductive reasoning, based on the mathematics of probability, to develop and apply the most effective methods for collecting, tabulating and interpreting quantitative information."

According to this definition, two characteristics distinguish the statistician. First, he is a specialist in the techniques for collecting, tabulating and interpreting data; second, he applies inductive reasoning in these functions. To illustrate, the statistician may be asked to draw a sample of the sugar-beet farms in this country, in order to provide, at minimum cost, certain information about sugar-beet growing which is needed for policy-making. He may be asked to help in the planning of an experiment to test the effectiveness of DDT as a killer of undesirable insects. He is likely to be consulted on the best machines and procedure for some complex tabulation. He may specify the conclusions which can be drawn validly from the results of the DDT experiment.

The role of inductive reasoning is vital. In practically all cases where statistical methods are used, the data which are collected or tabulated or interpreted are of interest only insofar as they represent some larger group of data which statisticians call a population. In the sugar-beet example, the information which is wanted is that which applies to the whole group of sugar-beet growers: the sample is used only as a saver of time and money. The population in this case is real and definite, assuming that one can define a sugar-beet grower without ambiguity. Similarly, in the DDT experiment, the object is likely to be to predict what will happen in future uses of DDT under certain specified conditions. The experiment is thought of as a representative of a whole series of future trials. Although the population in this example is less tangible, it nevertheless governs the statistician's thinking when he is engaged in planning the experiment.

Thus the theory of statistics deals essentially with the properties of samples drawn in a specified way from one or more specified populations, and with the inferences that can be made about the populations from a knowledge of the samples. The collection of data is regarded by the statistician as the operation of drawing from a population a sample with certain desired properties. Similarly, the first step in the interpretation of data is to consider whether the data can be assumed to be drawn from some specified population. Moreover, these populations are characterized by *variability* in that the elements which comprise the population are different from one another. Without such variability, statistical problems are either trivial or non-existent.

Within the profession of statistics, there is a wide range in the amount of theoretical knowledge that is used in a man's work. At one extreme, some statisticians confine themselves, apart from some teaching, to research in the mathematical theory. At the other, there are statisticians who are essentially administrators of action programs and who use their basic theoretical training only in that it gives them, so to speak, a statistical point of view in their work.

These considerations suggest that a well-rounded graduate program must be flexible and should make available to the student the following features:

1. A thorough training in the theory of sampling and in any supporting mathematics necessary for that study.

2. Some knowledge, at a fairly concrete level, of the applications of statistical theory.

3. For students who may undertake consulting work, some basic knowledge of an applied field in which statistics is used.

We consider now the various parts of the training program in more detail.

4. Training in statistical theory. Most present graduate programs start with the assumption that the student has little or no undergraduate preparation in statistics. The students come either from mathematics or from some applied field such as economics or biology. The latter students usually possess some acquaintance with the applied phases of the subject, have had their enthusiasm aroused during an undergraduate course. The mathematicians, on the other hand, may have elected statistics simply from an interest in applied rather than pure mathematics and may have attended at most a course in probability. This lack of undergraduate preparation, which limits the amount that can be achieved in the graduate training, is, I believe, temporary, since there has been a growth in the undergraduate training in statistics concurrent with the growth in graduate training. Some time will elapse, however, before good undergraduate training is available generally.

The three basic courses in statistical theory are probability and distribution theory; statistical estimation; and the testing of statistical hypotheses. Probability appropriately comes first, since it enters into all statistical techniques. The course in probability opens with a statement of the laws of probability for discrete and for continuous variables and usually proceeds to its main topic, distribution theory. The typical problem in distribution theory may be stated as follows. Given that the random variates $x_1, x_2, \cdots, x_n$ follow a known law of probability, find the law of probability followed by the known functions $\phi_i(x_1, x_2, \cdots, x_n)$, $i=1, 2, \cdots, k \leqq n$. This type of problem arises repeatedly from applications, where the x_i represent the data which are to be collected or interpreted, while the functions ϕ_i represent the summary statistics (*e.g.*, means, totals, percentages) on which inferences about the population are based. Further, the amount and type of information which a statistic ϕ_i can supply about the population clearly depends on the frequency distribution of ϕ_i.

Several methods are available for solving distribution problems. Expressions for the desired frequency distributions can always be written down either as complex summations or as multiple integrals. The problem therefore reduces to one of performing the summations or integrations so as to obtain a reasonably simple answer. Alternatively, the method of moment-generating functions or characteristic functions provides a solution by a double application of the Fourier integral theorem. In addition, a distribution problem can sometimes be interpreted as the problem of finding the volume of some solid in multi-dimensional euclidean space and resolved by metrical methods in geometry.

Many distribution problems in statistical theory are unsolved in the sense that the integrations and summations cannot be expressed in simple terms. A rather common situation is that a solution is possible if the sample is very small, *e.g.* has, two, three or four observations, and that a limiting form of the solution is easy to find when the sample is very large. In the intermediate range the solu-

tion is lacking. Accordingly, important parts of the course on distribution theory are those dealing with various forms of the law of large numbers and with methods leading to approximate solutions of distribution problems. The course should also include an account of the properties of those frequency distributions which occur commonly in the theory, such as the binomial, Poisson, Gaussian, Chi-square, Student's t, Fisher's z, and so on.

The second basic course, statistical estimation, deals with the problem of estimating the properties of the population from a knowledge of those of the sample. Given the type of population and sample and the characteristics to be estimated, the theory of estimation seeks foolproof mathematical rules which automatically lead to the best method of estimation. For certain types of population, these rules have been found. In general, the problem is much more difficult; indeed it is not entirely clear how to define a "best" estimate. Nevertheless, substantial progress has been made along productive lines.

In many uses of statistical methods, the point at issue is whether the population is of a certain specified type, or whether two or more populations are the same in certain characteristics. For example, knowing the cost of buying and applying a certain amount of artificial fertilizer to a crop, one can calculate the minimum increase in yield which must result from the fertilizer in order to make its use profitable at a given selling price for the crop. In an agricultural experiment on this question, the data would comprise two samples, one of crop yields without applied fertilizer and one with applied fertilizer. In the interpretation of these data, the interesting question might be: does the mean yield in the latter population exceed that in the former population by at least the amount required for the application to be remunerative? This is an example of a statistical hypothesis about two populations. The *test* of the hypothesis is some calculation which enables the experimenter to decide whether the data are or are not consistent with the hypothesis. Such decisions are, of course, subject to error. The theory of testing hypotheses deals with an analysis of the various types of error and attempts to discover mathematical rules which lead to a decision with the minimum risk of error.

Thus the main body of theory teaches the student how to solve distribution problems, how to estimate properties of the population and how to test statistical hypotheses. In addition, many special problems in estimation or in testing hypotheses occur so frequently in applications that the solutions to these problems should be part of the basic knowledge of the student. Sometimes these special techniques are taught as illustrations of the theory: sometimes they are given in special courses. Examples are the method of least squares, a technique for estimation of which the fundamental theorems were established by Gauss; and multivariate analysis, which deals with the relations among groups of variables.

5. Supporting mathematics. The essential courses in mathematics are (1) advanced calculus, particularly multiple integration and expansions in series, both of which arise repeatedly in the solution of distribution problems, and (2) the

algebra of determinants and matrices and of linear and quadratic forms, which forms the mathematical groundwork for the method of least squares and for much of the development in multivariate analysis. Both are standard courses in almost any mathematics department which gives advanced lectures. With a good grasp of these topics, the student should be able to follow most of the statistical theory.

In addition, a number of other branches of mathematics are useful in that they contribute to particular parts of statistical theory, although they do not form the thread running through all parts. How the statistician rates these depends on his predilections. Personally, I would place on the "most useful" level a course in numerical methods including interpolation, numerical integration and some finite differences and a course on complex variable as far as contour integration. Numerical methods are useful because the results of theoretical investigations generally require the construction of tables before they can be handled by the non-mathematical applied worker. Complex variable is a helpful tool in dealing with distribution problems.

On the next grade of usefulness, one might mention a number of topics: combinatory analysis, useful for discrete distribution problems; vector analysis and n-dimensional geometry, again useful for distribution problems: point-set and measure theory, which is needed for following some of the more fundamental developments in probability; finite groups, which has recently found applications to the theory of the design of experiments: and calculus of variations, which is the basis of the general theory of testing hypotheses. This list above covers a fairly extensive range in advanced mathematics and includes some topics which may not be taught by the mathematics department. Such topics can be dealt with either by indicating a short course of reading to the student at the appropriate time or by arranging seminars.

Since it is to be anticipated that other branches of mathematics will provide fruitful applications to statistics in the future, the list of useful supplementary mathematical knowledge is not likely to be static.

6. Training in applied statistics. Two types of training may be offered. The first consists of courses in which the relevant body of statistical theory is applied to some subject-matter field, such as economics, psychology, engineering or biology or to some special type of activity such as the conduct of sample surveys, the design of experiments, or the setting-up of a quality-control program in an industry. Such courses are usually attended also by workers in the applied field who wish to obtain some grasp of the contributions of statistics to that field.

Further, the student may receive some experience in the independent application of his statistical knowledge by answering letters containing queries, by assisting in consulting work and by helping to construct or criticize the plans for research projects. These tasks, in which the student must himself select the appropriate statistical theory for a problem, or develop new theory if needed, often present considerable difficulty even to the student who has a good grasp of theory.

7. Training in an applied field. The object is to give the student a sufficient acquaintance with the field so that he can cooperate intelligently with workers in the field and can translate statistical problems arising there into theoretical counterparts which really correspond to the situation in which the problems arose. For the mathematician, in view of time limitations, such training tends to consist of some courses which deal with the fundamental theory in the subject-matter field and some which describe current research in the field.

There is some doubt amongst statisticians as to the amount of time which should be allotted to the work in applications or in the applied field at the expense of mathematics or basic theory. The answer probably depends on the student. With a student of high general intelligence and good mathematical ability who is likely to be successful in most things that he undertakes, the best procedure may be to concentrate on theory and mathematics on the grounds that youth is the time to learn mathematics and that, should applications later attract him, he can learn what is needed by his own efforts. With an average graduate student whose development is less predictable, it should be remembered that the abilities required for theoretical statistics are considerably different from those required for applied statistics. The student's training should therefore give him the opportunity to learn whether his preferences and his skill lie in theory or in applications.

8. Concluding remarks. The program of study described above, if supplemented by the production of a thesis, is intended to constitute an adequate Ph.D. training. Courses for a master's degree, while naturally less comprehensive, should, I believe, follow the same general plan.

The administration of such a program requires coordinated effort from a number of departments. Where a cooperative spirit is present, this coordination can be secured in several ways: it is perhaps too early to be sure which is the best way. At both Columbia and Iowa State, for instance, the programs are at present under the general direction of a committee recruited from the departments concerned, the courses being given in the appropriate departments. At London, as has been indicated, there is a separate department of statistics. With the latter arrangement, it is advisable to have all teachers of courses in statistics at least part-time members of the department of statistics.

In conclusion, I should like to re-emphasize that statistics depends primarily on mathematics and mathematicians for its future development. Anyone interested in the progress of statistics cannot but hope that able young mathematicians will continue to be attracted into this field, which offers a wide range of useful and stimulating applications. Such mathematicians need not be regarded as lost or strayed from the fold. For while statistics has as yet contributed relatively little to repay its debt to mathematics, it is to be expected that new points of view and new problems arising in mathematical statistics will in course of time enrich the body of mathematical knowledge itself.

PAIRS OF TRIANGLES INSCRIBED IN A CIRCLE

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1. Introduction. The notion of *mean* triangle connected with a given triangle in the sense defined by Delens [1] plays an interesting part in some properties of the triangle. Particularly, Ramler [2] has considered pairs of triangles having a common mean, and use will here be made of the same notion in a study of *any* pair of triangles inscribed in the same circle; this leads to the generalization of some theorems given in Ramler's paper.

2. Mean triangle related to a given triangle. Using complex coordinates, let a_1, a_2, a_3 be the turns corresponding to the vertices of a triangle $A_1A_2A_3$ inscribed in the base circle having center O, while Ω is the unit point, and let us denote by

$$s_1 = a_1 + a_2 + a_3, \qquad s_2 = a_2a_3 + a_3a_1 + a_1a_2, \qquad s_3 = a_1a_2a_3$$

the symmetric functions of the a_i. The vertices of the mean triangle connected with $A_1A_2A_3$ are then $s_3^{1/3}$, $s_3^{1/3}\omega$, $s_3^{1/3}\omega^2$, where $1+\omega+\omega^2=0$.

The conjugate to a will be denoted by $\bar{a}$.

Van Horn [3] has given a construction of the mean triangle similar to the one contained in our solution of a question proposed by Boutin [4]; this last author has pointed out [5] the relations between the mean triangle and the isodynamic points.

A very simple construction is derived from the geometric meaning [6] of s_3: the equation* to the Simson line of a point t for the triangle $A_1A_2A_3$ being

$$tx - s_3\bar{x} = (t^3 + s_1t^2 - s_2t - s_3)/2t, \tag{1}$$

s_3 is the point for which the Simson line is parallel to the circumdiameter through Ω. Hence the following construction:

If XY is the circumdiameter parallel to the Simson line of any point Z, the vertices of the mean triangle are the points dividing in the ratio 1:2 the two arcs XZ and the two arcs YZ.

3. Angle associated with a pair of triangles. Let b_1, b_2, b_3 be the turns corresponding to the vertices of a second triangle $B_1B_2B_3$ inscribed in the base-circle, and denote by

$$\sigma_1 = b_1 + b_2 + b_3, \qquad \sigma_2 = b_2b_3 + b_3b_1 + b_1b_2, \qquad \sigma_3 = b_1b_2b_3$$

the symmetric functions of the b_i.

If A and B are the points s_3 and σ_3, *i.e.*, the points whose Simson lines in $A_1A_2A_3$ and $B_1B_2B_3$ respectively are parallel to the circumdiameter $O\Omega$, then the angle $\theta=(OA, OB)$ is given by

$$e^{i\theta} = \sigma_3/s_3;$$

* The equation shows that the vertices of the mean triangle have their Simson lines respectively parallel to the circumdiameters passing through these vertices; these Simson lines are the cusp-tangents to Steiner's tricusp, envelope of the Simson lines of $A_1A_2A_3$.

we will call it the *angle associated with $A_1A_2A_3$ and $B_1B_2B_3$.*

The equation (1) shows further that the Simson lines of any point on the circumcircle with respect to $A_1A_2A_3$ and $B_1B_2B_3$ form an angle ϕ such that

$$e^{2i\phi} = \sigma_3/s_3.$$

If A and B are the points having their Simson lines in $A_1A_2A_3$ and $B_1B_2B_3$ parallel to any given direction, the angle $\theta=(OA, OB)$ is a constant.

The sides of the mean triangle corresponding to $A_1A_2A_3$ form angles equal to $\theta/3$, $\theta/3+120°$, $\theta/3+240°$ with the sides of the mean triangle corresponding to $B_1B_2B_3$.

The Simson lines of any point on the circumcircle with respect to the triangles $A_1A_2A_3$ and $B_1B_2B_3$ form a constant angle equal to $\theta/2$.

When the triangles have a common mean, $\sigma_3=s_3$ and $\theta=0$.

4. Orthopoles. Murnaghan [7], Godeau [8] and Ramler [2] have given an interesting property* about the common orthopole of the sides of a triangle with respect to another triangle having the same mean: we will now generalize it.

Apply to $A_1A_2A_3$ a rotation about O, the rotation angle being θ and the new position of $A_1A_2A_3$ being $A_1'A_2'A_3'$; then the coordinates of the points A_i' are $a_i\sigma_3/s_3$.

But the orthopole of A_2A_3 with respect to the triangle $B_1B_2B_3$, *i.e.*, the intersection of the Simson lines of A_2 and A_3 in the triangle $B_1B_2B_3$, is

$$(b_1 + b_2 + b_3 + a_2 + a_3 + b_1b_2b_3/a_2a_3)/2 = (\sigma_1 + a_1\sigma_3/s_3 + a_2 + a_3)/2.$$

Hence the following theorem:

A rotation about O through angle θ sends $A_1A_2A_3$ into $A_1'A_2'A_3'$, and a rotation about O through angle $-\theta$ sends $B_1B_2B_3$ into $B_1'B_2'B_3'$. The orthopoles of the sides of $A_1A_2A_3$ with respect to $B_1B_2B_3$ are the midpoints of the segments joining the orthocenter H_b of $B_1B_2B_3$ to those of $A_1'A_2A_3$, $A_1A_2'A_3$, $A_1A_2A_3'$, and the orthopoles of the sides of $B_1B_2B_3$ with respect to $A_1A_2A_3$ are the midpoints of the segments joining the orthocenter H_a of $A_1A_2A_3$ to those of $B_1'B_2B_3$, $B_1B_2'B_3$, $B_1B_2B_3'$.

When $A_1A_2A_3$ and $B_1B_2B_3$ have a common mean, $\theta=0$, and we find that, in this case, the midpoint of H_aH_b is the common orthopole of the sides of each of the triangles with respect to the other.

5. Common line of images. As the line of images [10] of a point on the circumcircle, with respect to $A_1A_2A_3$, passes through H_a and is parallel to the Simson line, the only possible common line of images for the triangles $A_1A_2A_3$ and $B_1B_2B_3$ is the straight line H_aH_b with the condition $\theta=0$, a result given by Ramler [2]:

Two triangles $A_1A_2A_3$ and $B_1B_2B_3$ inscribed in a circle have a common line of images when they have a common mean; the common line of images joins the orthocenters H_a and H_b of the triangles.

* In fact, this property had been noted before in a somewhat different form by Neuberg [9].

According to a well known property, if the parallels to H_aH_b through A_1, A_2, A_3 and B_1, B_2, B_3 cut the circumcircle again at A_1'', A_2'', A_3'' and B_1'', B_2'', B_3'', then the perpendiculars dropped from A_1'', A_2'', A_3'' and B_1'', B_2'', B_3'' on A_2A_3, A_3A_1, A_1A_2 and B_2B_3, B_3B_1, B_1B_2 respectively meet at the point whose line of images is common to the given triangles.

6. Common Simson lines. The conditions for the coincidence of (1) and the equation

$$\tau x - \sigma_3\bar{x} = (\tau^3 + \sigma_1\tau^2 - \sigma_2\tau - \sigma_3)/2\tau$$

to the Simson line of a point τ with respect to $B_1B_2B_3$, are

$$t/\tau = s_3/\sigma_3,$$
$$t^2/s_3 + s_1t/s_3 - s_2/s_3 - 1/t = \tau^2/\sigma_3 + \sigma_1\tau/\sigma_3 - \sigma_2/\sigma_3 - 1/\tau.$$

The three possible values of t are therefore given by

$$(2) \qquad (s_3 - \sigma_3)t^3/s_3^2 + (s_1 - \sigma_1)t^2/s_3 - (\bar{s}_1 - \bar{\sigma}_1)t + (s_3 - \sigma_3)/\sigma_3 = 0.$$

There are three common Simson lines corresponding, in triangle $A_1A_2A_3$ to three points forming a triangle T_a, and in triangle $B_1B_2B_3$ to three points forming a triangle T_b; the triangle T_a is sent into triangle T_b by a rotation about O, through angle θ.

Let h_a and h_b be the coordinates of the orthocenters of T_a and T_b; then

$$h_a = t_1 + t_2 + t_3 = -(s_1 - \sigma_1)s_3/(s_3 - \sigma_3),$$
$$h_b = \tau_1 + \tau_2 + \tau_3 = -(s_1 - \sigma_1)\sigma_3/(s_3 - \sigma_3).$$

Hence

$$h_a - h_b = -(s_1 - \sigma_1).$$

The segment joining the orthocenters of T_a and T_b is equipollent to the segment joining those of $B_1B_2B_3$ and $A_1A_2A_3$.

As we further have

$$t_1t_2t_3/\tau_1\tau_2\tau_3 = s_3^3/\sigma_3^3,$$

the angle associated with the triangles T_a and T_b equals three times the angle associated with the triangles $A_1A_2A_3$ and $B_1B_2B_3$.

It may also be noted that, in (2), $t_1t_2t_3 = -s_3/\sigma_3$ and, as the Simson lines of t_1, t_2, t_3 are concurrent when $t_1t_2t_3 = s_3$, we may state the following:

The common Simson lines are concurrent when the given triangles have opposite means.

Equation (2) shows also that, if $A_1A_2A_3$ and $B_1B_2B_3$ have a common mean, there is only *one* common Simson line; it corresponds to the same point on the circumcircle as the common line of images.

7. An ellipse associated with a pair of triangles. We have already noted that the Simson lines of any point on the circumcircle with respect to $A_1A_2A_3$ and $B_1B_2B_3$ form an angle $\theta/2$. The locus of their intersection has the following parametric equation:

$$x = (\bar{s}_2 - \bar{\sigma}_2)/2(\bar{s}_3 - \bar{\sigma}_3) + t/2 - (\bar{s}_1 - \bar{\sigma}_1)/2t(\bar{s}_3 - \bar{\sigma}_3)$$

and is the ellipse having as center the point

$$(\bar{s}_2 - \bar{\sigma}_2)/2(\bar{s}_3 - \bar{\sigma}_3),$$

the lengths of the axes being

$$\left| 1 \pm (s_1 - \sigma_1)^{1/2}(\bar{s}_1 - \bar{\sigma}_1)^{1/2}/(s_3 - \sigma_3)^{1/2}(\bar{s}_3 - \bar{\sigma}_3)^{1/2} \right|.$$

The locus of the intersection of the Simson lines of a variable point on the circumcircle, with respect to the triangles $A_1A_2A_3$ and $B_1B_2B_3$, is an ellipse having as axes

$$\left| R \pm \frac{H_aH_b}{2 \sin \theta/2} \right|,$$

R being the radius of the circumcircle.

8. Feuerbach points of tangential triangles. The coordinates of the Feuerbach points of the tangential triangles of $A_1A_2A_3$ and $B_1B_2B_3$ are $x_a = s_2/s_1$ and $x_b = \sigma_2/\sigma_1$. But

$$x_a = \bar{s}_1 s_3/s_1, \qquad x_b = \bar{\sigma}_1\sigma_3/\sigma_1$$

and

$$\frac{x_b}{x_a} = \left(\frac{s_1}{\bar{s}_1} \Big/ \frac{\sigma_1}{\bar{\sigma}_1}\right)\frac{\sigma_3}{s_3}.$$

The angle formed by the circumdiameters passing through the Feuerbach points of the tangential triangles of $A_1A_2A_3$ and $B_1B_2B_3$ is equal to the difference between θ and twice the angle formed by the Euler lines of the given triangles.

9. Isogonal conjugates. We consider the isogonal conjugates M_a and M_b of the midpoint M of H_aH_b in the triangles $A_1A_2A_3$ and $B_1B_2B_3$. The coordinate β of the isogonal conjugate of a point α in the triangle $A_1A_2A_3$ is given by [11]

$$\beta(1 - \alpha\bar{\alpha}) = -\alpha + s_1 - s_2\bar{\alpha} + s_3\bar{\alpha}^2.$$

Hence the coordinates μ_a and μ_b of M_a and M_b are such that

$$2\mu_a(1 - \mu\bar{\mu}) = s_1 - \sigma_1 - s_2(\bar{s}_1 + \bar{\sigma}_1) + s_3(\bar{s}_1 + \bar{\sigma}_1)^2/2,$$
$$2\mu_b(1 - \mu\bar{\mu}) = \sigma_1 - s_1 - \sigma_2(\bar{s}_1 + \bar{\sigma}_1) + \sigma_3(\bar{s}_1 + \bar{\sigma}_1)^2/2.$$

If it is noted that $\bar{s}_1 = s_2/s_3$, and if ν is the coordinate of the mid-point N of M_aM_b, then

$$8\nu(1 - \mu\bar{\mu}) = (\bar{s}_1^2 - \bar{\sigma}_1^2)(\sigma_3 - s_3)$$

and

$$64\,\overline{ON}^2(1-\mu\bar{\mu})^2=(s_1-\sigma_1)(\bar{s}_1-\bar{\sigma}_1)(s_1+\sigma_1)(\bar{s}_1+\bar{\sigma}_1)(s_3-\sigma_3)(\bar{s}_3-\bar{\sigma}_3).$$

Hence the following theorem:

If ϖ is the power, with respect to the circumcircle, of the mid-point M of H_aH_b, and if N is the mid-point of the segment joining the isogonal conjugates of M in the triangles $A_1A_2A_3$ and $B_1B_2B_3$, then

$$ON=\frac{R}{2\varpi}H_aH_b\cdot OM\sin\frac{\theta}{2}\cdot$$

This generalizes a theorem given by Ramler [2]:

The isogonal conjugates, as to two triangles having a common mean, of the mid-point of the segment joining their orthocenters, are symmetric with respect to the circumcenter.

10. Pairs of triangles having a common orthocenter. When H_b coincides with H_a, the triangles $A_1A_2A_3$ and $B_1B_2B_3$ are circumscribed to a conic having O and H_a as foci.

Then the triangles T_a and T_b (section 5) are equilateral, and the locus (section 6) of the intersections of the Simson lines corresponding to a moving point on the circumcircle, with respect to $A_1A_2A_3$ and $B_1B_2B_3$, is a circle equal to the common nine-point circle of those triangles.

11. Triangles symmetric with respect to the circumcenter. In that case, $\sigma_1=-s_1$, $\sigma_2=s_2$, $\sigma_3=-s_3$ and equation (2) becomes

$$t^3+s_1t^2-s_2t-s_3=0.$$

When the given triangles are symmetric with respect to the circumcenter, the common Simson lines pass through the circumcenter.

The ellipse in section 6 is then inscribed in $A_1A_2A_3$ and $B_1B_2B_3$ and tangent to the sides of the triangles at the images of the feet of the altitudes with respect to the mid-points of the corresponding sides. The properties of that ellipse are are well known [12]; in this case, $\theta=\pi$ and the axes are $|R\pm OH_a|$.

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DISCUSSIONS AND NOTES

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The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

Beginning with the August-September number of the MONTHLY the editor of this department will be Professor E. F. Beckenbach, University of California at Los Angeles, Los Angeles 24, California. Manuscripts should henceforth be sent to him.

TWO THEOREMS ON MULTI-DEGREE EQUALITIES*

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THEOREM 1. *The multi-degree equality*

$$a_1, a_2, \cdots, a_p \overset{n-3}{=} b_1, b_2, \cdots, b_p \dagger \tag{1}$$

with

$$t = \frac{2(\sum b_i^{n-1} - \sum a_i^{n-1})}{(n-1)(\sum b_i^{n-2} - \sum a_i^{n-2})},$$

implies

$$\begin{cases} a_1, \cdots, a_p, t-b_1, \cdots, t-b_p \overset{n}{=} b_1, \cdots, b_p, t-a_1, \cdots, t-a_p, & n \text{ even}, \\ a_1, \cdots, a_p, t-a_1, \cdots, t-a_p \overset{n}{=} b_1, \cdots, b_p, t-b_1, \cdots, t-b_p, & n \text{ odd}. \end{cases} \tag{2}$$

Proof. By virtue of (1), the relations (2) are satisfied, regardless of the values of t, for all integral exponents from 1 to $n-2$ inclusive. For the exponents $n-1$ and n, the two resulting equations are both satisfied by the value of t given above.

A numerical example. From $1+5 \neq 2+3$ and $1^2+5^2 \neq 2^2+3^2$, with $t=13$, one immediately obtains

$$1, 5, 8, 12 \overset{3}{=} 2, 3, 10, 11.$$

I have previously shown the following theorem.‡

* Note by the translator, H. L. Dorwart. In 1938, Professor Gloden published a 90 page monograph on the Tarry-Escott Problem (see The Tarry-Escott Problem by Dorwart and Brown, this MONTHLY, vol. 44, 1937, pp. 613–626) under the title *Sur les Égalités Multigrades* (Beffort, Luxembourg). In 1944, shortly after his escape from Germany where he had been held in exile for forty months, he published a revised and enlarged German edition whose title is *Mehrgradige Gleichungen* (Nordhoff, Groningen, Holland).

† The notation means that the sum of the numbers on the left equals the sum of the numbers on the right for each of the first $n-3$ integral powers of the numbers.

‡ Euclides, Madrid, July 1944, pp. 518–519.

THEOREM 2. *The multi-degree equality*

$$(3) \qquad a_1, a_2, a_3 \overset{m}{=} b_1, b_2, b_3, \qquad m = 2, 4,$$

whose terms satisfy the additional relations

$$a_1 + a_2 - a_3 = 2(b_1 + b_2 - b_3), \qquad a_3 \neq a_1 + a_2, \qquad b_3 \neq b_1 + b_2,$$

implies the following multi-degree equality with five terms per member true for the first four odd degrees

$$5a_1 - 3a_2 + 3a_3,\ 7a_1 - a_2 + a_3,\ -3a_1 + 5a_2 + 3a_3,\ -a_1 + 7a_2 + a_3,$$

$$2(b_1 + b_2 + 3b_3) \overset{m}{=} a_1 + a_2 + 7a_3,\ 3a_1 + 3a_2 + 5a_3,\ 2(3b_1 - b_2 + b_3),$$

$$2(-b_1 + 3b_2 + b_3),\ 3(a_1 + a_2 - a_3) \qquad m = 1, 3, 5, 7.$$

To solve (3), I have now established the following formulas

$$[1, -4, -2],\ [2, 0, -3],\ [3, -1, 1] \overset{m}{=} [1, 1, 3],\ [2, -4, -1],\ [3, 0, -2],\ m = 2, 4,$$

where $[r, s, t]$ represents the quadratic form $ru^2 + suv + tv^2$.

A numerical example. For $u = 4$ and $v = 1$, we have

$$29, 45, (-2) \overset{2,4}{=} 15, 46, 23 \quad \text{with} \quad 29 + 45 - (-2) = 2(15 + 46 - 23).$$

Applying Theorem 2, we obtain (after dividing both members by 4)

$$1, 33, 39, 65, 71 \overset{1,3,5,7}{=} 11, 15, 53, 57, 73.$$

CONSTRUCTION OF THE IN-FEUERBACH POINT

S. M. KARMELKAR, Akola, Berar, India

CONSTRUCTION. *Let ABC be a triangle. Draw AP, the altitude to BC. Let D be the middle point of BC. Let I be the center of the inscribed circle. Draw DX perpendicular to BC. Join PI and produce it to meet DX in X. Join DI and produce it to meet AP in Y; join XY. Draw IL perpendicular to BC; draw LZ perpendicular to XY from L. Then Z is the in-Feuerbach point.*

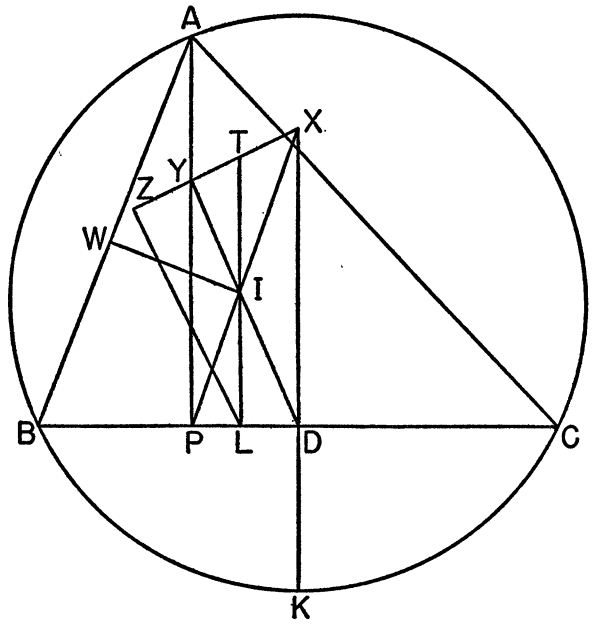
Proof. Use will be made of the following theorem: The angle formed by the altitude and the circumdiameter issued from the same vertex of a triangle is bisected by the bisector of the angle at the vertex considered. The angle equals half the difference of the base angles.*

Make the above construction and produce LI to meet XY in T. (See figure.) It is easy to prove the following properties of the construction: $PL/LD = YP/XD$; $IT = IL$; $\angle PYL = \angle LXD$; ZL bisects $\angle PZD$; the circle with TL as diameter passes through Z; ZI makes with ZL the same angle as ZL makes with the perpendicular from Z to PD, parallel to IL. The last statement and the above

* See Altschiller-Court, College Geometry, pp. 53, 59.

theorem show that ZI passes through the center of the circle PZD. The circle with center I and radius IL touches the circle PZD as they have the same tangent at Z, their common point.

It only remains to prove that the circle PZD is the nine-point circle. Draw the circle ABC. Draw DK, the perpendicular bisector of BC meeting the arc opposite to A in K. Join AK; I lies on AK. Draw IW perpendicular to AB.



Triangles AYI and IDK are similar. Hence $AY/DK=AI/IK$. Triangles AWI and CDK are similar and $CK=IK$. Hence $AI/CK=WI/DK=AI/IK$, and $AY=WI$, the radius of the inscribed circle. Hence AY is equal and parallel to IL; so $AYLI$ is a parallelogram. Hence $\angle PYL=\angle PAI$, and the latter equals $(\angle B-\angle C)/2$. Now PD subtends at Z an angle equal to double the angle which PL subtends at Z. Hence $\angle PZD=\angle B-\angle C$, and the circle PZD is the nine-point circle.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

The Philosophy of Bertrand Russell (The Library of Living Philosophers, vol. V). Edited by P. A. Schilpp. Evanston and Chicago, Northwestern University, 1944. 16+815 pages. $4.00.

This is the fifth volume of *The Library of Living Philosophers*, the previous volumes dealing with John Dewey, George Santayana, Alfred North Whitehead, and G. E. Moore. The basic thought behind the enterprise is to help remove one of the greatest obstacles to fruitful discussion in philosophy, namely "the curious etiquette which apparently taboos the asking of questions about a philosopher's meaning while he is alive" (F. S. C. Schiller). There is no denying the fact that different contemporary interpreters find different ideas in the writings of the same philosopher. Could not at least some clarification be brought about by putting these varying interpretations and critiques before the philosopher while he is still in our midst, and let him act both as defendant and judge? That is the way in which the articles in this volume approach Bertrand Russell's philosophy. It is opened by one of the most charming essays Mr. Russell ever wrote, a brief autobiography entitled "My Mental Development." There follow 21 descriptive and critical articles on R's philosophy, covering the same extremely wide range as the protagonist's work: mathematics and logic, theory of knowledge, the philosophical aspects of science, of language and history; social, political and educational philosophy; psychology, metaphysics, ethics and religion. In 61 pages Russell boldly faces and responds to this not always too consonant chorus of philosophical voices, answering their questions, meeting a fair part of their many criticisms, and rectifying misinterpretations. A carefully compiled bibliography of his publications (45 pages) greatly increases the usefulness of the book for those who are anxious to study Russell's philosophy at the source. Also the detailed index deserves a word of praise.

The articles—how could it be else?—vary greatly in tone and value. Some, as, *e.g.*, Reichenbach's on Logic, are content to outline R's essential ideas in the field under discussion and ask a few relevant questions. Others attempt a detailed critical evaluation. Such, as one would expect, is Gödel's article on Mathematical Logic. An example of the best one can hope to accomplish in a philosophical dispute is the exchange of views on Philosophy of Science between E. Nagel and R. Here the disputants, critical but fair-minded, meet as it were on the same level. Einstein's criticisms are less detailed but sustained by his profound experience in the actual creation and operation of physical theories,—an experience which Russell lacks. Einstein is a great admirer of Russell's work, to the reading

of which, as he confesses, he owes innumerable happy hours. Let us pass in silence other articles where the critics show themselves clearly incapable of doing justice to R's thoughts or are more anxious to ride their own hobby-horse into the arena than to face the issues raised by R's philosophy. In spite of such shortcomings, inevitable in a community where freedom of thought is treasured, the book proves that philosophical discussion can benefit greatly by concentration on the work of one philosopher, especially if that philosopher can still be asked for his own interpretation and is as versatile and frank as Russell in his answers. The book, however, does not provide, nor does it intend to provide, a substitute for first-hand contact with the original thought of the philosopher himself.

Great is the temptation for the reviewer to quote profusely from Russell's introduction which, in sketching his own life and intellectual adventures, conjures up a whole period. Speaking of his friends at Cambridge in the nineties and their activities he exclaims: "For those who have been young since 1914 it must be difficult to imagine the happiness of those days." Here is a little illuminating story: "I remember the precise moment, one day in 1894," so R tells us, "as I was walking along Trinity Lane, when I saw in a flash (or thought I saw) that the ontological argument is valid. I had gone out to buy a tin of tobacco; on my way back, I suddenly threw it up in the air and exclaimed, as I caught it: Great Scott, the ontological argument is sound." Can anyone, after this anecdote, doubt that Russell is constitutionally a philosopher and not a mathematician?

In his "Reply to Criticisms" Russell draws a sharp line between that part of philosophy which is concerned with facts, and that which depends on ethical considerations or value judgments. In his opinion ethical propositions should be expressed in the optative mood, not in the indicative. They express desires rather than facts, although he agrees with Kant that beyond expressing a desire "ethical judgments must have an element of universality. I should interpret '*A* is good' as 'Would that all men desired *A*'." For this reason, he continues, "I do not offer the same kind of defence for what I have said about values as I do for what I have said on logical or scientific questions." In his replies touching on social, ethical and educational matters Russell occasionally uses the weapon of irony, though more sparingly than one would perhaps expect. Even when he begins with a sentence like this: "Mr. Brightman's essay on my philosophy of religion is a model of truly Christian forbearance," he continues with a dispassionate and frank restatement of his religious attitude. In only one instance, Bode's paper on Educational Philosophy, he shoots back with sharp arrows.

From among the articles a few more which concern the theory of scientific knowledge ought to be cited, at least by title: G. E. Moore, to whom Russell in his youth owed his emancipation from Hegelianism, writes about R's Theory of Description, J. Feibleman on the Introduction to the Second Edition of the Principles of Mathematics, P. P. Wiener on the Method of R's (early) work on Leibnitz, A. P. Ushenko on his Critique of Empiricism, and R. M. Chisholm on the Foundations of Empirical Knowledge. In the face of such an *embarras de*

richesse the reviewer plans to focus his attention on the two articles of the most undeniable interest to mathematicians, Reichenbach's and Gödel's essays on Logic, and on E. Nagel's discussion of R's Philosophy of Science. If *Principia Mathematica* can be called a critique of naïve set-theory, then Gödel's article is a critique of this critique, and what I, as the reviewer, am expected to write now, is a critique of a critique of a critique. This makes my task so difficult that I see no other way out than to base my review on a brief independent survey of the situation; it adds one more feature to the picture: the historical perspective.*

In the essay on Logic Reichenbach gives Russell his due credit for improving the formal apparatus of symbolic logic, in particular for the consistent use of propositional functions with their variables, and for his interpretation of the "If then" as material implication $\rightarrow$. Indeed, it is used in this sense throughout mathematics, and only material implication is a primitive logical operator like $\sim$, $\cap$, $\cup$. However, Reichenbach's account of the ramified theory of types misses the essential point. In all questions concerning the logical foundations of mathematics he has the tendency to oversimplify the issues. His remarks on pp. 43–44 turn Brouwer's wine into water. Neither Reichenbach nor Nagel challenges the Frege-Russell thesis that logic and pure mathematics are essentially identical and that the theory of natural numbers with its definitions and inferences by complete induction, is reducible to logic. The Survey shows why I do not believe in this thesis at all; I find, on the contrary, that since Burali-Forti's and Cantor's discovery of the antinomies, the whole drift of research has been away from it.

In his reply Russell argues for the law of excluded middle, and against the definition of truth in terms of verifiability. But he argues on a level that has little to do with Brouwer's deep insight into the fallacy of that law with regard to existential propositions. Russell wishes to keep room for such unverifiable truths as he feels are "necessary for the interpretation of beliefs which none of us, if we are sincere, are prepared to abandon." The following dictum of his should be taken to heart: "As logic improves, less and less can be proved." Reichenbach ventures beyond deductive logic by opening a discussion with Russell on the principles of inductive logic on which the inductive methods of natural science depend; but they find little common ground.

Kurt Gödel's paper on R's Mathematical Logic is the work of a pointillist: a delicate pattern of partly disconnected, partly interrelated, critical remarks and suggestions. It must have been very difficult to arrange them in the one-dimensional sequence to which the writer, unlike the painter, is forced by his medium of words. With caution and circumspection the argument touches on a great variety of possibilities (and pitfalls) of interpretation. Russell varies the formulation of his vicious circle principle as follows: "No totality can contain members *definable* only in terms of this totality, or members *involving* or *presupposing* this totality." He himself obviously thinks that it makes little differ-

* Mathematics and Logic, this MONTHLY, January 1946 [quoted as *Survey*].

ence whether one says "definable" or "involving" or "presupposing." But Gödel sees here three different principles, and the acceptance or rejection of one or the other depends above all on whether the members are *given* or *constructed.* I think, with the latter contrast he touches on a very significant point. As far as the Survey preceding this review elaborates the same point, it is nothing but a comment on Gödel's paper. The constructive definition of the lowest level in the hierarchy of levels, first given in my book *Das Kontinuum* some thirty years ago, is in full accord with this view. About Russell's axiom of reducibility Gödel says that "it is demonstrably false unless one assumes either the existence of classes or of infinitely many 'qualitates occultae'."

In addition to Russell's own work concerning the solution of the paradoxes, two other possible directions are mentioned in which, as Russell himself had pointed out, a "criterion for the existence of classes" might be sought, namely the theory of limitation of size exemplified by the Zermelo-Fraenkel axioms, and a zigzag theory toward which Gödel sees Quine's recent system moving.* After disposing of Ramsay's escape by means of propositions of infinite length, he envisages as a further possibility, though somewhat vaguely, a theory in which "every concept is significant everywhere except for certain 'singular points' . . . so that the paradoxes would appear as something analogous to dividing by zero. . . . Our logical intuitions would then remain correct up to certain minor corrections, *i.e.*, they could then be considered to give an essentially correct, only somewhat 'blurred' picture of the real state of affairs".

In order to appreciate his last sentence one must know that Gödel takes the paradoxes very seriously; they reveal to him "the amazing fact that our logical intuitions are self-contradictory." This attitude toward the paradoxes is of course at complete variance with the view of Brouwer who blames the paradoxes not on some transcendental logical intuition which deceives us, but on a gross error inadvertently committed in the passage from finite to infinite sets. I confess that in this respect I remain steadfastly on the side of Brouwer.

On the ground of all his experience Gödel makes a strong plea for the realistic standpoint where classes are conceived as real objects, namely as "pluralities of things," or as structures consisting of such pluralities, and he adds: "It seems to me that the assumption of such objects [classes or concepts] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions and in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about the 'data.' Logic and mathematics (just as physics) are built upon axioms with a real content which cannot be 'explained away'." In his opinion the two major attempts in this realistic direction are "the simple theory of types (in an appropriate interpretation) and axiomatic set theory." He adds the warning: "Many symptoms show only too clearly however that the primitive concepts need further clarification."

* This MONTHLY, vol. 44, 1937, pp. 70–80.

Gödel hardly mentions Brouwer and Hilbert, and the last remarks sound as if he were prepared to retire from their advanced positions to some axiomatic setup of the Zermelo type. Does this mean that he has given up all hope of settling things along the lines of Hilbert's approach, and is content to adjourn the problem of consistency *ad calendas Graecas*? It is impossible to discuss realism in logic without drawing in the empirical sciences. In defending his program against Brouwer, Hilbert pointed emphatically to the situation in theoretical physics. The individual physical statements and laws have no "meaning" verifiable in immediate observation; only the system as a whole can be confronted with experience. Here consistency is absorbed into the farther-reaching requirement of "*concordance.*" Every indirect determination of the value of a physical quantity in a concrete case (*e.g.*, the charge of an electron) is based on theories establishing functional relations between that quantity and others amenable to direct observation (as, *e.g.*, position of a pointer on a scale). Concordance means that, with due regard to the inaccuracy of measurements, all such determinations must lead to the same value. We never can be sure whether such concordance, however complete for the moment, will still survive when our observations expand and become more accurate. Why then should we wish that consistency, the mathematical part of concordance, be assured *a priori* for all future? Is it not enough that the system has met the test of all our elaborate mathematical experiments so far? Will it not be early enough to change the foundations when, at a later stage, discrepancies appear? That is a position against which I cannot find much to say. But what shall be the guiding principles of our theoretical constructions? Gödel, with his basic trust in transcendental logic, likes to think that our logical optics is only slightly out of focus and hopes that after some minor correction of it we shall see *sharp*, and then everybody will agree that we see *right*. But he who does not share this trust will be disturbed by the high degree of arbitrariness involved in a system like Z, or even in Hilbert's system. What is the supporting faith? Success alone cannot be the answer. How much more convincing and closer to facts are the heuristic arguments and the subsequent systematic constructions in Einstein's general relativity theory, or the Heisenberg-Schrödinger quantum mechanics! A truly realistic mathematics should be conceived, in line with physics, as a branch of the theoretical construction of the one real world, and should adopt the same sober and cautious attitude toward hypothetic extensions of its foundations as physics exhibits. Theoretical physics today is in a healthier state than mathematics. But neither Gödel nor I can offer concrete suggestions for a cure, although Gödel seems to hope for clues from a careful study of Leibnitz's notes on his project of a *characteristica universalis*.

By talking about physics we have already passed into the domain of Nagel's controversy with Russell on Philosophy of Science. Unfamiliar with R's pertinent systematic publications, I must limit my discussion to this controversy as it reflects itself in my own mind. The following is typical for the pointed arguments by which Russell sometimes likes to reduce an intricate epistemological

problem to a simple formula: "Common sense says: 'I see a brown table.' It will agree to both the statements: 'I see a table' and 'I see something brown.' Since, according to physics, tables have no color, we must either (a) deny physics, or (b) deny that I see a table, or (c) deny that I see something brown. It is a painful choice; I have chosen (b), but (a) or (c) would lead to at least equal paradoxes." In looking upon things as "bundles of qualities" and letting physics tell us that tables have no color, Russell remains pretty close to the favored epistemological slogan of the 19th century that only the raw material of sensations is immediately given to us, and to the doctrine, as old as Democritus and Huyghens, that nothing but "the atoms and their motions exist in reality" or that light and color are "in reality" oscillations of the ether. But in the development of physics the physical concepts have revealed themselves more and more as free constructions, mere *symbols* handled according to certain rules; theoretical physics becomes a system as thoroughly formalized as Hilbert's mathematics. Concerning the other side, the "given," many philosophers are now willing to agree that the sensory data, far from being given in their naked purity, are theoretical abstractions, while the true raw material is the manifest disclosed world in which man finds himself in his everyday life. It is quite characteristic that Hilbert bases his mathematics on the practical manipulation of concrete symbols rather than on some "pure consciousness" and its data. Thus quantum mechanics sets the physical phenomena symbolized by the Schrödinger-Dirac ψ's against the actual measurements for whose description our common sense "workshop" language is sufficient and adequate.

"Occam's razor," the search for a "minimum vocabulary," are some of the favored terms in which Russell describes the gist of his method. He pronounces as his supreme maxim of scientific philosophizing: "When possible substitute constructions out of known entities for inferences to unknown entities." Russell's goal is "to find an interpretation of physics which gives a due place to perceptions." Nagel criticizes in detail the steps by which he arrives (allegedly) at the twofold conclusion that (1) all the objects of common sense and developed science are logical constructions out of *events*, and that (2) our perceptions are events, but there exist many events which are not perceptions. I am inclined to side with Nagel, in particular when he says (p. 345, footnote) with regard to R's definition of point-instances in terms of events (dubious even from a purely mathematical standpoint): "One need only compare [this definition] with such analyses as those of Mach concerning mass and temperature, or those of N. R. Campbell concerning physical measurements to appreciate the difference between an analysis which is quasi-mathematics and an analysis which is directed toward actual usage." When Russell speaks of perceptions as events he does not seem to use the term in its "immanent" sense for which it is true that an actual perception is always *my* perception.

The physicist of today sees no need to use as material of his constructions some such ultimate entities as events; the pure symbols are enough for him. But he can handle these symbols and carry out his measurements only as a man

living in and accepting our common sense world. Of course the connection between theory and observation must be understood, and it is the main task of an epistemological analysis to make this understanding explicit. Nagel comes to a similar conclusion: "It seems to me, therefore, that, instead of making the elimination of symbols for constructs the goal of the logical analysis of physics, a more reasonable and fruitful objective would be the following: to render explicit the pattern of interconnections between constructs and observations, on the strength of which these latter can function as relevant evidence for theories about the former." With this sober statement I think both Gödel and Einstein would agree.

But the debate on questions of this sort will probably go on forever. The volume under review is likely to stir up a host of thoughts in the mind of every reader. Even a mathematician must at times grapple with the problem to understand "what it is all about."

HERMANN WEYL

Analytic Geometry. By R. R. Middlemiss. New York, The McGraw-Hill Book Company, 1945. 7+306 pages. $2.75.

The author has presented a well organized text in analytic geometry which is directed toward the needs and interests of not only the future major in mathematics, but also the engineer, chemist, or physicist. Excellent taste has been shown in the selection of the topics—especially those which forsehadow the requirements of calculus. For example, as the author points out in the preface, the amount of emphasis upon formal study of conic sections has been minimized to those essentials which may be useful to the technical student. Moreover, greater emphasis has resulted in the treatment of algebraic functions including a well chosen chapter upon polynomial curves, a chapter upon rational fractional functions, and a chapter upon other algebraic curves. Since the student of the calculus encounters the circle, ellipse, or hyperbola less frequently than he does other algebraic functions, this shift of emphasis is fortunate. Also greater attention is given to the treatment of transcendental functions and curves (including a section on hyperbolic functions) which are more common than the conic sections in calculus. As a result, the student is able to grasp better the actual properties of mathematical functions, the concepts of which are introduced in the first chapter and fortunately repeated throughout the text.

Dr. Middlemiss has included many other noteworthy principles which too frequently have been neglected in a student's preparation for the calculus. He has wisely emphasized addition and multiplication of ordinates as a means of ascertaining through a quick sketch the general graphical representation of a mathematics function, another valuable forerunner to the student's needs of calculus. Another feature of interest to the future calculus student is the introduction of the limit concept in a section concerned with horizontal asymptotes and the presentation of a few elementary problems in mechanics including the concept of moment of force.

Aside from the topics mentioned, polar coordinates and parametric equations are included in separate chapters. Of interest to any person in experimental work is an adequate chapter about curve fitting. Such a chapter will be valuable to the research worker in biological sciences, psychology, education, and statistics as well as to the student in the physical sciences. The last section of the book is devoted to solid analytic geometry. In the three chapters upon solid analytics, the author appears to reach his greatest clarity in exposition. The excellent figures which represent three dimensional properties are extremely helpful to the student who may have trouble in visualizing the geometry of space. In fact, all the figures throughout the text are noteworthy for their aid in explanation and for their ease of interpretation. At the end of the text are tables of logarithms (four place) of trigonometric functions (both natural and logarithmic values), of powers and roots, and of natural logarithms. The author should be complimented for giving answers to nearly all problems, both "even" and "odd" numbered examples.

Other meritorious features of the text are numerous. An introductory review chapter of the essential concepts of algebra and trigonometry required in analytics is helpful. The explanation of directed line segments, projection, and point of division is particularly good. Other topics which are especially well presented to the student include the normal form, systems of lines, and algebraic curves.

The only criticism that this review can offer is that in certain sections of the text a greater number of illustrative examples would be helpful. The explanation does seem somewhat brief with respect to translation and rotation—especially in the way of illustrative problems. Concerning the ellipse no illustrative example is presented other than that for finding certain properties of an ellipse given its equations. A few illustrative problems for the parabola, ellipse, and hyperbola in which properties of the curve are given in order to find the equation seem to be very much needed. Although the author is wise not to overemphasize conics, a greater number of illustrative problems at this place would make the student's and teacher's job somewhat easier. Moreover, no illustrative problems accompany the explanation of the hyperbola. Also a problem is needed to show the application of the rules presented for the identification of a conic.

Aside from the lack of a few illustrative problems at certain places, this text represents a distinct contribution to the needs of students and teachers of analytic geometry. The organization of the text, the style of writing, the numerous well chosen exercises, the pleasing format—all these are outstanding features. Most important of all, Dr. Middlemiss appears to have written the text for the student, whose viewpoint too frequently has been overlooked in the past. In some instances, writers—perhaps almost afraid that students might understand mathematics—have too often confused and discouraged those students who might have been not only reasonably good technicians, but also potential mathematicians of considerable merit. In short, this reviewer highly recommends the text to all teachers and students of analytic geometry.

W. B. MICHAEL

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

NOTE ON LOGARITHMS AND COMPOUND INTEREST

J. S. FRAME, Michigan State College

Can the construction of a table of logarithms be explained to high school students or college freshmen without the use of integral calculus? Yes, if we admit the use of the binomial series for non-integral exponents, namely,

$$(1) \qquad (1+x)^r = 1 + rx + r(r-1)x^2/2! + r(r-1)(r-2)x^3/3! + \cdots,$$

which is valid for $|x| < 1$.

Let $y = nr$ be the simple interest on one dollar for n periods at the rate r per period, and let $A = (1+r)^n$ be the compound amount for the same period of time. Then $x = A - 1$ is the compound interest. For each choice of r we define a base number $b = (1+r)^{1/r}$, and find

$$(2) \qquad 1 + x = A = b^y, \qquad y = \log_b (1+x).$$

The quantity A^r may not only be expanded as in equation (1), but also, if $|r| < 1$, in the form

$$(3) \qquad (1+r)^y = 1 + yr + y(y-1)r^2/2! + y(y-1)(y-2)r^3/3! + \cdots.$$

Equating the expressions in (3) and (1), subtracting 1 from each side of the equality, and dividing by r, gives the following equation:

$$(4) \qquad y + y(y-1)r/2! + y(y-1)(y-2)r^2/3! + \cdots = x + (r-1)x^2/2! + (r-1)(r-2)x^3/3! + \cdots.$$

The essential idea of a limit enters at this point, when we let r approach zero on both sides of (4). We assume that both sides are still equal when $r = 0$. The left-hand side reduces to y, and the right-hand side to $x - x^2/2 + x^3/3 - \cdots$. The base $b = (1+r)^{1/r}$ approaches a limiting value $e = 2.718281828 \cdots$, which is called the natural base, and we have

$$(5) \qquad y = \log_e (1+x) = x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \cdots.$$

By studying simple and compound discount instead of simple and compound interest, formula (5) can be shown to hold for negative as well as positive values of x, provided that $|x| < 1$. Substituting in (5) the values $x = 1/5$, $-1/5$, $-1/10$, $1/125$, *etc.*, in turn, we can compute the numbers

$$(6) \qquad A = \log_e 6/5, \quad B = -\log_e 4/5, \quad C = -\log_e 9/10, \quad D = \log_e 126/125.$$

Then we use the laws of logarithms to obtain the following natural logarithms:

$$\log_e 2 = 2A + B + C, \quad \log_e 3 = 3A + 2B + C, \quad \log_e 5 = 4A + 3B + 2C, \tag{7}$$
$$\log_e 7 = 4A + 4B + 3C + D, \quad \log_e 10 = 6A + 4B + 3C.$$

Other natural logarithms of integers can be found in a similar manner, but care should be taken to compute the auxiliary logarithms A, B, C, D, *etc.*, to one more decimal place then is required in the final answer, to allow for the multiplication of errors by the coefficients in (7).

Interpolation from a known $\log_e N$ to an unknown $\log_e(N+f)$, where $N>10$ and $|f|<1$, is given accurately to six decimals by the formula

$$\log_e (N + f) = \log_e N + \frac{2f}{2N + f} + \frac{1}{12}\left(\frac{2f}{2N + f}\right)^3, \tag{8}$$

as can be shown by setting $f=xN$ and using the series (5).

To convert from natural logarithms to logarithms to the base 10, all that is required is to divide each natural logarithm, as computed above, by the value of $\log_e 10$ found from (7).

CLUB REPORTS 1944–45

Regis College Mathematics Club, Weston, Massachusetts

A formal candlelight installation ceremony for the new officers was held in September at our log cabin, Carondolet. New members received at this time the key charm, symbol of membership in the club. Evening socials were held on festive occasions throughout the year. Topics discussed at the monthly meetings included

Mathematics as a future
Air navigation
Soap bubbles
Methods of apportionment in Congress.

The mechanism of the seismograph and the tabulation of the data from this instrument were explained by Reverend Daniel Linehan, S.J., to the members of *RCMC* who visited the Weston College Seismic Station in January.

Plans were carried out at the monthly meetings for the construction of mathematical models and for a multiplex of pictures of the *RCMC* alumnae at their present occupations. The models, the multiplex, and the honor problems of the department were displayed at the close of the second semester in an exhibit sponsored by the Club. Among the models were (1) the regular solids, (2) stringed models of Desargues' configuration in a plane and in space, of a cylinder and a tangent plane convertible into a hyperboloid of one sheet with its tangent plane, of two intersecting oblique cones having their bases in the same plane and having equal altitudes, (3) knots, Peaucellier cells, and many constructions made with yarn and poster paper. Miss Marion Stark aided us in this project with many valuable suggestions.

The quarterly publication of the *RCMC*, *The Mathematical Angle*, had an increased circulation during this year. The fourth number became a summer *round robin*, in which each member wrote of her summer activities.

During the year, a representative number of members from the *RCMC* attended the biweekly meetings of the Harvard Mathematical Club.

Officers for the year were: President, Marguerite Carell; Vice-President, Mary O'Sullivan; Secretary, Annemarie Bailey; Treasurer, Jacqueline Penez. The Faculty Adviser was Sister Leonardo.

Kappa Mu Epsilon, Chicago Teachers College

The activities of the Illinois Gamma Chapter were varied, consisting of talks by faculty and students, a picnic, a party, and the installation of the Illinois Delta Chapter at St. Francis College, Joliet, Illinois. Our own chapter initiated 23 members. Among the topics discussed were:

Time—the fourth dimension, by Dr. J. T. Johnson

The oriental game of Nim, by Bernard Malina

The law of probability and combinations applied to biology, by Dr. E. C. Colin of the Biology Department

Mathematical recreations, by Professor Joseph Urbancek

Officers for the year 1944–45 were: President Archimedes, Ruth Mary Thometz; Vice-President Euler, Elaine Drews; Secretary Galileo, Veronica Stuart; Treasurer Kepler, Gertrude Koehl. Officers for 1945–46 are: President Archimedes, Patricia Powers; Vice-President Euler, Jeanne Anderson; Secretary Galileo, Alice Martinson; Treasurer Kepler, Dorothy Tisevich. During both years Professor Joseph Urbancek acted as Corresponding Secretary Descartes and Faculty Sponsor.

Mathematics Club, Immaculate Heart College, Hollywood

Monthly meetings were held from October, 1944 to May, 1945, at which papers were given by a faculty member, a guest speaker, and three student speakers. Their titles were as follows:

Differential equations, by Peggy Haws

The solar system, by Lucille Kos

Mathematics in music, by Mr. Frank Darvas of the Department of Music

Non-euclidean geometry, by Sister Margaret Ann

The future of geometry—wishful thinking, by Dr. P. H. Daus, Head of the Mathematics Department at the University of California at Los Angeles.

The officers for the year 1944–45 were: President, Zilda Cross; Vice-President, Kitty Campion; Secretary-Treasurer, Sister Margaret Ann; Program Chairman, Peggy Haws; Faculty Adviser, Peggy Masterson. Officers for 1945–46 are: President, Kitty Campion; Vice-President, Sarah Doherty; Secretary-Treasurer, Patricia Goodbody; Faculty Adviser, Dr. Myrtle Collier.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 716. *Proposed by H. E. G. P.*

On April 1, 1946, the Erewhon Daily Howler carried the following item: "The famous astrologer and numerologist of Guayazuela, the Professor Euclide Paracelso Bombasto Umbugio, predicts the end of the world for the year 2141. His prediction is based on profound mathematical and historical investigations. Professor Umbugio computed the value of the formula

$$1492^n - 1770^n - 1863^n + 2141^n$$

for $n=0, 1, 2, 3$, and so on, up to 1945 and found that all the numbers which he so obtained in many months of laborious computation are divisible by 1946. Now, the numbers 1492, 1770, and 1863 represent memorable dates: the Discovery of the New World, the Boston Massacre, and the Gettysburg Address. What important date may 2141 be? That of the end of the world, obviously."

Deflate the Professor! That is, show with little computation that the formula proposed is divisible by 1946 for $n=0, 1, 2, 3, \cdots$.

E 717. *Proposed by Orrin Frink, Jr., State College, Pa.*

In the game called "matching chips" each of the two players selects, independently of the other, either a red or a white chip from his store of chips. If both chips are found to be of the same color, then player A keeps both chips; otherwise player B keeps both chips. Red chips are worth r cents each, and white chips are worth w cents each. Develop the theory of this game.

E 718. *Proposed by Roy Dubisch, Montana State University*

Let A be a positive integer of r digits given by $A=\sum_{i=1}^{r} x_i 10^{i-1}$ and define $D(A)=(\sum_{i=1}^{r} x_i)^2$. By $D^n(A)$, where n is a positive integer, we mean the result of applying the operator, D, n successive times to A. Prove that for every A there exists an n such that $D^n(A)=1$, 81, or 169. (*Cf.* Porges, *A Set of Eight Numbers*, this MONTHLY [1945, 379–384].)

E 719. *Proposed by George Grossman, New York City*

A man addresses n envelopes and writes n checks in payment of n bills.

(a) If the n bills are placed at random in the n envelopes, what is the probability that each bill is placed in a wrong envelope?

(b) If the n bills and the n checks are placed at random in the n envelopes, one of each in each envelope, what is the probability that in no instance are the enclosures completely correct?

(c) In (b), what is the probability that each bill and each check will be in a wrong envelope?

E 720. *Proposed by Victor Thébault, Tennie, Sarthe, France*

At the vertices of an equilateral triangle three equal circles are drawn externally tangent to the circumcircle. Show that one of the three tangents to these equal circles, from any point whatever on the circumcircle, is equal to the sum of the other two.

SOLUTIONS

A Maximal Determinant

E 680 [1945, 341]. *Proposed by Gordon Pall, McGill University*

Prove that a real determinant of order 6, with elements numerically not exceeding unity, cannot have a value greater than 160.

I. *Solution by the Proposer.* It is sufficient to prove the conclusion for determinants of order 6 having all elements equal to ± 1. For, denoting the elements of the determinant by a_{ij}, if $-1 < a_{ij} < 1$, and the cofactor A_{ij} is positive, replace a_{ij} by $+1$; but if A_{ij} is negative, replace a_{ij} by -1; and if $A_{ij} = 0$ use either value. In any case, the determinant is either increased or unaltered by such changes, and we can keep on in this way until every element is replaced by ± 1.

Now, any determinant of order 6 with all elements ± 1 is divisible by 2^5. For, on reducing in the elementary way to order 5, every element becomes $\pm 1 \cdot 1 \pm 1 \cdot 1$, which is even, and 2 factors out of each row.

Consider the Laplace expansion by means of the 20 minor determinants of order 3 in the first three rows. Each such minor determinant either has two columns proportional (and so is zero), or can be gotten by permuting rows or columns or by changing their signs from

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = -4, \qquad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix} = -4.$$

Let a, b, c, d denote distinct non-proportional columns of three elements ± 1. Then, for example, if the first three rows of the determinant have the form $aaaabc$, only four determinants $|abc|$ do not vanish, and the Laplace expansion

will not exceed $4\cdot 4\cdot 4=64$. Similarly, in the case of *aaabbc*, there are six non-zero minor determinants, and the expansion cannot exceed 96. The case *aaabcd* permits at most $(1+3\cdot 3)16=160$. The case *aabbcc* gives 128.

There remains only the possibility *aabbcd*. Apparently we might then have $(4+2\cdot 4)16=192$. This implies, however, that the last three rows are in some order *xxyyzw*, and that the complementary minor of each non-zero determinant such as *abc* is a non-zero determinant such as *xzw*. Unless *xx* is under *cd*, a non-zero determinant *abc* or *acd* or *bcd* will be multiplied by one of the zero determinants having two columns *xx*. And if *xx* is under *cd*, then *yy* is not under *cd*, and a similar remark applies. Hence an expansion equal to 192 does not exist.

An example with determinant equal to 160 is

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{vmatrix}.$$

II. *Note by John Williamson, Queens College.* It was proved by Hadamard* that, if $R=(r_{ij})$ is a square matrix of order n with real elements r_{ij}, where $-1\leqq r_{ij}\leqq 1$, then the absolute value of $|R|\leqq n^{n/2}$. Further, if $|R|$ has its maximum value $n^{n/2}$, each element $r_{ij}=\pm 1$ and matrix R satisfies the orthogonality conditions $\sum_{j=1}^{n} r_{ij}r_{kj}=0$, $i\neq k$. It easily follows† that, if $n>2$, then $n\equiv 0 \bmod 4$. The question whether or not an Hadamard matrix, *i.e.*, a matrix R with the maximum value $n^{n/2}$ for its determinant, exists for every $n\equiv 0 \bmod 4$ has not yet been answered. In 1933 Paley† discussed the known results in this connection and extended them considerably, proving, in particular, that an Hadamard matrix exists for all values of $n=4m\leqq 200$ with the possible exception of the six numbers 92, 116, 156, 172, 184, 188. A list of the known values of n for which an Hadamard matrix exists is given in a paper by Williamson.‡

As remarked in this MONTHLY,§ not much is known about the maximum value of a determinant each element of which is ± 1 when $n\not\equiv 0 \bmod 4$, except for values of n less than eight.‖

* Jacques Hadamard, "Résolution d'une question relative aux déterminants," Bulletin des Sciences Mathématiques, (2), vol. 17, 1893, part 1, pp. 240–246.

† R. E. A. C. Paley, "On orthogonal matrices," Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 12, 1933, pp. 311–320.

‡ John Williamson, "Hadamard's determinant theorem and the sum of four squares," Duke Mathematical Journal, vol. 11, 1944, pp. 65–81.

§ *Editorial Note* to problem E 264, this MONTHLY, [1945, 342].

‖ John Williamson, "Hadamard's determinant theorem," (abstract), this MONTHLY, [1945, 417].

Let K_n be a determinant of order n each element of which is ± 1. It is easily shown that $K_n = 2^{n-1}\Delta_{n-1}$, where Δ_{n-1} is a determinant of order $n-1$ each element of which is 1 or 0. Conversely, if Δ_{n-1} is known, the corresponding K_n may be written down, and K_n has a maximum value if and only if Δ_{n-1} has. If D_{n-1} denotes any Δ_{n-1} with a maximum value, then the problem under consideration is to show that $D_5 = 5$.

Let the matrix of D_{n-1} be $A = (a_{ij})$ and let $D_{n-1} = |A|$ be positive. Then, if $a_{ij} = 1$, its cofactor A_{ij} is positive or zero, and if $a_{ij} = 0$, then A_{ij} is negative or zero, as otherwise D_{n-1} would not have a maximum value. If

$$\Delta_n = \begin{vmatrix} q & e_1 & e_2 \cdots e_{n-1} \\ d_1 & & \\ d_2 & & A \\ d_{n-1} & & \end{vmatrix},$$

then

$$\Delta_n = q|A| - d_1\sum_{j=1}^{n} e_j A_{1j} - d_2\sum_{j=1}^{n} e_j A_{2j} - \cdots - d_{n-1}\sum_{j=1}^{n} e_j A_{n-1,j}$$

$$= q\Delta_{n-1} - \sum_{i,j=1}^{n} d_1 e_j A_{ij}.$$

Since each d_i or e_j has the value 1 or 0,

$$\Delta_n = q\Delta_{n-1} - \Sigma' A_{ij}, \tag{1}$$

where $\Sigma' A_{ij}$ is the summation of the cofactors A_{ij} of all the elements of a submatrix of A. The submatrix is formed from the rows for which the corresponding d_i have the value 1 and from the columns for which the corresponding e_j have the value 1. If $q = 1$, Δ_n will have its largest absolute value when $\Sigma' A_{ij}$ has the largest negative value. If $q = 0$, Δ_n will have its largest absolute value when $\Sigma' A_{ij}$ has the largest positive value. For small values of n it is easy to determine at a glance what submatrices of A to choose to make the absolute value of Δ_n a maximum. Since it can be shown that for $n \leqq 6$ any D_n contains a first minor which is a D_{n-1}, starting with $D_1 = 1$ and applying the above process successively we arrive at two types of $D_2 = 1$,

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix},$$

one type of $D_3 = 2$,

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix},$$

two types of $D_4 = 3$,

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix},$$

and three types of $D_5=5$,

$$\begin{vmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{vmatrix}.$$

By applying the same process to each of the three types of D_5 the largest value of Δ_6 obtained was 9. Therefore $D_6=9$.

This last result is interesting for two reasons. First, by Hadamard's theorem, $K_7=2^6m\leqq 7^{7/2}$, and m might have been as large as 14 instead of being only 9. Second, any first minor of an Hadamard determinant H_8 gives rise to a Δ_7 of value 8. Therefore, *no D_7 contains a first minor which is a D_6.*

The following unpublished results may all be obtained by suitable applications of (1). There exist a Δ_8 of value 56, a Δ_9 of value 120, and a Δ_{10} of value 264, which fall far short of the possible maximum values 76, 195, and 521 respectively given by Hadamard's theorem. However, the value 264 for Δ_{10} is larger than the value 243 of Δ_{10} obtained from any first minor of an Hadamard determinant H_{12}. Since $D_{10}\geqq 264>243$, *no D_{11} contains a first minor which is a D_{10}.*

Finally, if an Hadamard determinant H_{4m} exists, there exists a Δ_{4m} with the value $(5m-3)m^{2m-1}$. Although no larger value for Δ_{4m} was found when $m=2$, 3, 4, the existence of a Δ_{20} with the value $25(5^9)$ shows that the value $(5m-3)m^{2m-1}$ is by no means the best for larger values of m.

Also solved by Margaret Olmsted.

Three Fractions Whose Sum is an Integer

E 682 [1945, 395]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find three positive integers x, y, z, such that the sum of ratios

$$\frac{y}{z}+\frac{z}{x}+\frac{x}{y}$$

is an integer. (Hint: suppose $x+y+z=23$.)

Solution by Paul Erdös, Stanford University, and Ivan Niven, Purdue University. We prove that there are infinitely many solutions in integers x, y, z with $(x, y, z)=1$. The sum of the fractions equals

$$\frac{x^2z + y^2x + z^2y}{xyz}.$$

Any prime p which divides one of x, y, z clearly must divide another. For otherwise two and only two terms of the above numerator would be divisible by p. Suppose $p^\alpha \| x$ (this means that p^α divides x, but $p^{\alpha+1}$ does not) and $p^\beta \| y$, and $(p, z) = 1$. Then the fraction has the form

$$\frac{k_1p^{2\alpha} + k_2p^{\alpha+2\beta} + k_3p^\beta}{k_4p^{\alpha+\beta}},$$

where the k's are prime to p. If $2\alpha \neq \beta$ then the numerator is divisible by p to the power min $(2\alpha, \beta)$, which is less than $\alpha+\beta$. Hence $2\alpha = \beta$.

Thus if $(x, y) = e$, it follows that e^2 divides y. Consequently, denoting (y, z) and (z, x) by f and g respectively, we know that $x = eg^2$, $y = fe^2$, $z = gf^2$. The original fraction becomes

$$\frac{e^3 + f^3 + g^3}{efg}, \tag{1}$$

and we show that this is an integer for infinitely many integral values e, f, g with no common factor, in fact with $g = 1$.

For, if we have one solution of the congruence

$$e^3 + f^3 + 1 \equiv 0 \pmod{ef},$$

another solution is the pair of integers f, $(f^3+1)/e$, and this is a new solution if $e < f$. The solution suggested by the proposer is given by $e = 2$, $f = 3$, $g = 1$, whence $x = 2$, $y = 12$, $z = 9$. Our procedure, however, does not give all solutions of the congruence from this basic solution, *e.g.* $e = 5$, $f = 9$.

The above congruence is equivalent to the two simultaneous congruences $e^3 \equiv -1 \pmod{f}$ and $f^3 \equiv -1 \pmod{e}$. Congruences of this sort have been studied by Jacobsthal (*Composito Math.*, 6 (1939), pp. 407–427).

Also solved by Murray Barbour, W. G. Brady, H. N. Carleton, P. W. A. Raine, W. J. Robinson, Joseph Rosenbaum, E. P. Starke, and the proposer. Robinson, Rosenbaum, and Starke found condition (1) above. Robinson computed the following particular primitive solutions (x, y, z): (1, 1, 1), (1, 2, 4), (2, 12, 9), (3, 18, 4), (14, 960, 25), (5, 350, 196), (3, 126, 196), (14, 588, 9), (225, 4941, 16605), (405, 1525, 33489). Rosenbaum gave the particular solutions corresponding to $e = 5$, $f = 7$, $g = 18, 78$; $e = 1$, $f = 14$, $g = 3, 5, 45, 61, 549, 915$. Starke gave the solutions corresponding to $(e, f, g) = (1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 3)$, $(1, 2, 9)$, $(1, 5, 9)$, $(1, 5, 14)$, $(1, 9, 146)$, $(1, 9, 365)$, $(1, 14, 915)$, $(1, 365, 5403014)$, $(2, 3, 7)$, $(2, 5, 133)$, $(2, 7, 13)$, $(2, 7, 27)$, $(2, 7, 117)$, $(2, 13, 21)$, $(2, 13, 63)$, $(2, 13, 735)$, $(9, 19, 3794)$. Barbour, Brady, Carleton, and Raine found the solution (1, 2, 4). The proposer found the solution (2, 12, 9) and pointed out that the solution (1539, 1369634, 129549924) appeared in the *Gazeta Matematica*, Bucarest, 1931, p. 73.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4198. *Proposed by C. D. Olds, San Jose State College*

Prove that

$$\frac{1}{(n-1)!}\int_n^\infty w(t)e^{-t}dt < \left(\frac{2}{e}\right)^n,$$

where t is real, n is a positive integer, and $w(t)=(t-1)(t-2)\cdots(t-n+1)$.

4199. *Proposed by N. J. Fine, Indianapolis, Ind.*

Let r be any integer greater than unity, n a non-negative integer, and α a non-negative integer less than r. Let ν be the number of digits in the expression of n in the scale r which are not less than $r-\alpha$, and let σ be the sum of the digits. Show that

$$\sum_{k=1}^{\infty}\left[\frac{n+\alpha r^{k-1}}{r^k}\right] = \nu + \frac{n-\sigma}{r-1},$$

where the square brackets denote the greatest integer function.

4200. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given in a plane the triangle ABC and the fixed point P which is center of a variable circle (P). Find the locus of the radical center of the circles passing through A, B, C, respectively, which have with (P) the sides BC, CA, AB as radical axes. Consider the analogous problem for a tetrahedron and a sphere with fixed center, and show that the locus is a twisted cubic through the vertices and centroid of the tetrahedron.

4201. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A tetrahedron is given for which the difference of squares of opposite edges is the same for the three pairs. (1) The three medians are equal and the line joining the circumcenter to the centroid is perpendicular to one of the faces. (2) One of the altitudes passes through the symmetric of the orthocenter of the face corresponding with respect to the circumcenter of the face. (3) The sum, or the difference, of the cosines of two opposite dihedrals of the tangential tetrahedron is the same for the three pairs.

$$x = 2\lambda^2 A_1 A_0,$$

and this proves the first part of the theorem.

The resultant of the nine forces is equipollent to the resultant F of three forces equipollent to the altitudes applied to O.

Let t_1, t_2, t_3 be the complex coördinates of the vertices, the circumcircle being taken as base-circle, and denote $t_1+t_2+t_3$ by σ. Then σ is the coördinate of H and the points A_2, B_2, C_2, where AH, BH, CH cut the circumcircle again, are $-t_2t_3/t_1$, $\cdots$. Hence, h being the coördinate of the orthocenter of $A_2B_2C_2$,

$$F = \sum [(\sigma - t_2t_3/t_1)/2 - t_1] = [\sigma - \sum (t_2t_3)/t_1]/2 = (\sigma + h)/2,$$

which proves the second part of the theorem, as $A_2B_2C_2$ is similar to the orthic triangle, the center of similitude being H and the ratio 2.

Centers of Similitude

4147 [1945, 47]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Two circles varying in magnitude and position roll on two fixed circles. Find the loci of their centers of similitude. If the straight line of their centers has a constant direction, the midpoint of the segment of their centers describes a straight line.

Partial Solution by Otto Dunkel, Washington University. The two fixed circles in a plane (C_1), (C_2) have the indicated centers and the radii R_1, R_2, $R_1>R_2$, and we may suppose that the segment $C_1C_2=2c$ entends horizontally from left to right. The point P is in their plane so that $C_1P=r\pm R_1$ and $C_2P=r\pm R_2$. There are two cases such that in case I there is a subcase for which the upper signs are used and in the second subcase the lower signs are used. In the first subcase of I we have $C_1P-C_2P=R_1-R_2=2a$, and in the second $C_2P-C_1P=R_1-R_2$. Hence as r and P vary, P describes in the first subcase the right branch of a hyperbola γ with foci C_1 and C_2, center at the midpoint O of C_1C_2, and diameter of length $2a$. In the second subcase P describes the left branch of γ. There are boundary cases given by the two straight lines each tangent to C_1 and C_2 and meeting in the external center of symmetry of these two circles. If T_1 and T_2 are the points of tangency of one such common tangent then the straight line through O and the midpoint of T_1T_2 is an asymptote of γ. Let x, y be the rectangular coordinates of P with the origin at O and the positive x axis in the direction C_1C_2. Then in case I for the right branch of γ we have

$$(1) \qquad \begin{gathered} (r + R_1)^2 = (x + c)^2 + y^2; \qquad (r + R_2)^2 = (x - c)^2 + y^2; \\ 2r(R_1 - R_2) + (R_1^2 - R_2^2) = 4cx. \end{gathered}$$

Now let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be two points on this branch of γ and let $E_{12}(x_e, y_e)$ be the external center of symmetry of (P_1), (P_2), circles with the indicated centers and radii r_1, r_2. Then $(r_2-r_1)x_e=r_2x_1-r_1x_2$ and the equation of the radical axis of (C_1) and (C_2) is $4cx-(R_1^2-R_2^2)=0$; and the last equation in (1) gives two

equations corresponding to P_1 and P_2. The elimination of the first term from these two equations gives

$$(2) \qquad \begin{aligned} (r_2 - r_1)(R_1^2 - R_2^2) &= 4c(r_2x_1 - r_1x_2) = 4c(r_2 - r_1)x_e, \\ 4cx_e &= (R_1^2 - R_2^2), \qquad r_2 \neq r_1, \end{aligned}$$

and hence E_{12} lies on the radical axis of the two fixed circles (C_1) and (C_2). If $r_2 = r_1$ and $x_2 \neq x_1$, it is obvious that this conclusion is again true. For the left branch we have merely to change the sign of both R_1 and R_2 in the above, and hence the final result is the same.

In case II there are also two subcases; in the first $C_1P = r + R_1$, $C_2P = r - R_2$, and the second subcase is obtained by altering the sign of both R_1 and R_2. It will suffice to consider the first. The last equation in (1) gives the corresponding equation needed here by changing the sign of R_2 only, and we then have

$$(1)' \qquad 2r(R_1 + R_2) + (R_1^2 - R_2^2) = 4cx.$$

In case II we have a hyperbola γ' if $R_1 + R_2 = 2a < 2c = C_1C_2$, and the first subcase gives the right branch while the other subcase gives left. Since the next step eliminates the first term of (1)′ the equations corresponding to (2) will be the same. Thus, finally, in both cases I and II the external center of symmetry of (P_1) and (P_2) lies on the radical axis of the fixed circles (C_1) and (C_2).

For a complete discussion the internal centers of symmetry of (P_1) and (P_2) should be considered; but in this study no results of interest have been obtained.

The remaining part of the problem is obvious from elementary theorems. For, if there are several pairs of points P_1 and P_2 such that the lines P_1P_2 are parallel, then since P_1 and P_2 lie on γ, or γ', the midpoints of the chords P_1P_2 lie on a diameter of the corresponding hyperbola.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Associate Professor Garrett Birkhoff of Harvard University has been elected as the Academico Correspondiente of the Academia Nacional de Ciencias Exactas Fisicas y Naturales de Lima and Socio Honorario of the Mexican Mathematical Society.

At the University of California, Assistant Professor A. P. Morse has been promoted to an associate professorship and Associate Professor C. B. Morrey to a professorship. Professor J. H. McDonald has retired.

At the University of Illinois, Drs. M. M. Day, C. W. Mendel, J. W. Peters, L. L. Steimley, and H. E. Vaughan have been promoted to assistant professorships; Assistant Professor H. J. Miles has been promoted to an associate professorship.

Dr. P. H. Anderson has been appointed Economic Analyst with the War Assets Corporation in Washington, D. C.

Dr. R. H. Bing of the University of Texas has been promoted to an assistant professorship.

Dr. W. B. Caton of Duluth State Teachers College has been promoted to an assistant professorship.

Assistant Professor Claude Chevalley of Princeton University has been promoted to an associate professorship.

Assistant Professor Helen E. Clarkson of Hardin-Simmons University has been appointed to an adjunct professorship at the University of South Carolina.

Assistant Professor J. L. Dorroh of Louisiana State University has been promoted to an associate professorship.

Dr. Benjamin Epstein has been appointed mathematician of the staff of the Cost Research Laboratory of the Carnegie Institute of Technology.

Dr. Rufus Isaacs has been appointed to an assistant professorship at the University of Notre Dame.

Dr. W. J. Kirkham has returned to Oregon State College after service in the United States Naval Reserve.

Dr. Szolem Mandelbrojt has been appointed to a professorship at the Rice Institute.

Dr. E. W. Montroll of the Polytechnic Institute of Brooklyn has been appointed to an assistant professorship at the University of Pittsburgh.

Assistant Professor Hermance Mullemeister of the University of Washington has been promoted to an associate professorship.

Dr. E. A. Nordhaus has been reappointed to an assistant professorship at Michigan State College.

G. P. Oakland of the University of Saskatchewan has been appointed Assistant Professor of Actuarial Science at the University of Manitoba.

Dr. I. E. Perlin has been appointed to an assistant professorship at the Georgia School of Technology.

Associate Professor F. A. Rickey of Louisiana State University has been promoted to a professorship.

Associate Professor L. V. Robinson of Emory University has been appointed to an associate professorship at the University of South Carolina.

Dr. P. C. Rosenbloom of Brown University has been promoted to an assistant professorship.

Assistant Professor I. J. Schoenberg of the University of Pennsylvania has been promoted to an associate professorship.

Jack Silber has returned to civilian life and is doing graduate work at the University of Chicago.

Helen F. Smith of Iowa State College of Agriculture and Mechanic Arts has been promoted to an assistant professorship.

Dr. F. E. Ulrich of the Rice Institute has been promoted to an assistant professorship.

Professor J. H. M. Wedderburn of Princeton University has retired with the title of professor emeritus.

Assistant Professor P. A. White of Louisiana State University has been appointed to an assistant professorship at the University of Southern California.

Dr. P. M. Whitman has been appointed to an assistant professorship at Tufts College.

Assistant Professor J. W. T. Youngs of Purdue University has been appointed to an associate professorship at Indiana University.

Professor Antoni Zygmund of Mount Holyoke College has been apointed to a professorship at the University of Pennsylvania.

The following appointments to instructorships are announced:
Brown University: Dr. Herbert Federer
College of the City of New York: Dr. S. F. Barber
Columbia University: Dr. E. R. Kolchin
Iowa State College: O. C. Carpenter, R. E. Carr, A. E. Engelbrecht, C. J. Maloney, R. M. Robinson
Ohio State University: Dr. V. F. Cowling
Purdue University: Dr. B. H. Arnold, Dr. W. F. Eberlin, Dr. A. L. Whiteman
The University of Chicago: A. R. Jacoby
The University of California: Dr. Abraham Seidenberg
The University of Illinois: Dr. P. W. Carruth, Dr. M. E. Munroe
The University of Michigan: Dr. L. V. Toralballa
Yale University: Dr. R. E. Fullerton, L. R. Norwood, Dr. B. J. Pettis

Professor Harry Bateman of California Institute of Technology died January 21, 1946. He was a charter member of the Association.

A. A. Blumberg of the Agricultural and Mechanical College of Texas died October 21, 1945.

Professor Emeritus A. L. Candy of the University of Nebraska died July 18, 1945.

Professor Emeritus F. H. Clutz of Gettysburg College died December 30, 1945.

Professor Emeritus A. R. Congdon of the University of Nebraska died November 11, 1945.

Professor C. W. Emmons of Simpson College, Indianola, Iowa, died December 29, 1945. He was a charter member of the Association.

Dr. Hans Fried of Swarthmore College died December 23, 1945.

Professor C. F. Gummer of Queen's University, Kingston, Canada, died January 21, 1946. He had been a charter member of the Association.

SUMMER COURSES

The following institutions announce courses in mathematics for the summer of 1946:

Columbia University. From July 8 to August 16 the following graduate courses will be offered: By Professor Artin: theory of rings, introduction to algebra. By Professor Kasner: survey of modern mathematics, differential geometry. By Professor Koopman: theory of functions of a complex variable. By Professor Lorch: the calculus of finite differences.

Ohio State University. From June 18 to August 30 the following advanced courses will be offered: By Professor Bamforth: vector analysis. By Mr. Barlas: advanced calculus. By Professor Blumberg: projective geometry, Fourier series and spherical harmonics. By Professor Mann: theory of equations.

The State University of Iowa. From June 12 to August 8 the following graduate courses will be offered: By Professor Chittenden: theory of determinants. By Professor Knowler: mathematics of business and industry. By Professor Oberg: integral equations.

The University of Chicago. From June 25 to August 31 the following graduate courses will be offered. By Professor Albert: introduction to modern higher algebra, division algebras. By Professor Barnard: linear functional operators. By Professor Hartung: advanced problems in teaching mathematics in secondary school and junior college. By Dr. Kaplansky: lattice theory. By Professor Kelley: theory of functions of a complex variable. By Professor Logsdon: analytic projective geometry. By Professor Schilling: continuous groups.

The University of Michigan. From July 1 to August 23 the following courses will be offered in addition to the standard courses in differential equations, theory of equations, solid analytic geometry and advanced calculus: By Professor Bartels: hydrodynamics. By Professor Churchill: Fourier series, operational mathematics. By Professor Craig: quality control, advanced statistics. By Professor Dwyer: statistics. By Professor Eilenberg: axiomatic set theory topology. By Professor Fischer: finite differences, elementary life insurance. By Professor Hay: elasticity. By Professor Hildebrandt: functions of a complex variable, theory of differential equations. By Professor Karpinski: teaching of algebra, history of geometry. By Professor Nesbitt: theory of matrices, finite differences. By Professor Rothe: topics in mathematical physics. By Professor Thrall: finite groups, algebraic geometry. By Professor Wilder: foundations of mathematics, studies in foundations.

The University of Minnesota. From June 19 to June 27 the following advanced courses will be given: By Professor Cameron: theory of probability, tutorial course in advanced mathematics. By Professor Carlson: modern analytic geometry. By Professor Wegner: differential equations. From July 29 to August 31: By Professor Campaigne: introduction to modern algebra, tutorial course in advanced mathematics.

The University of Pennsylvania. From July 2 to August 23 advanced courses will be given in modern analytic geometry, differential equations, introduction to modern higher algebra, and the calculus of variations.

The University of Wyoming. From June 20 to July 24 the following advanced courses will be given: By Professor Barr: differential equations. By Professor Neubauer: advanced college algebra. By Professor Rechard: differential equations. From July 25 to August 23: By Professor Alden: advanced college algebra. By Professor Barr: differential equations.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

RELEASE OF FACULTY PERSONNEL FROM THE SERVICES

In view of the critical need for university and college faculty personnel in many institutions, the Office of War Mobilization and Reconversion has arranged with the services to expedite the release of personnel desired by the higher institutions in the United States. Dr. M. H. Trytten, Director of the Office of Scientific Personnel, has announced the following criteria and procedures for the release of faculty personnel from the services.

A. PURPOSE.

The shortage of college and university teachers is so acute that many veterans may be denied the right to educational benefits provided under the Servicemen's Readjustment Act unless steps are taken to increase the supply of competent teachers.

To aid in relieving this shortage, the War and Navy Departments have from time to time released teachers who did not have enough points for discharge. The American Council on Education has heretofore served as the clearing house for these released in the manner described in emergency supplements (dated August 27 and September 6, 1945) to *Higher Education and National Defense.*

In order to regularize and extend this practice, the Office of War Mobilization and Reconversion has created an Interagency Committee on Manpower Shortages, one of whose duties is to establish criteria, standards, and procedures for clearing requests for release from military duties. The college portion of this program will be administered by the U. S. Office of Education. No new Army and Navy policies on release are involved.

B. STANDARDS AND CRITERIA.

1. Any duly accredited institution of higher education may request the Office of Education to certify for release any member of the Armed Forces who is qualified for college teaching and the auxiliary duties related to teaching.

2. It is not necessary for the man to have been a member of the faculty of the college requesting his release.

3. Only men on duty within the continental United States may be requested from the Army. (This does not preclude a college from making representation direct to the War Department for the release of personnel on duty outside the continental United States. Men may be released who are deemed "essential to the national health, safety, or interest.") This restriction does not apply to men who are in the Navy.

4. The man whose release from the Army is sought must make his own request through military channels in the manner described in the next section of this statement. The individual whose release is sought should be warned that unless he has had overseas service he will lose mustering-out pay. These conditions are not applicable to Navy personnel.

5. The man whose release is requested must be employed full time. He may engage in teaching and the auxiliary duties related to teaching. His case will be strengthened if the college can show that his return to the campus will directly or indirectly enrich the educational services provided for veterans.

6. The college must show that the man whose release is requested is needed and that the college has not been able to secure a suitable individual from civilian sources.

7. The request for release should include a comprehensive factual statement of the duties to be performed by the individual, and an outline of his qualifications.

C. PROCEDURES.

The President or his duly constituted representative should address the request for release of military personnel that is needed for college staff members *in duplicate* to Dr. Ernest V. Hollis, Division of Higher Education, U. S. Office of Education, Washington 25, D. C. The information to be supplied differs in some particulars for Army and Navy personnel. This is due to the fact that Army policy requires the man to initiate his own request for release when it is determined that he is eligible. In turn, by law a person who requests his own release from military duty loses mustering-out pay unless he has served outside the continental United States. By Navy procedure the man who is eligible does not request his own release and hence does not take all of the steps required of soldiers. The Marine Corps follows Navy policy.

1. As a preliminary to writing the Office of Education requesting the certification of a man for release, the college should furnish the following information which the man himself usually must supply:

(a) A statement that an individual from the Army is willing to request his own release through military channels; that Navy personnel is willing to be released.

(b) Name, rank, and serial number.

(c) Branch of military service, length of service, and nature of present duty.

(d) Current Armed Forces address.

(e) If the individual is in the Army, the name and address of his commanding officer.

The letter to the Office of Education containing the above information should also present factual statement of the duties to be performed, and the date on which this service will begin, and an outline of his qualifications.

2. A copy of the statement which is sent in duplicate to the Office of Education should be furnished to the man whose release is sought. If the man is in the Army he must attach a signed copy to the request he initiates. Army personnel should not initiate this request until the college advises him of his elgibility. Navy personnel does not take this step.

3. The Office of Education will review the information submitted by a duly constituted college official for its conformity to the criteria and standards for release which have been established by the Interagency Committee of OWMR.

4. The Office of Education will certify to individuals who meet standards for release to (a) the commanding officer of Army personnel, and (b) to the Bureau of Naval Personnel for Navy and Marine Corps personnel. Copies of the certification will go to the individual, to the college, and for Army personnel to the War Department.

5. Send separate request for the release of individuals even though the release of more than one man may be requested under the same cover.

6. Any Government agency or private association which receives from a university or an individual a request for release under the conditions set forth in this circular should forward the information to the U. S. Office of Education or advise the inquirer of the approved procedures for securing consideration.

7. Certification of an individual by the Office of Education for release does not guarantee that the Army or the Navy will release the individual in question; the Armed Services necessarily reserve the right to retain men whom they consider essential for military purposes.

MATHEMATICS REQUIRED FOR GRADUATION IN KANSAS

For about twenty years there has been no state requirement in Kansas that high school graduates must have studied some mathematics. So on April 15, 1944, the Kansas Association of Teachers of Mathematics and the Kansas Section of the M.A.A. joined in requesting the State Department of Education to establish a requirement of at least one unit of mathematics for graduation from accredited high schools of Kansas. This action was taken after the adoption of the recommendation which follows.

"The K. A. T. M. Committee on the Improvement of Instruction recommends that the Kansas Association of Teachers of Mathematics and the Kansas Section of the Mathematical Association of America unite in requesting the Kansas State Department of Education to require a minimum of one unit of mathematics for graduation from accredited high schools of the state. The Committee recommends that the secretaries of the two organizations submit this request to the State Department and that their communication include the following explanations and suggestions.

"This request is made because the members of these organizations believe that mathematics study is of fundamental importance in the general education of every individual. This request does not imply that every high school pupil should study algebra. All high schools should provide, for boys and girls who are unable to profit from the study of algebra, a course which is based on principles and applications of arithmetic which nearly everyone can use to advantage in everyday living. It is urged, however, that superior students should not be enrolled in this course. Students who have sufficient ability and a good understanding of arithmetical principles when they enter high school should be given an opportunity to extend their knowledge of mathematics through the study of algebra.

"The administrators of some small high schools may feel that they cannot have two ninth grade courses in their schools and that if this requirement were made they would be forced to choose between the basic arithmetic course and elementary algebra. Even in the smallest schools there are two classes of ninth grade mathematics. One could be arithmetic and the other algebra. In very small schools these two courses could be offered in alternate years.

"Perhaps one-third to one-half of entering ninth grade pupils should be enrolled in the arithmetic course, and the remainder in the algebra course in the ninth year. The placement of pupils in these two courses should be on the basis of general ability and knowledge of fundamental arithmetic, rather than on any plans which the students may have regarding future college attendance. There are at least two reasons for this. First, boys and girls cannot know for certain whether they will go to college. Second, if the high school is to be the student's last opportunity at formal education, it is all the more important that he be given an opportunity there to gain some basic understanding of mathematical methods that are used by persons in many important fields.

"At the end of the ninth year, it is probable that some of the pupils who have done at least reasonably good work in the arithmetic course should continue with mathematics study, and they should be enrolled in algebra the tenth year. Some of them should continue on through plane geometry and perhaps even another year of mathematics.

"Every school, regardless of size, should offer a year of plane geometry and either require or encourage enrollment in this course of all students who are capable of pursuing it with understanding. It is highly desirable that at least one additional year of mathematics be offered. In very small schools, the plan of offering courses in alternate years can be used to make available more advanced courses.

"This is not a wartime proposal. Although the needs of the armed forces and war industries have focused the attention of the schools and the public on deficiencies in the mathematics instruction of our youth, the need for sound basic training in mathematics will always be present as long as we are concerned with educating boys and girls to understand their environment and to become useful citizens."

As a result of this united effort on the part of the mathematical organizations of Kansas, the Kansas High School Handbook for 1945 lists a minimum of one unit of mathematics among the requirements for high school graduation. At the time of the negotiations with the State Department of Education, the mathematical organizations also sent a letter containing some suggestions for the high school mathematics program to all the high school principals in the state.

MATHEMATICAL ASSOCIATION OF AMERICA

REPORT OF THE TREASURER FOR THE YEAR 1945

The following report of the Secretary-Treasurer as Treasurer for the year 1945 has been approved by the Finance Committee and accepted by vote of the Board of Governors.

I. Total Funds of the Association on December 31, 1944

(See Treasurer's report, pp. 230–233 of the Monthly for April, 1945)

Current Fund (checking account)		$ 5,474.49
Savings Account (Ithaca Savings Bank)		1,000.00
Savings Account (Oberlin Savings Bank)		648.96
Invested Funds (Cleveland Trust Company)		
Carus Fund	$ 8,353.37	
Chace Fund	9,347.93	
Houck Fund	8,676.14	
Chauvenet Fund	614.04	
Life Membership Fund	755.03	
General Fund	21,916.26	49,662.77
		$56,786.22

II. Current Fund Account for 1945

Receipts		Expenditures	
Balance, Jan. 1, 1945	$ 5,474.49	Monthly	
Dues	9,001.16	Publication	$ 5,094.19
Initiation fees	248.00	Reprints	70.79
Subscriptions	2,744.37	Editor-in-Chief's office	422.36
Sale of back numbers Monthly	540.91	Secretary-Treasurer's office	
Advertisements	543.50	Clerical help	3,268.95
Sale of Archibald's Outline	156.75	Postage	387.66
Interest on General Fund	791.66	Printing	264.29
Interest on Carus Fund	301.58	Office supplies	243.14
Interest on Chace Fund	337.49	Bank charges	108.75
Interest on Houck Fund	314.15	Exec. and Finance Committees	102.38
Interest on Chauvenet Fund	23.34	Regional Governors	184.51
Interest on Life Membership Fund	26.93	Sections	86.39
Interest from Hardy Fund	120.00	Subventions	
Sale of monographs (Carus)	523.35	Amer. Math. Society	100.00
Sale of Papyrus (Chace)	137.95	Mathematical Reviews	350.00
Miscellaneous sources	4.14	National Math. Magazine	200.00
Transferred from General Fund	1,264.79	Math. Cuneiform Texts (Chace)	400.00
		Back numbers Monthly	118.15
		Meetings	167.33
		Coordinating Committee	122.64
		Coop. Committee on Sci. Teaching	39.78
		To General Fund	6,078.70
		Subscriptions to Annals	12.50
		B. F. Finkel (Hardy Fund)	120.00
		Transferred to Carus Fund	824.93
		Transferred to Chace Fund	75.44
		Transferred to Houck Fund	314.15
		Transferred to Chauvenet Fund	23.34
		Transferred to Life Memb. Fund	26.93
		Balance, Dec. 31, 1945	3,347.26
	$22,554.56		$22,554.56

III. Savings Account, Ithaca Savings Bank

Balance, Jan. 1, 1945	$1,000.00		
Interest	7.50	Balance, Dec. 31, 1945	$1,007.50
	$1,007.50		$1,007.50

IV. Invested Funds, Cleveland Trust Company

Cash Balance, Jan. 1, 1945	$ 40.77	Market value of securities, Dec. 31, 1945	$57,113.00
Market value of securities, Dec. 31, 1944	49,622.00	Cash balance, Dec. 31, 1945	279.92
From Current Fund	6,078.70		
Increase in value of securities	1,651.45		
	$57,392.92		$57,392.92

List of Securities

	Par Value	Market Value Dec. 31, 1945
U. S. Savings Bonds, 1947	$ 700.00	$ 658.00
U. S. Treasury Bonds, 2%, 1950	3,000.00	3,060.00
U. S. Treasury Bonds, 2%, 1954	2,000.00	2,080.00
U. S. Treasury Bonds, 2½%, 1969	2,000.00	2,060.00
U. S. Treasury Bonds, 2½%, 1972	1,000.00	1,010.00
U. S. Treasury Bonds, 2¼%, 1952	5,000.00	5,050.00
U. S. Treasury Bonds, 1¾%, 1948	2,000.00	2,040.00
U. S. Savings Bonds, Ser. G, 2½%, 1953	3,000.00	2,850.00
U. S. Savings Bonds, Ser. G, 2½%, 1954	8,200.00	7,790.00
Canadian Nat. Ry. Co. Bonds, 4½%, 1956	2,000.00	2,420.00
Gatineau Power Co. 1st Mort. Bonds, Ser. A, 4½%, 1970	2,000.00	2,120.00
C. and O. Ry. Co. Ref. Bonds, Ser. D, 3½%, 1996	3,000.00	3,180.00
Union Pacific R.R. Co. Deb. Bonds, 3½%, 1971	3,000.00	3,210.00
Amer. Tel. & Tel. Co. Conv. Deb. Bonds, 3%, 1956	2,000.00	3,020.00
Columbus and So. Ohio Elec. Co. Bonds, 3¼%, 1970	2,000.00	2,200.00
Commonwealth Edison Co. Conv. Deb. Bonds, 3½%, 1958	2,000.00	2,680.00
New York Steam Corp. 1st Mort. Bond, 3½%, 1963	1,000.00	1,060.00
Amer. Tobacco Co. Bonds, 3%, 1969	4,000.00	4,240.00
C. and O. Ry. Co. common stock, 25 shares		1,400.00
Amer. Tel. & Tel. Co. common stock, 10 shares		1,910.00
Standard Oil Co. New Jersey common stock, 20 shares		1,320.00
Atch. Top., Santa Fe R.R. non cum. pfd. stock, 15 sh.		1,755.00
		$57,113.00

V. Carus Fund

Balance, Jan. 1, 1945	$8,353.37		
Sale of monographs	523.35		
Interest	301.58		
Increase in value of securities	277.45	Balance, Dec. 31, 1945	$9,455.75
	$9,455.75		$9,455.75

VI. Chace Fund

Balance, Jan. 1, 1945	$ 9,347.93	Mathematical Cuneiform Texts	$ 400.00
Sale of Papyrus	137.95		
Interest	337.49		
Increase in value of securities	310.47	Balance, Dec. 31, 1945	$ 9,733.84
	$10,133.84		$10,133.84

VII. Houck Fund

Balance, Jan. 1, 1945	$8,676.14		
Interest	314.15		
Increase in value of securities	289.00	Balance, Dec. 31, 1945	$9,279.29
	$9,279.29		$9,279.29

VIII. Chauvenet Fund

Balance, Jan. 1, 1945	$ 614.04		
Interest	23.34		
Increase in value of securities	21.47	Balance, Dec. 31, 1945	$ 658.85
	$ 658.85		$ 658.85

IX. Life Membership Fund

Balance, Jan. 1, 1945	$ 755.03	To General Fund	$ 88.64
Interest	26.93		
Increase in value of securities	24.77	Balance, Dec. 31, 1945	718.09
	$ 806.73		$ 806.73

X. General Fund

Balance, Jan. 1, 1945	$21,916.26	Transferred to Current Fund	$ 1,264.79
From Current Fund	6,078.70		
Increase in value of securities	728.29		
From Life Membership Fund	88.64	Balance, Dec. 31, 1945	27,547.10
	$28,811.89		$28,811.89

XI. TOTAL FUNDS OF THE ASSOCIATION, DECEMBER 31, 1945

Current Fund (checking account)		$ 3,347.26
Savings Account (Ithaca Savings Bank)		1,007.50
Savings Account (Oberlin Savings Bank)		648.96
Invested Funds (Cleveland Trust Company)		
Carus Fund	$ 9,455.75	
Chace Fund	9,733.84	
Houck Fund	9,279.29	
Chauvenet Fund	658.85	
Life Membership Fund	718.09	
General Fund	27,547.10	57,392.92
		$62,396.64

THE NEW EDITOR-IN-CHIEF

By recent action of the Board of Governors Professor C. V. Newsom of Oberlin College was elected Editor-in-Chief of the MONTHLY for a five year period beginning January 1, 1947. Professor Newsom has been an Associate Editor since 1943. During this time he has conducted the interesting and valuable department known first as War Information and later, with the coming of peace, as General Information. He is a member of the Board of Governors and of various committees. He brings to his new tasks a rich experience in the affairs of the Association.

CALENDAR OF FUTURE MEETINGS

Twenty-eighth Summer Meeting, Ithaca, New York, August 19–20, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., October 1946
ILLINOIS, NORMAL, May 10–11, 1946
INDIANA
IOWA
KANSAS
KENTUCKY, Louisville, April 27, 1946
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, N. Y., May 4, 1946
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Omaha, May 4, 1946
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Claremont, March 8, 1947
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May 4, 1946

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"SLOPE" IN SOLID ANALYTIC GEOMETRY

C. B. ALLENDOERFER, Haverford College

1. Introduction. For some time it has concerned the author that the usual treatment of the analytic geometry of lines in space is not an immediate generalization of their treatment in the plane. The particular point at issue is the use of "slope" in the plane and of "direction cosines" in space. It is unsound pedagogy to treat two dimensional concepts and three dimensional concepts as separate entities when they are in fact two special cases of a general theory. This situation has been recognized by others and has been overcome in several recent texts which use direction cosines in the plane as well as in space. From the purely geometrical point of view this is ideal; but it raises another difficulty when the analytic geometry is applied to calculus. For slope is essential to the calculus as it is usually presented, and many courses in analytic geometry may not have the time to include discussions of both direction cosines and slope.

It is not generally recognized that the notion of slope can be extended to solid analytic geometry, and the purpose of this paper is to show how this can be accomplished. The use of slope in both the plane and space has the advantages: (1) it provides a unified treatment of plane and solid analytic geometry, (2) it provides a direct approach to the derivative in the plane and to partial derivatives in space. It has the disadvantage of lack of symmetry in the variables, thus losing elegance, and further it suffers from the need of considering special cases (usually glossed over in the plane) which arise from the infinite values introduced by the use of the tangent function. These disadvantages are serious, but the author feels that the advantages are sufficiently strong for this approach to merit the attention of teachers of college mathematics.

2. Slopes of lines. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two points on a line, l; then the slopes p and q of l are given by the definition:

DEFINITION.

$$\text{The } x\text{-slope, } p, \text{ of } l \text{ is } p = \frac{z_2 - z_1}{x_2 - x_1}$$

$$\text{The } y\text{-slope, } q, \text{ of } l \text{ is } q = \frac{z_2 - z_1}{y_2 - y_1}.$$

(The reader will note that p corresponds to $\partial z/\partial x$ and q to $\partial z/\partial y$ as is customary in treatments of partial differential equations.)

It is clear that p is the ordinary (two-dimensional) slope of the projection of l in the XZ plane, and similarly that q is the slope of the projection of l in the YZ plane. It follows, then, from plane analytic geometry that p and q are independent of the two points chosen to define them. If l is parallel to the YZ plane, $x_2 - x_1 = 0$; in this case we say that $1/p = 0$, or loosely that $p = \infty$. Similarly when l is parallel to the XZ plane, $y_2 - y_1 = 0$ and $1/q = 0$ or $q = \infty$.

Occasionally it is useful to consider the slope of the projection of l into the XY plane. The slope of this projection will be called the xy-slope of l and is given by the definition:

DEFINITION.

$$\text{The } xy\text{-slope, } m, \text{ of } l \text{ is } m = \frac{y_2 - y_1}{x_2 - x_1}.$$

For lines parallel to the XY plane it follows that $p=0$ and $q=0$, and m is the usual slope of a line as defined in plane geometry. For lines parallel to the YZ plane, $1/m=0$ or $m=\infty$.

It is useful to point out that:

For lines parallel to the X axis: $p=0$, q is undefined $(=0/0)$. $m=0$.

For lines parallel to the Y axis: p is undefined $(=0/0)$, $q=0$, $m=\infty$.

For lines parallel to the Z axis: $p=\infty$, $q=\infty$, m is undefined $(=0/0)$.

It further follows that p/q is defined except for lines parallel to the XY plane, and that unless p/q is undefined: $m=p/q$. This shows that except for the special case just mentioned, it is sufficient to deal with p and q exclusively and to omit reference to m.

3. The equations of a line. In this section the equations of a line not parallel to the XY plane will be developed first. Such a line will be called a general line. Lines parallel to the XY plane will be called special lines.

Point-slope Equations. To find the equations of a general line given a point on it, say (x_1, y_1, z_1) and its slopes p and q, let (x, y, z) be an arbitrary point on the line. Then if $1/p\neq 0$ and $1/q\neq 0$:

$$\frac{z - z_1}{x - x_1} = p; \qquad \frac{z - z_1}{y - y_1} = q$$

or

$$z - z_1 = p(x - x_1); \qquad z - z_1 = q(y - y_1)$$

are the equations of the line. If $1/p=0$, the first of these equations becomes $x-x_1=0$; if $1/q=0$, the second of these equations becomes $y-y_1=0$.

Two-point Equations. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two given points, we seek to find the equations of the line defined by these two points. By substituting the values of p and q given by their definitions into the point-slope equations of the line, we have the desired result:

When $x_2-x_1\neq 0$ and $y_2-y_1\neq 0$:

$$z - z_1 = \frac{z_2 - z_1}{x_2 - x_1}(x - x_1); \qquad z - z_1 = \frac{z_2 - z_1}{y_2 - y_1}(y - y_1).$$

When $x_2-x_1=0$, the first of these equations becomes $x=x_1$. When $y_2-y_1=0$, the second of these equations becomes $y=y_1$. All of these possibilities are included by writing the equations in the form:

$(z - z_1)(x_2 - x_1) = (z_2 - z_1)(x - x_1)$; $(z - z_1)(y_2 - y_1) = (z_2 - z_1)(y - y_1)$.

Special lines. A special line is one parallel to the XY plane. Its equations follow from plane geometry:

Point-slope form: $z = z_1$; $y - y_1 = m(x - x_1)$

Two-point form: $z = z_1$; $(y - y_1)(x_2 - x_1) = (y_2 - y_1)(x - x_1)$.

4. Condition that two lines be perpendicular. We consider first two general lines. These lines will be perpendicular if the lines l_1 and l_2 passing through the origin and parallel to them are perpendicular. If the slopes of the given lines are p_1, q_1 and p_2, q_2 respectively, it follows that l_1 and l_2 have these same slopes since parallel lines have equal slopes. Since the lines are general, none of the quantities p_1, q_1, p_2, q_2 are zero. It is easy to show that the points $P_1(1/p_1, 1/q_1, 1)$ and $P_2(1/p_2, 1/q_2, 1)$ lie on l_1 and l_2 respectively. We do not exclude the cases where any of $1/p_1$, $1/q_1$, $1/p_2$, $1/q_2$ are zero. In order that l_1 be perpendicular to l_2 it is necessary that the triangle P_1OP_2 have a right angle at O. By applying Pythagoras' Theorem, we find the relation:

$$1/p_1p_2 + 1/q_1q_2 + 1 = 0.$$

This result is readily seen to be an appropriate generalization of the condition for the perpendicularity of two lines in a plane.

If either l_1 or l_2 is a special line, the condition for perpendicularity is $m_1m_2 = -1$ provided that neither m_1 nor m_2 is ∞. If $m_1 = \infty$, m_2 must be zero; and if $m_2 = \infty$, m_1 must be zero.

More generally, to find the angle, θ, between two general lines, we apply the law of cosines to the triangle P_1OP_2, the angle θ being at the vertex O. The result is:

$$\cos\theta = \frac{\dfrac{1}{p_1p_2} + \dfrac{1}{q_1q_2} + 1}{\sqrt{\dfrac{1}{p_1^2} + \dfrac{1}{q_1^2} + 1}\sqrt{\dfrac{1}{p_2^2} + \dfrac{1}{q_2^2} + 1}}.$$

5. The slopes of planes. For our purposes planes will be divided into two classes as follows: planes parallel to the Z axis will be called special planes; other planes are general planes. A general plane, π, will intersect the XZ plane and the YZ plane in two lines called the traces of π, whose slopes in these planes are p and q respectively. A plane, π, then has two slopes which are defined as follows:

DEFINITION. The x-slope, P, of π is the slope, p, of its trace in the XZ plane.
The y-slope, Q, of π is the slope, q, of its trace in the YZ plane.

These slopes are always defined for general planes, although in certain cases they may be infinite.

It is useful to introduce the xy-slope of a plane as follows:

DEFINITION. The xy-slope, M, of π is the slope, m, of its trace in the XY plane.

This slope will seldom be used except for special planes.

The slopes of certain planes have particular values which are of some interest:

A plane parallel to the XY plane has: $P=0$, $Q=0$, and M is undefined.
A plane parallel to the XZ plane has: P is undefined ,$Q=\infty$, and $M=0$.
A plane parallel to the YZ plane has: $P=\infty$, Q is undefined, and $M=\infty$.

6. The equations of a plane. It is proved in standard texts that every equation of the first degree: $Ax+By+Cz+D=0$ is the equation of a plane, and every plane has an equation of this form.

Three-point Form. Let $P_1(x_1, y_1, z_1)$; $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$ be any three non-collinear points. Also let:

$$T_{yz} = 1/2 \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}; \quad T_{zx} = 1/2 \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix};$$

$$T_{xy} = 1/2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & x_3 & 1 \end{vmatrix}; \quad T_{xyz} = 1/2 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

It will be noted that T_{yz}, T_{zx}, T_{xy} are the areas (with a specified algebraic sign) of the projections of the triangle $P_1P_2P_3$ upon the YZ plane, the XZ plane, and the XY plane respectively. Not all of these are simultaneously zero. Then it is proved in many standard texts that the plane, π, passing through these three points has the equation:

$$T_{yz}x + T_{zx}y + T_{xy}z - T_{xyz} = 0.$$

This is the three-point form of the equation of a plane.

Slopes of a Plane in Terms of Three Non-collinear Points on it. The trace of π in the XZ plane is obtained by putting $y=0$ in the equation just above. The x-slope, P, of the plane equals the x-slope, p, of this trace. Therefore: $P=p=-T_{yz}/T_{xy}$. Similarly $Q=-T_{zx}/T_{xy}$; and $M=-T_{yz}/T_{zx}$. The analogy with the slope of a line in plane geometry is now clear. For if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points on a line, its slope is $m=y_2-y_1/x_2-x_1$. Here y_2-y_1 is the length of the projection of the segment P_2P_1 on the Y axis and x_2-x_1 is the length of the projection of P_2P_1 on the X axis. The slope is then the ratio of the lengths of these projections.

Reference should be made to section 5 for a statement of the cases in which P, Q, and M are not defined. Unless P/Q is undefined it follows that $M=-P/Q$; and consequently we can deal exclusively with P and Q except for this special case.

Slope, Z-intercept Form. For general planes $T_{xy} \neq 0$ and hence the three-point form of the equation of a plane may be solved for z (taking into account the values for P and Q) as follows:

$$z = Px + Qy + T_{xyz}/T_{xy}.$$

The last term of this equation does not involve any unknown and is easily seen to equal the Z-intercept, c, of the plane. Hence we write the equation in the form:

$$z = Px + Qy + c.$$

Special planes do not have an equation of this form.

This result enables us to find the slopes of a general plane as well as its Z-intercept from its equation. The rule is: solve the equation for z; the resulting coefficient of x is the x-slope, P; the coefficient of y is the y-slope, Q, and the constant term is the Z-intercept, c. The reader will note the similarity of this procedure to that used for finding the slope and Y-intercept of a line in the plane.

Point-slope Form. Given the slopes of a plane and a point (x_1, y_1, z_1) on it we seek to find its equation. The slope, Z-intercept equation is

$$z = Px + Qy + c.$$

Since the point lies on this plane: $z_1 = Px_1 + Qy_1 + c$. Subtracting these two equations we obtain the desired result:

$$z - z_1 = P(x - x_1) + Q(y - y_1).$$

If a special plane is not parallel to the YZ plane, its equation is readily seen to be:

$$y - y_1 = M(x - x_1).$$

A plane parallel to the YZ plane has the equation $x = x_1$.

7. Conditions that a plane and a line be perpendicular. It is an elementary result that a line is perpendicular to a plane if and only if it is perpendicular to two non-parallel lines in that plane. If π is a general plane, its trace in the XZ plane has slopes: $p = P$ and $1/q = 0$; and its trace in the YZ plane has the slopes $1/p = 0$ and $q = Q$. A general line l whose slopes are p and q is perpendicular to π provided it is perpendicular to these two traces. The conditions follow immediately from the results of section 4, namely:

$$\frac{1}{pP} + 1 = 0; \qquad \frac{1}{qQ} + 1 = 0.$$

Or: A general line is perpendicular to a general plane if and only if the slopes of the line are the negative reciprocals of the slopes of the plane. This may also be worded: the slopes of the normals to a general plane are the negative reciprocals of those of the plane.

In the interests of brevity the treatment of special lines and planes is omitted. The chief facts are: (1) A special line is never perpendicular to a general plane,

a special plane is never perpendicular to a general line; (2) A special line is perpendicular to a special plane when $mM=-1$.

8. The distance from a plane to a point. Let π be the general plane $Ax+By+Cz+D=0$ $(C\neq 0)$ and $R(x_1, y_1, z_1)$ be any point. The slopes of π are $P=-A/C$ and $Q=-B/C$. The normal to π therefore has the slopes: $p=C/A$ and $q=C/B$. So the equations of the normal to π passing through R are:

$$z - z_1 = C/A(x - x_1); \qquad z - z_1 = C/B(y - y_1).$$

To find the coordinates of the intersection of this normal with π, we first write the above equations in the form:

$$x = x_1 + A/C(z - z_1); \qquad y = y_1 + B/C(z - z_1).$$

Since $C\neq 0$, these equations are always possible. Substituting these values in the equation of the plane and solving for z, we have the result:

$$z = z_1 - C\left(\frac{Ax_1 + By_1 + Cz_1 + D}{A^2 + B^2 + C^2}\right).$$

The value of z thus determined is the z coordinate of the point S which is the foot of the perpendicular to π from R. The other coordinates of S are found to be:

$$y = y_1 - B\left(\frac{Ax_1 + By_1 + Cz_1 + D}{A^2 + B^2 + C^2}\right); \quad \text{and}$$

$$x = x_1 - A\left(\frac{Ax_1 + By_1 + Cz_1 + D}{A^2 + B^2 + C^2}\right).$$

The distance, d, from π to R is the distance SR and hence:

$$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

According to the customary convention, the sign of the radical is chosen opposite to that of D. This result can be extended without difficulty to special planes.

9. Application to partial differentiation. Let us consider a surface $z=f(x, y)$ and a point R (x_1, y_1, z_1) on it. Let S be the point $(x_1+\Delta x, y_1, z_1+\Delta z)$ where $z_1+\Delta z=f(x_1+\Delta x, y_1)$. And let T be the point $(x_1, y_1+\delta y, z_1+\delta z)$ where $z_1+\delta z=f(x_1, y_1+\delta y)$. The plane passing through R, S, and T has the slopes: $P=\Delta z/\Delta x$ and $Q=\delta z/\delta y$. Thus the equation of this plane is:

$$z - z_1 = \frac{\Delta z}{\Delta x}(x - x_1) + \frac{\delta z}{\delta y}(y - y_1).$$

The tangent plane to the surface at R is defined to be the limiting position of the RST plane when $\Delta x \to 0$ and $\delta y \to 0$ independently. The slopes of the limit plane are given by

$$P = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \frac{\partial z}{\partial x} \quad \text{and} \quad Q = \lim_{\delta y \to 0} \frac{\delta z}{\partial y} = \frac{\partial z}{\partial y}.$$

Thus the geometrical interpretation of $\partial z/\partial x$ and $\partial z/\partial y$ is that they are the slopes of the tangent plane to the surface at the given point. In this fashion partial derivatives may be introduced by a geometrical argument parallel to that used for ordinary derivatives.

Another, less conventional, approach to partial derivatives is the following. Let the surface and the point R be defined as above. But now let S be the point $(x_1+\Delta x,\ y_1+\Delta y,\ z_1+\Delta z)$ where $z_1+\Delta z = f(x_1+\Delta x,\ y_1+\Delta y)$; and let T be the point $(x_1+\delta x,\ y_1+\delta y,\ z_1+\delta z)$ where $z_1+\delta z = f(x_1+\delta x,\ y_1+\delta y)$. From the formulas given in section 6 it follows that the slopes of the plane through R, S, and T are as follows:

$$P = \frac{\Delta y \delta z - \Delta z \delta y}{\Delta y \delta x - \Delta x \delta y}; \quad \text{and} \quad Q = \frac{\Delta z \delta x - \Delta x \delta z}{\Delta y \delta x - \Delta x \delta y}.$$

The limits of these quantities as S and T approach R are then the slopes of the tangent plane, and are equal to the corresponding partial derivatives. Certain restrictions, however, must be placed on this limit process, for there are cases for which the limiting position of the plane RST is not the tangent plane. For example, let us suppose that R, S, and T all lie on a circular cross section of a right circular cylinder. If S and T approach R along arcs of this circle, the limit plane is a normal plane! A restriction sufficient to avoid this difficulty is the following. Let S approach R along a differentiable curve C_1, and let T approach R along a differentiable curve C_2. The limiting position of the plane RST will be a tangent plane if the curves C_1 and C_2 have distinct tangent lines at R.

We recall that the numerators and denominators of the above expressions for P and Q are equal to twice the signed areas of the projections of the triangle RST upon the corresponding coordinate planes. The partial derivatives are therefore equal to the limits of the ratios of the areas of the appropriate pairs of these projections. It will be noticed that this interpretation is particularly useful in establishing the formula for the element of surface area of the surface $z=f(x, y)$:

$$d\sigma = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dy dx.$$

ON SETS OF DISTANCES OF n POINTS

P. ERDÖS, Stanford University

1. The function f(n). Let $[P_n]$ be the class of all planar subsets P_n of n points and denote by $f(n)$ the minimum number of different distances determined by its n points for P_n an element of $\{P_n\}$. Clearly, $f(3)=1$ (with the three points forming the vertices of an equilateral triangle) $f(4)=2$, $f(5)=2$. The following theorem establishes rough bounds for arbitrary n. Though I have sought to improve this result for many years, I have not been able to do so.

THEOREM 1. *The minimum number $f(n)$ of distances determined by n points of a plane satisfies the inequalities*

$$(n - 3/4)^{1/2} - 1/2 \leqq f(n) \leqq cn/(\log n)^{1/2}.$$

Proof. Let P_1 be an arbitrary vertex of the least convex polygon determined by the n points, and denote by K the number of different distances occurring among the distances P_1P_i $(i=2, 3, \cdots, n)$. If N is the maximum number of times the same distance occurs, then clearly $KN \geqq n-1$.

If r is a distance that occurs N times then there are N points on the circle with center P_1 and radius r, which all lie on the same semi-circle (since P_1 is a vertex of the least convex polygon). Denoting these points by $Q_1, Q_2, \cdots, Q_N$, we have $Q_1Q_2 < Q_1Q_3 < \cdots < Q_1Q_N$, and these $N-1$ distances are pairwise distinct. Thus $f(n) \geqq \max(N-1, (n-1)/N)$, which is a minimum when $N(N-1) = n-1$. This yields the first part of the theorem.

Considering now the points (x, y) with *integer* coordinates for $0 \leqq x, y \leqq n^{1/2}$, we obtain at least n points P_i which pairwise have distances of the form $(u^2+v^2)^{1/2}$, $0 \leqq u \leqq n^{1/2}$, $0 \leqq v \leqq n^{1/2}$. Now it is well-known that the number of different integers not exceeding $2n$ which are of the form u^2+v^2 is less than $cn/(\log n)^{1/2}$, and the proof is complete.*

For n points in k-dimensional space the same method yields $c_1 n^{1/k} < f(n) < c_2 n^{2/k}$.

2. Some conjectures concerning f(n). Let us assume that our n points form a convex polygon. Then I conjecture that $f(n) \geqq [n/2]$, with the equality sign valid when the n points are vertices of a regular n-gon. I am unfortunately unable to prove this. The following conjecture is stronger: In every convex polygon there is at least one vertex with the property that no three vertices of the polygon are equally distant from it. If this is the case, then clearly we would obtain $[n/2]$ different distances by considering all the distances from such a vertex.

A still stronger conjecture is that on every convex curve there exists a point P such that every circle with center P intersects the curve in at most 2 points.

3. The function g(n; r). Denoting by $g(n; r)$ the maximum number of times a given distance r can occur among n points of a plane we establish

* Landau, Verteilung der Primzahlen, vol. 2.

THEOREM 2. $n^{1+c/\log\log n} < g(n;r) < n^{3/2}$.

Proof. Assuming that there are x_i points at distance r from P_i, clearly $g(n;r) = \max \frac{1}{2}\sum_{i=1}^{n} x_i$. We suppose that $x_1 \geqq x_2 \geqq \cdots \geqq x_n$. Now the x_i points at distance r from P_i can contain at most two points with distance r from P_j. Hence

$$\sum_{i=1}^{j} (x_i - 2i + 2) \leqq n \quad \text{for} \quad j = 1, 2, \cdots, n. \tag{1}$$

Put $[n^{1/2}] = a$, $n^{1/2} - a = \epsilon$, $0 \leqq \epsilon < 1$. We have from (1)

$$x_1 + x_2 + \cdots + x_a \leqq n + 2\binom{a}{2} = 2n - 2\epsilon n^{1/2} + \epsilon^2 - n^{1/2} + \epsilon < 2n - 2\epsilon n^{1/2} \tag{2}$$

for $n \geqq 4$. Thus

$$x_a < \frac{1}{a}(2n - 2\epsilon n^{1/2}) = 2n^{1/2}. \tag{3}$$

Hence from (2) and (3)

$$\sum_{i=1}^{n} x_i < 2n - 2\epsilon n^{1/2} + (n - a)2n^{1/2} = 2n^{3/2}$$

or

$$g(n; r) < n^{3/2}.$$

By again considering the set of points (x, y), $0 \leqq x, y \leqq a$ we easily obtain (using well known theorems about the number of solutions of $u^2 + v^2 = m$)*

$$g(n) > n^{1+c/\log\log n}$$

which completes the proof.

It seems likely that $g(n) < n^{1+\epsilon}$.

4. Maximum and minimum distances. If r is the diameter of the points P_i, it is well known that r can occur only n times.† This follows almost immediately from the fact that if $\overline{P_1P_2} = r$ and $\overline{P_3P_4} = r$ the lines P_1P_2 and P_3P_4 must intersect, for otherwise a simple argument shows that the diameter of $P_1P_2P_3P_4$ would be greater than r. Connect P_i with P_j if and only if their distance is r. We distinguish two cases. In Case 1, every P_i is connected with at most two other P's. In this case the number of lines, *i.e.*, of pairs of points at distance r is clearly $\leqq n$.

* See *e.g.* P. Erdös, London Math. Soc. Journal, 1937, vol. 12, p. 133. The proof would depend on the prime number theorem for primes of the form $4k+1$ (or on some weaker elementary result concerning the distribution of primes of the form $4k+1$).

† Jahresbericht der Deutschen Math. Vereinigung, vol. 43, 1934, p. 114.

If P_1 would be connected with three vertices say P_2, P_3, P_4 where P_1P_3 is between P_1P_2 and P_1P_4 then P_3 can not be connected with any other P_i, since P_3P_i would have to intersect both P_1P_2 and P_1P_4 (the angle $P_2P_1P_4$ is of course $\leqq\pi/3$), and thus be greater than r. Now we can just omit P_3 and since both the number of points and the number of distances are reduced by 1, the proof can be completed by induction.

It would be interesting to have an analogous result for n points in k dimensional space. Vázsonyi* conjectured that in three-dimensional space the maximum distance can not occur more than $2n-2$ times.

If one could prove that in k-dimensional space the maximum distance can not occur more than kn times, the following conjecture of Borsuk would be established: *Each k-dimensional* subset of diameter 1 can be decomposed into $k+1$ summands each having diameter <1.

Let now r' denote the minimal distance between any two P's. First it is easy to see that r' can not occur more often than $3n$ times. This is immediately clear from the fact that since r' was the minimal distance between any two P's, there can be no more than 6 P's at distance r' from any given P.

Connect P_i with P_j if and only if their distance is r'. A simple argument shows that no two such lines P_1P_2 and P_3P_4 can intersect (otherwise there would be two P's at distance $<r'$). Thus the graph we obtain is planar, and from Euler's theorem it follows that the number of edges of such a graph is not greater than $3n-6$. Thus we have proved the following

THEOREM 3. *Let the maximum and minimum distances determined by n points in a plane be denoted by r and r', respectively. Then r can occur at most n times and r' at most $3n-6$ times.*

It is easy to give n points where the maximum distance occurs exactly n times. By more complicated arguments we can prove that the minimal distance r' can occur not more than $3n-cn^{1/2}$ times, where c is a constant. On the other hand the example of the triangular lattice shows that r' can occur $3n-c_1n^{1/2}$ times. I did not succeed in determining exactly how often r' can occur.

One could try to generalize Theorem 3 to higher dimensions. But already the case of three-dimensional space presents great difficulties. It would be of some interest to determine the maximum number of points on the unit sphere of k dimensions such that the distance of any two is $\geqq 1$.

* Oral communication.

SUMMABILITY OF POWER SERIES

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1. Introduction. Let b_{nk} denote a matrix of real or complex numbers. This matrix determines a series-to-sequence transformation B,

$$\sigma_n = \sum_{k=0}^{\infty} b_{nk}u_k, \tag{1}$$

by means of which a given series $u_0+u_1+\cdots$ is said to be *summable* to the value σ if the series in (1) converge and define numbers $\sigma_0, \sigma_1, \cdots$ such that

$$\lim_{n\to\infty} \sigma_n = \sigma. \tag{2}$$

The transformation B is said to be *regular* if each convergent series $\sum u_n$ is summable B to the value to which it converges.

We present four theorems characterizing the matrices b_{nk} for which the transformation B has stated properties. Theorem 1 is widely known and we shall give references and historical remarks. In particular, we shall point out that Harald Bohr originated the theorem. Theorem 2 is known. It is believed that Theorems 3 and 4 are new. Finally, in §8, it is shown that a theorem due to I. Schur [1918] is a corollary of Theorem 4. Here and hereafter, the name of an author followed by a date in square brackets indicates that the reference is to be found in the bibliography at the end of this paper.

2. Regularity. The following theorem gives the test which one applies to a matrix b_{nk} to decide whether it determines a regular series-to-sequence transformation.

THEOREM 1. *In order that the series-to-sequence transformation B be regular, it is necessary and sufficient that*

$$\lim_{n\to\infty} b_{nk} = 1 \qquad k = 0, 1, 2, \cdots \tag{3}$$

and there exist a constant M such that

$$\sum_{k=0}^{\infty} |\, b_{nk} - b_{n,k+1} \,| < M \qquad n = 0, 1, 2, \cdots. \tag{4}$$

In case $b_{nk}=0$ when $k>n$, the nonvanishing elements of the matrix

$$\begin{array}{lll} b_{00} & b_{01} & b_{02}\cdots \\ b_{10} & b_{11} & b_{12}\cdots \\ b_{20} & b_{21} & b_{22}\cdots \\ \cdots & & \end{array} \tag{5}$$

lie in a triangle and the matrix is said to be triangular. Harald Bohr [1909]

proved that if b_{nk} is a triangular matrix satisfying (3) and (4), then B is regular. This pioneering paper has never received the attention and credit it deserves. The author has never seen a reference to it in the literature of the theory of summability; he first learned of the existence of the paper through a conversation with Bohr in 1931 when the author was a Fellow at Princeton. This paper of Bohr antedates both the work of Silverman [1913], a thesis written at the University of Missouri under the direction of E. R. Hedrick and accepted by the Graduate Faculty in May 1910, and of Toeplitz [1911]. Silverman proved sufficiency, and Toeplitz proved both necessity and sufficiency, of the well known Silverman-Toeplitz conditions that a matrix a_{nk} be such that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}s_k = \lim_{n\to\infty} s_n \tag{6}$$

for each sequence s_n for which $\lim s_n$ exists. For early expositions of the Silverman-Toeplitz theory, see Carmichael [1919] and Hurwitz [1922].

The Silverman-Toeplitz theory was widely publicized for several years before serious attention was given to the series-to-sequence transformation (1). Carmichael [1919] proved sufficiency of the conditions in Theorem 1, and indicated on page 120 that both he and T. H. Hildebrandt then knew that the conditions are necessary as well as sufficient. Perron [1920] proved sufficiency of the conditions. It seems that Hahn [1922] and Takenaka [1922] were the first to publish complete proofs of Theorem 1.

The remainder of this section is inserted in response to the suggestion of a referee that further discussion and a simple example be presented for readers unfamiliar with the theory of summability. A method of summability for series is, by definition, a method for assigning values to series. *Convergence* is the standard example. A series $u_0+u_1+u_2+\cdots$ with partial sums $s_0=u_0$, $s_1=u_0+u_1$, $s_2=u_0+u_1+u_2, \cdots$ is *convergent* (or is *summable by the method of convergence*) to s if $s_n\to\infty$ as $n\to\infty$. Another simple and useful method is the method known as the *arithmetic mean method* and as the *Cesaro method* C_1 *of order* 1. A series $u_0+u_1+\cdots$ is *summable* C_1 to σ if $\sigma_n\to\sigma$ as $n\to\infty$ when

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1}.$$

The series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

is not convergent because its sequence 1, 0, 1, 0, 1, 0, $\cdots$ of partial sums does not have a limit. But the C_1 transform

$$1, \frac{1}{2}, \frac{2}{3}, \frac{2}{4}, \frac{3}{5}, \frac{3}{6}, \frac{4}{7}, \frac{4}{8}, \cdots$$

has the limit $\frac{1}{2}$ and accordingly the series is summable C_1 to $\frac{1}{2}$. This illustrates

the fact that series are summable (evaluable) by methods other than convergence.

Writing the C_1 transformation in the form

$$\sigma_n = \sum_{k=0}^{n} \frac{1}{n+1} s_k,$$

one sees that it has the form

$$\sigma_n = \sum_{k=0}^{\infty} a_{nk} s_k$$

where, for each $n=0, 1, \cdots$, $a_{nk}=1/(n+1)$ when $0 \leqq k \leqq n$ and $a_{nk}=0$ when $k>n$. Replacing s_0 by u_0, s_1 by u_0+u_1, $\cdots$ in the above sequence-to-sequence C_1 transformation and simplifying the result gives the C_1 series-to-sequence transformation

$$\sigma_n = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) u_k.$$

This has the form (1) where, for each $n=0, 1, 2, \cdots$

$$b_{nk} = \left(1 - \frac{k}{n+1}\right) \qquad 0 \leqq k \leqq n$$

and $b_{nk}=0$ when $k>n$.

3. Summability of absolutely convergent series. We shall abbreviate the statement that a series $\sum u_n$ is summable by a method B by saying that $B\{\sum u_n\}$ exists. The statement that $\sum u_n$ is summable B to the value σ will be abbreviated to $B\{\sum u_n\}=\sigma$.

Theorem 1 characterizes the matrices b_{nk} such that $B\{\sum u_n\}=\sum u_n$ whenever $\sum u_n$ converges; these matrices are the regular matrices. Since each absolutely convergent series is convergent, it is of course true that each regular matrix is such that $B\{\sum u_n\}=\sum u_n$ whenever $\sum u_n$ converges absolutely. Since there are nonabsolutely (conditionally) convergent series, one must admit the possibility that there may be nonregular matrices b_{nk} such that $B\{\sum u_n\}=\sum u_n$ whenever $\sum u_n$ converges absolutely. In fact, when one decreases the class of series for which the formula $B\{\sum u_n\}=\sum u_n$ is required to hold, one has the problem of determining whether this enlarges the class of methods B having the required property.

If B^* is the particular matrix with elements b^*_{nk} defined by

$$b^*_{nk} = \cos \frac{k}{n+1} \qquad n, k = 0, 1, 2, \cdots, \tag{7}$$

the oscillatory character of the cosine implies that (4) fails to hold and hence that B^* is not regular. However the matrix of B^* satisfies the conditions (8) and (9) below; hence, by Theorem 2, $B^*\{\sum u_n\}=\sum u_n$ whenever $\sum u_n$ converges absolutely.

THEOREM 2. *In order that the series-to-sequence transformation B be such that $B\{\sum u_n\} = \sum u_n$ whenever $\sum u_n$ converges absolutely, it is necessary and sufficient that*

$$\lim_{n\to\infty} b_{nk} = 1 \qquad k = 0, 1, 2, \cdots \tag{8}$$

and there exists a constant M such that

$$|b_{nk}| \leqq M \qquad n, k = 0, 1, 2, \cdots . \tag{9}$$

Observe that if one modifies the statement of Theorem 2 by deleting the word absolutely and replacing (9) by (4), one obtains another way of phrasing the content of Theorem 1. Theorem 2 and several related theorems were first proved by Hahn [1922]; they have been rediscovered several times by other authors who apparently were unaware of the broad scope of Hahn's work.

4. Summability of a power series inside the circle of convergence. It is a fundamental fact of the theory of power series that $\sum c_n z^n$ converges absolutely for each z_0 lying inside the circle of convergence. If we set $u_n = c_n z_0^n$, the statement that z_0 lies inside the circle of convergence of $\sum c_n z^n$ becomes equivalent to the statement that $\sum u_n z^n$ has radius of convergence greater than 1. Moreover, the question whether $B\{\sum a_n z^n\}$ exists whenever z lies inside the circle of convergence of $\sum a_n z^n$ becomes equivalent to the question whether $B\{\sum u_n\}$ exists whenever $\sum u_n z^n$ has radius of convergence greater than 1. We select the latter formulation of the question, and characterize in the next theorem (case $R=1$) the class of transformations B for which $B\{\sum u_n\}$ exists whenever $\sum u_n z^n$ has radius of convergence greater than 1. The class is larger than the class for which $B\{\sum u_n\}$ exists whenever $\sum u_n$ converges absolutely.

THEOREM 3. *Let $R \geqq 0$. In order that the series-to-sequence transformation B be such that $B\{\sum u_n\}$ exists whenever the power series $\sum u_n z^n$ has radius of convergence greater than R, it is necessary and sufficient that there exist constants $\beta_0, \beta_1, \cdots$ such that*

$$\lim_{n\to\infty} b_{nk} = \beta_k \qquad k = 0, 1, 2, \cdots \tag{10}$$

and, corresponding to each number r for which $r > R$, a constant $M(r)$ such that

$$|b_{nk}| \leqq M(r) r^k \qquad n, k = 0, 1, \cdots . \tag{11}$$

If (10) and (11) hold and $\sum u_n z^n$ has radius of convergence greater than R, then $\sum \beta_k u_k$ converges absolutely and

$$B\{\sum u_k\} = \beta_0 u_0 + \beta_1 u_1 + \cdots . \tag{12}$$

5. Proof of necessity of the conditions in Theorem 3. Let (1) be a transformation such that $B\{\sum u_n\}$ exists whenever $\sum u_n z^n$ has radius of convergence greater than R. Proof of necessity of (10) is very easy. Let q be a nonnegative integer and let $\sum u_n$ be the particular series for which $u_q = 1$ and $u_n = 0$ when

$n \neq q$. Then $\sum u_n z^n$ has radius of convergence greater than R and accordingly the transform, which is b_{nq}, must be convergent to a limit which we may call β_q. Thus $b_{nq} \to \beta_q$ as $n \to \infty$, and we have proved necessity of (10).

A condition of the nature of (11), but weaker than (11), is obtained very easily. Let r be a fixed number such that $r > R$. Since $\sum r^{-n} z^n$ has radius of convergence $r > R$, the transform

$$\sigma_n = \sum_{k=0}^{\infty} b_{nk} r^{-k} \tag{13}$$

must exist, that is, the series on the right must be convergent for each $n = 0, 1, 2, \cdots$. Since, for each n, the series $\sum b_{nk} r^{-k}$ must be convergent, the terms of the series must be bounded. Thus there is a constant $M_n(r)$, depending on n and r, such that

$$| b_{nk} r^{-k} | \leqq M_n(r) \qquad k = 0, 1, 2, \cdots . \tag{14}$$

We now suppose, intending to establish a contradiction of our hypothesis, that the stronger condition (11) does not hold. Then, for some r_1 for which $r_1 > R$, there is no constant M such that $| b_{nk} r_1^{-k} | \leqq M$ for each pair of integers n and k. From (10) it follows that, for each fixed k, the numbers $| b_{nk} r_1^{-k} |$ form a bounded sequence; and from (14) it follows that, for each fixed n, the numbers $| b_{nk} r_1^{-k} |$ form a bounded sequence. Hence the numbers $| b_{nk} r_1^{-k} |$ form an unbounded double sequence of which each row is bounded and each column is bounded. Henceforth, when α, β and γ are symbols involving subscripts, we write $b(\alpha, \beta)$ for $b_{\alpha\beta}$ and $r_1(\gamma)$ for $r_1^{-\gamma}$. Let $q_1, q_2, \cdots$ and $Q_1, Q_2, \cdots$ be sequences of positive numbers such that $q_n \to 0$ and $Q_n \to \infty$ as $n \to \infty$. Choose positive integers n_1 and k_1 such that

$$| b(n_1, k_1) r_1(k_1) | > Q_1. \tag{15}$$

Choose an integer j_2 so great that $j_2 > k_1$ and

$$\sum_{k=j(2)}^{\infty} | b(n_1, k) r_1(k) | < q_1; \tag{16}$$

this is possible since $r_1 > R$ and the series in (13) is, for each n, absolutely convergent when $r > R$. Then choose n_2 and k_2 such that $k_2 > j_2$ and

$$| b(n_2, k_2) r_1(k_2) | > \sum_{k=0}^{k(1)} | b(n_2, k) r_1(k) | + Q_2. \tag{17}$$

When $n_1, k_1, n_2, k_2, \cdots, n_{h-1}, k_{h-1}$ have been determined, choose $j_h > k_{h-1}$ so great that

$$\sum_{k=j(h)}^{\infty} | b(n_{h-1,k}) r_1(k) | < q_{h-1}. \tag{18}$$

Then choose n_r and k_r such that $k_r > j_r$, $n_r > n_{r-1}$, and

$$|b(n_h, k_h)r_1(k_h)| > \sum_{k=0}^{k(h-1)} |b(n_h, k)r_1(k)| + Q_h. \tag{19}$$

We thus define, by induction, increasing sequences $n_1, n_2, \cdots$ and $k_1, k_2, \cdots$ of integers such that (19) and, by virtue of (18),

$$\sum_{k=k(h+1)}^{\infty} |b(n_h, k)r_1(k)| < q_h \tag{20}$$

hold for each $h=1, 2, \cdots$. Let $\sum u_k$ be the special series for which $u_k = r_1(k_h)$ when $k=k_h$, $h=1, 2, 3, \cdots$ and $u_k=0$ when $k \neq k_1, k_2, \cdots$. Then $\sum u_k z^k$ has radius of convergence $r_1 > R$ and hence must, by our hypothesis, be summable B. But the transform σ_n of this series is such that, for each index h,

$$\begin{aligned} |\sigma(n_h)| &= \left| \sum_{k=0}^{\infty} b(n_h, k)u_k \right| \\ &= \left| b(n_h, k_h)r_1(k_h) + \sum_{k=0}^{k(h-1)} b(n_h, k)u_k + \sum_{k=k(h+1)}^{\infty} b(n_h, k)u_k \right| \\ &\geqq |b(n_h, k_h)r_1(k_h)| - \sum_{k=0}^{k(h-1)} |b(n_h, k)r_1(k)| - \sum_{k=k(h+1)}^{\infty} |b(n_h, k)r_1(k)| \end{aligned} \tag{21}$$

and hence, because of (19) and (20), such that

$$|\sigma(n_h)| \geqq Q_h - q_h. \tag{22}$$

Therefore $|\sigma(n_h)| \to \infty$ as $h \to \infty$, and this contradiction of the conclusion that σ_n must be convergent completes the proof of the necessity of the conditions in Theorem 3.

In case the sequence $k_1, k_2, \cdots$ becomes infinite sufficiently rapidly, the series $\sum u_n$ which we constructed to prove necessity of (11) generates a gap power series $\sum u_n z^n$ whose circle of convergence is its natural boundary. It would be interesting to have criteria that $B\{\sum u_n\}$ exists whenever $\sum u_n z^n$ generates an analytic function with singularities belonging to smaller prescribed sets.

6. Proof of sufficiency of the conditions in Theorem 3. We now assume that (10) and (11) hold, and that $\sum u_n$ is a series for which $\sum u_n z^n$ has radius of convergence r_1 greater than R. We shall show that $B\{\sum u_n\}$ exists by proving (12).

Choose r such that

$$R < r < r_1. \tag{23}$$

Then $\sum u_n r^n$ converges absolutely. Let $M(r)$ be the constant in (11). Using (11), we obtain

$$|b_{nk}u_k| \leqq |b_{nk}r^{-k}|\,|u_k r^k| \leqq M(r)\,|u_k r^k| \tag{24}$$

and hence, using (10), also

$$|\beta_k u_k| \leqq M(r)\,|u_k r^k|. \tag{25}$$

Hence $\sum b_{nk}u_k$ and $\sum \beta_k u_k$ converge absolutely. Hence the B transform σ_n of $\sum u_n$ exists. Let $\sigma=\sum \beta_k u_k$. To show that $\sigma_n \to \sigma$, let $\epsilon>0$ and choose an index q such that

$$M(r)\sum_{k=q}^{\infty}|u_k r^k| < \frac{1}{4}\epsilon. \tag{26}$$

Then choose N such that

$$\sum_{k=0}^{q-1}|b_{nk}-\beta_k||u_k| < \frac{1}{2}\epsilon \qquad n > N; \tag{27}$$

this choice of N is made possible by (10). Then

$$\begin{aligned}|\sigma_n-\sigma| &\leqq \sum_{k=0}^{\theta-1}|b_{nk}-\beta_k||u_k| + \sum_{k=q}^{\infty}|b_{nk}-\beta_k||u_k| \\ &\leqq \sum_{k=0}^{q-1}|b_{nk}-\beta_k||u_k| + \sum_{k=q}^{\infty}|b_{nk}||u_k| + \sum_{k=q}^{\infty}|\beta_k||u_k|\end{aligned} \tag{28}$$

and hence $|\sigma_n-\sigma|<\epsilon$ when $n>N$. This completes the proof of Theorem 3.

7. Conditions for $B\{\sum u_n\}=\sum u_n$. We shall show that the following theorem follows easily from Theorem 3.

THEOREM 4. *Let $R\geqq 1$. In order that the series-to-sequence transformation B be such that $B\{\sum u_n\}=\sum u_n$ whenever the power series $\sum u_n z^n$ has radius of convergence greater than R, it is necessary and sufficient that*

$$\lim_{n\to\infty} b_{nk} = 1 \qquad k = 0, 1, 2, \cdots \tag{29}$$

and that, corresponding to each number r for which $r>R$, there exist a constant $M(r)$ such that

$$|b_{nk}| \leqq M(r)r^k \qquad n, k = 0, 1, 2, \cdots. \tag{30}$$

If B has the property in question, then Theorem 3 implies that (10) and (11) must hold. Application of B to the series $0+\cdots+0+1+0+\cdots$ treated in the first paragraph of section 5 shows that (29) holds, and necessity is proved. Sufficiency of the conditions in Theorem 4 is proved by setting $\beta_0=\beta_1=\cdots=1$ in the latter part of the statement of Theorem 3.

If, in the statement of Theorem 4, the condition $R\geqq 1$ were replaced by the condition $R\geqq 0$ of Theorem 3, the result would be illusory when $0\leqq R<1$. In the first place, if $0\leqq R<1$, the hypothesis that $\sum u_n$ has radius of convergence greater than R would not imply convergence of $\sum u_n$, and accordingly there could be no B such that $\sum u_n$ is summable B to the number to which it converges. In the second place, if $0\leqq R<1$, the requirement that (30) hold for each r greater than R would be inconsistent with the requirement that (29) hold.

8. A theorem of Schur and Szász. Schur [1918] proposed the problem of proving sufficiency of the condition (32) in the following theorem, and Szász [1922] proved the theorem. Okada [1929] proved sufficiency by use of contour integration.

THEOREM 5. *In order that a sequence* $b_0, b_1, \cdots$ *be such that*

$$\lim_{n\to\infty} \frac{b_n u_0 + b_{n-1}u_1 + \cdots + b_0 u_n}{b_n} = \sum u_n \tag{31}$$

whenever $\sum u_n z^n$ *has radius of convergence greater than* 1, *it is necessary and sufficient that*

$$\lim_{n\to\infty} b_{n-1}/b_n = 1. \tag{32}$$

The condition (32) implies and is implied by the set of conditions

$$\lim_{n\to\infty} b_{n-k}/b_n = 1 \qquad k = 0, 1, 2, \cdots. \tag{33}$$

To show that Theorem 5 is a corollary of Theorem 4, let

$$b_{nk} = b_{n-k}/b_n \qquad 0 \leqq k \leqq n \tag{34}$$

and $b_{nk}=0$ when $k>n$. By Theorem 4, case $R=1$, (31) holds for the stated class of series if and only if (33) holds and there corresponds to each θ in the interval $0<\theta<1$ a number $M(\theta)$ such that

$$\left| (b_{n-k}/b_n)\theta^k \right| \leqq M(\theta) \qquad 0 \leqq k \leqq n. \tag{35}$$

However, as we shall show, (32) implies (35); hence (32) alone serves as a necessary and sufficient condition. Let $b_{n-1}/b_n \to 1$, let $0<\theta<1$, and let $x_n=(b_{n-1}/b_n)\theta$. Then $x_n\to\theta$. Hence there is a number $A(\theta)$ such that $|x_n|<A(\theta)$ for all n. Moreover there is at most a finite set of values of n for which $|x_n|>1$; let $B(\theta)$ denote the number of such n's. Then, when $0\leqq k\leqq n$,

$$\left| (b_{n-k}/b_n)\theta^k \right| = \left| x_{n-k+1}x_{n-k+2}\cdots x_n \right| \leqq [A(\theta)]^{B(\theta)} \tag{36}$$

and (35) holds when $M(\theta)$ is the last member of (36). Thus (32) implies (35) and Theorem 5 is proved.

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AN INTERPOLATION FORMULA

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1. Introduction and preliminary material. In almost any discussion of interpolation formulas a certain collection of formulas are derived which express the interpolated value of a function in terms of certain of its values and its differences. These standard formulas are all expressions for the polynomial which coincides with the given function at certain values of the independent variable, and therefore must be transformable into each other. The purpose of this note is to show that they are all in fact instances of one and the same formula; that is, it is possible to write and prove a formula involving a number of parameters which, for various choices of the parameters, reduces to various standard formulas. Since the proof of this general formula is quite simple, one thus obtains the standard set of interpolation formulas (for functions tabulated at equal intervals, in terms of differences) without much effort.

In order to avoid questions of convergence, we shall suppose that a finite number of values of a function $F(z)$ are given corresponding to equally spaced values of the argument z, say $z=a, a+h, \cdots, a+mh$. Let $f(z)$ be the polynomial which coincides with $F(z)$ at these tabulated values. Our problem is to find convenient expressions for the polynomial $f(z)$ in terms of the tabulated values and their differences. We shall not discuss how well $f(z)$ represents the original function $F(z)$.

If we make the transformation $z=a+xh$, the polynomial $f(z)$ transforms into $g(x)=f(a+xh)$, and the tabulated values correspond to consecutive integral values of x. Thus our problem takes the form: Given the values, for $m+1$ consecutive integers, of a polynomial of degree m, to find convenient expressions for $g(x)$ in terms of the tabulated values and their differences. This form of the problem is notationally more convenient than the previous form, and is no less general

We shall use the notation of forward differences. We define $\Delta^{(0)}g(x)$ to be $g(x)$ for each integer x, $\Delta^{(1)}g(x)=g(x+1)-g(x)$, and by induction

$$\Delta^{(k)}g(x) = \Delta^{(k-1)}g(x+1) - \Delta^{(k-1)}g(x). \tag{1}$$

A set of polynomials useful to us is defined by the equations

$$P_k(x) = x(x-1)(x-2)\cdots(x-k+1)/k!, \qquad k = 1, 2, \cdots,$$

(2) $$P_0(x) = 1,$$

$$P_k(x) = 0, \qquad k = -1, -2, \cdots.$$

These evidently satisfy

(3) $$P_k(x) = 0 \qquad (k > 0,\ x = 0, \cdots, k-1).$$

Another useful property of these polynomials, which will now be established, is

(4) $$\Delta^{(1)}P_k(x) = P_{k-1}(x).$$

If $k>0$, by (1) and (2)

$$\begin{aligned}\Delta^{(1)}P_k(x) &= P_k(x+1) - P_k(x)\\ &= [(x+1)(x)\cdots(x-k+2) - x(x-1)\cdots(x-k+1)]/k!\\ &= kx(x-1)\cdots(x-k+2)/k!\\ &= P_{k-1}(x),\end{aligned}$$

proving (4) for $k>0$. If $k\leqq 0$, $P_k(x)$ is constant, so its first difference is zero, and $k-1$ is negative, so $P_{k-1}(x)$ is also zero. Thus (4) holds for all integers k.

By repeated applications of (4), we find that for all nonnegative integers j the equation

(5) $$\Delta^{(j)}P_k(x) = P_{k-j}(x) \qquad (k = 0, \pm 1, \pm 2, \cdots)$$

is satisfied.

2. A lemma equivalent to Sheppard's Rule. It is now easy to establish the following lemma.

LEMMA. *Let $c_0, c_1, c_2, \cdots$ be a sequence of integers such that each difference $c_{h+1}-c_h$ $(h=1, 2, \cdots)$ is either 0 or 1. Let $g(x)$ be a polynomial, and m its degree. Then for all x*

(6) $$g(x) = \sum_{j=0}^{m} P_j(x + c_j)\Delta^{(j)}g(-c_{j+1}).$$

Let $g(x)$ be written in the form

(7) $$g(x) = a_0 + a_1P_1(x + c_1) + \cdots + a_mP_m(x + c_m).$$

This is evidently possible; the coefficient of x^m on the right is $a_m/m!$, and this can be made equal to the coefficient of x^m in the left member by proper choice of a_m. Transposing the term $a_mP_m(x+c_m)$ to the left and repeating the argument gives the value of a_{m-1}, and so on. Taking kth differences of both members of (7) yields, with (5),

(8) $$\Delta^{(k)}g(x) = \sum_{j=0}^{m} a_jP_{j-k}(x + c_j).$$

In the sum on the right, all terms with $j<k$ vanish by (2). The term with $j=k$ is a_k, also by (2). In (8) we now set $x=-c_{k+1}$. By the choice of the integers c_h, each term of the sum

$$(c_j - c_{j-1}) + (c_{j-1} - c_{j-2}) + \cdots + (c_{k+2} - c_{k+1}) \qquad (j > k+1)$$

is either zero or 1, so that

$$0 \leqq c_j - c_{k+1} \leqq j - k - 1 \qquad (j > k).$$

But this, together with (3), implies that

$$P_{j-k}(-c_{k+1}+c_j) = 0 \qquad (j > k).$$

Thus all that remains of (8) is

$$\Delta^{(k)}g(-c_{k+1}) = a_k,$$

and (7) takes the form (6), establishing the lemma.

Equation (6) can be stated as a simple rule for writing a class of interpolation formulas. Write down the successive differences of $g(x)$, the difference of order 0 (that is, g itself) being chosen on any line and each succeeding difference being either on the same line as its predecessor or else on the line above. Multiply each $\Delta^{(k)}g$ by a coefficient whose denominator is $k!$ and whose numerator is the product of k factors, the first being $x-$(argument of $\Delta^{(k-1)}g$) and the others decreasing by steps of 1. This is essentially the same as Sheppard's rule.* Among others, it yields the Newton-Gregory "forward" and "backward" formulas and the Newton-Gauss "forward" and "backward" formulas, as is known and will again be shown in the collection of special cases following the next theorem.

3. A more general formula. Although the lemma yields several interesting special cases, there are others which it does not cover, so we now proceed to establish a more general formulation.

THEOREM. *Let $c_{-1}, c_0, c_1, \cdots$ be a sequence of integers such that each difference $c_{h+1}-c_h$ is either 0 or 1. Let $g(x)$ be a polynomial, and m its degree. Then the equation*

$$(9) \qquad g(x) = \sum_{j=0}^{m} [P_j(x+c_j)+y_jP_{j-1}(x+c_{j-1})][\Delta^{(j)}g(-c_{j+1})-y_{j+1}\Delta^{(j+1)}g(-c_{j+2})]$$

is valid for all numbers $x, y_0, y_1, \cdots, y_{m+1}$ such that no two consecutive y_j are both different from zero.

Upon expansion of its right member, equation (9) becomes

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* H. Freeman, An Elementary Treatise on Actuarial Mathematics, page 73.

$$g(x) = \sum_{j=0}^{m} P_j(x + c_j)\Delta^{(j)}g(-c_{j+1})$$
$$+ y_0 P_{-1}(x + c_{-1})g(-c_1) - y_{m+1}P_m(x + c_m)\Delta^{(m+1)}g(-c_{m+2})$$
$$+ \sum_{j=1}^{m} y_j[-P_{j-1}(x + c_{j-1})\Delta^{(j)}g(-c_{j+1}) + P_{j-1}(x + c_{j-1})\Delta^{(j)}g(-c_{j+1})]$$
$$- \sum_{j=0}^{m} y_j y_{j+1} P_{j-1}(x + c_{j-1})\Delta^{(j+1)}g(-c_{j+2}).$$

But $P_{-1}(x+c_{-1})$ is zero by definition (2), $\Delta^{(m+1)}g(x)=0$ because $g(x)$ is of degree m, and $y_j y_{j+1}=0$ by the hypotheses of the theorem. Also, each term of the second summation in the right member of the expanded form of equation (9) vanishes identically. Hence the y_j all drop out of formula (9), and the theorem is seen to be equivalent to the lemma, which has already been established.*

4. Applications. We now proceed to write some special cases of equation (9). If we choose all $c_h=0$, then by the lemma or the theorem with all $y_j=0$ we find

$$g(x) = g(0) + x\Delta^{(1)}g(0) + \cdots + [x(x-1)\cdots(x-k+1)/k!]\Delta^{(k)}g(0) + \cdots, \tag{10}$$

which is the Newton-Gregory formula, or more specifically the Newton-Gregory "forward" formula. If we choose $c_h=h-1$, by the lemma or by the theorem with all $y_j=0$

$$g(x) = g(0) + x\Delta^{(1)}g(-1) + \cdots + [(x+k-1)\cdots(x)/k!]\Delta^{(k)}g(-k) + \cdots, \tag{11}$$

which is the Newton-Gregory "backward" formula.

If we choose $c_{2k}=c_{2k+1}=k$, equation (6), or equation (9) with all y_j set equal to 0, yields

$$g(x) = g(0) + x\Delta^{(1)}g(-1) + [(x+1)x/2!]\Delta^{(2)}g(-1) + \cdots + [(x+k)\cdots(x-k+1)/(2k)!]\Delta^{(2k)}g(-k) + [(x+k)\cdots(x-k)/(2k+1)!]\Delta^{(2k+1)}g(-k-1) + \cdots, \tag{12}$$

which is the Newton-Gauss "backward" formula.

The remaining five formulas which we shall discuss all are based on the choice

$$c_{2k-1} = c_{2k} = k - 1. \tag{13}$$

Substitution in (6) yields

* This arrangement of the proof was suggested by the referee.

$$
\begin{aligned}
g(x) &= g(0) + x\Delta^{(1)}g(0) + [x(x-1)/2!]\Delta^{(2)}g(-1) + \cdots \\
&\quad + [(x+k-1)\cdots(x-k)/(2k)!]\Delta^{(2k)}g(-k) \\
&\quad + [(x+k)\cdots(x-{}^{x})/(2k+1)!]\Delta^{(2k+1)}g(-k) + \cdots,
\end{aligned} \tag{14}
$$

which is the Newton-Gauss "forward" formula.

In order to apply (9) with non-zero values of y_j, it is convenient first to perform a certain amount of algebraic manipulation. The first of the following four formulas holds for $k>0$, the others for $k\geqq 0$.

$$
\begin{aligned}
&P_{2k}(x+k-1) + y_{2k}P_{2k-1}(x+k-1) \\
&\qquad = (x+k-1)\cdots(x-k+1)[x-k+2ky_{2k}]/(2k)!,
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&P_{2k+1}(x+k) + y_{2k+1}P_{2k}(x+k-1) \\
&\qquad = (x+k-1)\cdots(x-k)[x+k+(2k+1)y_{2k+1}]/(2k+1)!,
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&\Delta^{(2k)}g(-k) - y_{2k+1}\Delta^{(2k+1)}g(-k) \\
&\qquad = -y_{2k+1}\Delta^{(2k)}g(-k+1) + (1+y_{2k+1})\Delta^{(2k)}g(-k),
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&\Delta^{(2k+1)}g(-k) - y_{2k+2}\Delta^{(2k+2)}g(-k-1) \\
&\qquad = (1-y_{2k+k})\Delta^{(2k+1)}g(-k) + y_{2k+2}\Delta^{(2k+1)}g(-k-1).
\end{aligned} \tag{18}
$$

The first two of these follow at once from (2); the second two are immediate consequences of the definition (1).

The c_k being as in (13), let us select $y_{2k}=1/2$, $y_{2k+1}=0$. With the help of (15) and (18), (9) reduces to

$$
\begin{aligned}
g(x) &= g(0) + x[\Delta^{(1)}g(0) + \Delta^{(1)}g(-1)]/2 + \cdots \\
&\quad + \frac{(x+k-1)\cdots(x-k+1)[x]}{(2k)!}\Delta^{(2k)}g(-k) \\
&\quad + P_{2k+1}(x+k)[\Delta^{(2k+1)}g(-k) + \Delta^{(2k+1)}g(-k-1)]/2 \\
&\quad + \cdots \\
&= g(0) + \cdots + \frac{x^2(x^2-1^2)\cdots(x^2-\overline{k-1^2})}{(2k)!}\Delta^{(2k)}g(-k) \\
&\quad + P_{2k+1}(x+k)[\Delta^{(2k+1)}g(-k) + \Delta^{(2k+1)}g(-k-1)]/2 \\
&\quad + \cdots,
\end{aligned} \tag{19}
$$

which is Stirling's formula.

If we select $y_{2k}=0$, $y_{2k+1}=-1/2$, with the help of (16) and (17) equation (9) reduces to

$$
\begin{aligned}
g(x) &= [g(0) + g(1)]/2 + [x-1/2]\Delta^{(1)}g(0) + \cdots \\
&\quad + P_{2k}(x+k-1)[\Delta^{(2k)}g(-k) + \Delta^{(2k)}g(-k+1)]/2 \\
&\quad + \frac{[x-1/2](x+k-1)\cdots(x-k)}{(2k+1)!}\Delta^{(2k+1)}g(-k) + \cdots,
\end{aligned} \tag{20}
$$

which is Bessel's formula.

If we select $y_{2k}=0$, $y_{2k+1}=-(x+k)/(2k+1)$, the right member of (16) vanishes, so the terms in (9) corresponding to odd j all disappear, and (9) takes the form

$$\begin{aligned} g(x) &= xg(1) + (1-x)g(0) + \cdots \\ &\quad + P_{2k}(x+k-1)\left[\frac{(x+k)}{2k+1}\Delta^{(2k)}g(-k+1)\right. \\ &\quad \left. + \left\{1 - \frac{x+k}{2k+1}\right\}\Delta^{(2k)}g(-k)\right] \\ &\quad + \cdots . \end{aligned} \tag{21}$$

This expansion can be given a more symmetric form if we first establish the identity

$$P_n(x+m) = (-1)^n P_n(n-m-1-x). \tag{22}$$

The truth of this statement is easily verified, since the factors on the left are the negatives of those on the right taken in the reversed order. Using this,

$$\begin{aligned} P_{2k}(x+k-1)\left\{1 - \frac{x+k}{2k+1}\right\} &= -P_{2k+1}(x+k-1) \\ &= +P_{2k+1}(k+1-x). \end{aligned}$$

It is customary to introduce a symbol, say ξ, for $1-x$. If this is done, equation (21) becomes

$$\begin{aligned} g(x) &= xg(1) + \xi g(0) + \cdots \\ &\quad + P_{2k+1}(x+k)\Delta^{(2k)}g(-k+1) + P_{2k+1}(\xi+k)\Delta^{(2k)}g(-k) \\ &\quad + \cdots . \end{aligned} \tag{23}$$

This is Everett's formula.

Finally we select $y_{2k+1}=0$, $y_0=0$, $y_{2k}=(k-x)/2k$. By (15), all terms in (9) corresponding to even values of $j>0$ disappear, and (9) takes the form

$$\begin{aligned} g(x) = g(0) + \sum_{k=0} P_{2k+1}(x+k)\Bigg[&\left\{1 - \frac{k+1-x}{2k+2}\right\}\Delta^{(2k+1)}g(-k) \\ &+ \frac{k+1-x}{2k+2}\Delta^{(2k+1)}g(-k-1)\Bigg]. \end{aligned} \tag{24}$$

With (22), this becomes

$$\begin{aligned} g(x) = g(0) + \sum_{k=0} [&P_{2k+2}(x+k+1)\Delta^{(2k+1)}g(-k) \\ &- P_{2k+2}(k+1-x)\Delta^{(2k+1)}g(-k-1)]. \end{aligned} \tag{25}$$

This is Steffensen's formula.

PRIME-REPRESENTING FUNCTIONS

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If $f(x)$ has the property that $f(n)$ is a prime number for $n=1, 2, \cdots$, then $f(x)$ is called a prime-representing function. If further we require that there be an infinite number of *distinct* primes among the values $f(n)$, then we may call $f(x)$ a *proper* prime-representing function. There are analytic functions having this property. However, there is reason to believe that no simple function, finitely expressible, has this property. Fermat conjectured falsely that $f(x)=2^{2^x}+1$ was such a function. It is well known that no polynomial can represent primes, unless it is constant and therefore improper. The object of this note is to offer a simple proof that no rational function can represent primes.

LEMMA. *If $P(x)$ is a polynomial, and $P(n)=A_n+\gamma_n$ where A_n is integral and* $\lim \gamma_n=0$, *then for large n, all the γ_n vanish.*

Let $P(x)$ be of degree $m-1$, so that $\Delta^m P(x)\equiv 0$. Putting $x=n$, we have $\Delta^m\gamma_n=-\Delta^m A_n$, for each n. Since A_n is integral, so is $\Delta^m A_n$ and therefore $\Delta^m\gamma_n$. Writing this as $\Delta^m\gamma_n=\sum_0^m\binom{m}{k}(-1)^{m-k}\gamma_{n+k}$, and using $\lim \gamma_n=0$, we have $\lim_{n\to\infty}\Delta^m\gamma_n=0$. But, $\Delta^m\gamma_n$ is integral and consequently for some n_0, we have $\Delta^m\gamma_n=0$ for all $n>n_0$. Now set

$$Q(x)=\sum_{k=0}^{\infty}\frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}\Delta^k\gamma_{n_0}.$$

This is a polynomial of degree m at most, for if $k\geqq 1$,

$$\begin{aligned}\Delta^{m+k}\gamma_{n_0} &= \Delta^k\Delta^m\gamma_{n_0}\\ &= \sum_{\lambda=0}^{k}\binom{k}{\lambda}(-1)^{k-\lambda}\Delta^m\gamma_{n_0+k}\\ &= 0.\end{aligned}$$

Then, $Q(n)=\gamma_{n+n_0}$ and since $\lim \gamma_n=0$, we must have $Q(x)\equiv 0$; thus, we have proved that $\gamma_n=0$ for $n>n_0$.

THEOREM. *If a rational function takes prime values at the integers, it must be constant.*

Let $R(x)=P(x)+g(x)/h(x)$ with $P(x)$, $g(x)$, $h(x)$ polynomials, and the degree of $g(x)$ less than that of $h(x)$. Let $R(n)=A_n$, a prime, for each $n\geqq 1$. If we set γ_n equal to $g(n)/h(n)$, then $\lim \gamma_n=0$. Then, we have $P(n)=A_n-\gamma_n$. Applying the lemma, we have $\gamma_n=0$ for large n, and thus $P(n)=A_n$, a prime, for all $n>n_0$. But, as we have noted before, this is impossible unless $P(x)$ is a constant. Finally, since $g(n)/h(n)=0$ for $n>n_0$, we see that $R(x)$ reduces to a constant.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

The River Mathematics. By A. Hooper. Henry Holt and Co., 1945. 8+401 pages. $3.75.

"The object of this book," writes the author in his opening line, "is to trace in a simple way the development of mathematical ideas and processes from their insignificant source to the mighty flood on which now sail the proud ships of Science, Engineering, and Aeronautics, ..." And again, a few lines later, "Both the casual visitor and more serious newcomer will find in this book a bird's-eye view of mathematical ideas."

Unfortunately the reader, whether casual or serious, will find in this book little more than a revised and enlarged version of the author's earlier and popular *Mathematics Refresher*. And the material of that book, successful though it may have been in the urgent task of providing many young men with a stock of elementary manipulative skills, is hardly the stuff to use in tracing the development of mathematical ideas.

Many of the chapters start in a relatively promising way with a discussion of the derivation of terms and with similar matters of general historical interest. But most of them, after too few pages of exposition of this sort, revert to collections of miscellaneous facts, rules, and illustrative examples, much of which material—even to certain identical methods of presentation and identical examples—is to be found in the *Refresher*. (It may be added that the educational effectiveness of the earlier work is lost, in the present volume, through the omission of exercises and self-tests.)

The reason for the inclusion of some of the new material is not clear. Chapter 14, for example, is an abbreviated conventional presentation of the analytic geometry of the straight line and the conic sections. It contributes no more to the development of mathematical ideas than does any ordinary text which includes the obvious applications of the conics to searchlights, telescopes, bridges, and such. In the same way, the all too brief and again conventional treatment of the calculus in Chapter 15 does little more than any ordinary text to increase the general reader's understanding of the development, nature, and significance of that branch of mathematics.

Many exciting promises are made in the opening chapter, which discusses in attractive detail the objectives of the book. But the remaining chapters leave these promises largely unfulfilled. Indeed, there are several good books, already on the market, which come closer than this book comes to meeting the stated objectives.

E. P. NORTHROP

An Introduction to Mathematics for Teachers. By L. E. Boyer. New York, Henry Holt and Co., Inc., 1945. 17+478 pages. $3.25.

One of the commonest problems of those who prepare teachers of elementary and secondary school mathematics is that the teacher candidates are found to have a very limited understanding of the subjects they hope to teach. Students may exhibit considerable skill in algebraic manipulation, formal differentiation, and integration, and yet be pitifully unprepared to explain "what mathematics is all about" to pupils in the lower schools. The author of this book has sought to help in solving this problem by providing under one cover several different sorts of material which should be mastered by teachers. In the preface he says: "The purpose of this book is to provide prospective elementary and secondary school teachers . . . some means of learning about the nature, significance and use of mathematics from early times to the present." Although apparently intended for use in colleges, it presupposes only the background possessed by students who "ended their formal study of mathematics in grade nine . . . ."

This is not a textbook for courses on methods of teaching. The content includes topics from arithmetic, elementary algebra, geometry, and trigonometry. The book exemplifies the tendency of recent books on general mathematics at the college level to provide generous expository discussion. Both the historical setting and practical applications of many of the topics are developed in the text. The exercises for students are numerous and varied for this sort of book, and the practical type of problem is more plentiful than usual. Although it is not entirely free from questionable statements and minor blemishes of style, none will seriously interfere with the usefulness of the book. On the whole, the author has done an interesting and effective piece of writing for the type of student he has in mind.

The popularity of books of this kind seems to be growing rapidly. In the case of this text, nearly all of the content is definitely of the secondary school level, and ideally prospective teachers (and perhaps college students in general) should be expected to be quite familiar with the ideas before they *enter* college. Normally the notions would be developed slowly and study would be extended over several years of high school. To achieve so many objectives in a single course in college is a formidable task. At present, far too many students have little understanding of some of these topics when they are graduated from college and are permitted to teach. When this situation has ultimately been remedied, in part by courses based on books like this one, there will be a better chance for the high schools to do this job properly. The colleges may then devote their efforts to a broader range of content and to a treatment of many of the same ideas suitable for students of greater mathematical maturity.

M. L. HARTUNG

Curso de Análisis Matemático. Tomo I. Por Cristóbal de Losada y Puga. Lima, Universidad Católica del Perú, 1945. 26+632 pp. $8.00.

This is the first volume of a treatise which the author intends to be comparable with the classical French *Traité d'Analyse.* At the same time, he has attempted to organize it in such a way that selected portions can be used for a first course in calculus. To this end he has adopted an approach in which a number of topics are treated from an elementary and from an advanced point of view in consecutive sections. The North American reader will be particularly interested in the work as a source of alternative approaches to the teaching of familiar topics. It is salutary to realize that neither the arrangement of material, the method of proof, nor the illustrative examples have to be those to which one has become accustomed.

This volume is divided into four parts: Introduction, Differentiation, Integration, Introduction to Differential Equations. The first part begins with a rapid introduction to the number concept and point sets. Chapters on variables and functions and on limits follow; these are elementary in character, though going beyond most North American introductions to the calculus in proving, for example, that the limit usually taken as the definition of e actually exists. A supplementary chapter discusses continuous functions from the advanced point of view, with careful proofs.

The second part discusses differentiation of the elementary functions, properties of derivatives, and infinitesimals, in the same fashion. (It is unfortunate that the author gives Duhamel's theorem in the traditional fallacious form.) The usual applications of differentiation (tangents, maxima and minima, velocities) follow; the exposition is not particularly detailed. Functions of several variables and partial derivatives are studied next, with a careful discussion of continuity and a proof of the implicit function theorem.

The third part opens with two chapters on integration, containing the topics customary in a first course in calculus. The elegant geometric interpretation of integration by parts was new to the reviewer. A supplementary chapter proves an existence theorem for the Riemann integral, introduces the Stieltjes integral, and gives a brief account of Lebesgue integration. It seems unfortunate that the author does not give more explanation of why these more general integrals are useful; it is to be hoped that he will do so in a later volume in connection with the study of series. A chapter on systematic integration culminates in a detailed presentation of elliptic integrals and their transformation. A chapter on applications of integration contains, besides the customary material, a discussion of the length of a rectifiable curve as the limit of inscribed polygons.

The fourth part carries the elementary theory of differential equations as far as the solution of the inhomogeneous linear equation with constant coefficients. Emphasis is placed on the geometric interpretation of a differential equation and on physical applications (outside of dynamics: the author believes that dynamical applications are better left until the student has begun a systematic study of mechanics).

In this first volume, the author has not gone into as great detail as the treatises which he has taken as his inspiration, but his exposition is more leisurely. In general, he has succeeded admirably in his aim of combining clarity with rigor. There are numerous historical remarks, and extended discussion of such interesting historical topics as Archimedes' quadrature of the parabola and Wren's rectification of the cycloid. The book is well printed, and the numerous diagrams deserve special mention for their high quality.

R. P. BOAS, JR.

Higher Algebra for Schools. By W. L. Ferrar. Oxford University Press, 1945. 7+214 pages. $3.75.

The subject matter of this book corresponds roughly to that treated by numerous American textbooks bearing the title College Algebra. However, the book differs from our college algebras in that it presents fewer topics but develops them at greater length.

Fully one-third of the book is devoted to the theory of polynomials and equations. There are two chapters on graphs, one on mathematical induction, and two chapters on infinite series with emphasis on the binomial series and the exponential and logarithmic functions. The concluding two chapters on linear equations and determinants do not go beyond determinants of order 4, but include such advanced topics as the rule for differentiating a determinant and the rule for multiplying two determinants.

It seems odd to an American that an algebra book which presupposes a knowledge of differentiation and integration of easy functions should omit Demoivre's theorem and allied topics. Again, the theory of permutations and combinations is treated briefly, and then primarily for the purpose of presenting the combinatorial proof of the binomial theorem; and there is no mention of probability. One also misses Descartes' rule and Horner's method, although Newton's method is included. These topics will presumably be treated in a subsequent volume promised by the author.

The book contains an abundance of illustrative examples and exercises involving polynomial identities, undetermined coefficients, relations among binomial coefficients, factorization of determinants and many formal devices. An ambitious student will find these problems a valuable aid in cultivating algebraic techniques. Some of the exercises are labeled "hard" and others "harder." In the reviewer's opinion many of the exercises not so labeled would drive our average freshman to despair. But this remark is no reflection on the author, who has written an excellent book for a selected group of British students whose mathematical preparation and traditions are different from those of our freshmen.

LOUIS WEISNER

NEW BOOKS RECEIVED

Analytic Geometry. Third Edition. By F. S. Nowlan. New York, McGraw-Hill Book Co., 1946. 14+355 pages. $2.75.

Elements of Calculus. By W. A. Granville, P. F. Smith, and W. R. Longley. Boston, Ginn and Co., 1946. 11+549 pages. $3.75.

The Parallel Postulate of Euclid. By G. H. Draper. Arlington, Draper, 1946. 8 pages. $0.25.

A Collection of Papers in Memory of Sir William Rowan Hamilton. Scripta Mathematica Studies, No. 2. New York, Scripta Mathematica, 1945. 82 pages. $1.00.

International Payments; A Science. By R. O. Hall. Washington, D. C., Storage Bookshop, 1946. 11+124 pages.

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PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 721. *Proposed by Joseph Rosenbaum, Bloomfield, Conn.*

Given two equal regular dodecagons. Show how to dissect one of them into twelve congruent pieces which can be fitted to the other to form another larger regular dodecagon.

E 722. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find a square of four digits such that if one interchanges the two middle digits the new number is a square for radix 9.

E 723. *Proposed by A. W. Goodman, Republic Aviation Corporation*

Let A and B be any real quantities, and let k be a positive integer. Show that

$$\left|\frac{\cos kB \cos A - \cos kA \cos B}{\cos B - \cos A}\right| \leqq k^2 - 1, \tag{1}$$

$$\left|\frac{\cos kB - \cos kA}{\cos B - \cos A}\right| \leqq k^2, \tag{2}$$

equality holding only in the limit as B and A approach zero.

E 724. *Proposed by N. J. Fine, University of Pennsylvania, and Ivan Niven, Purdue University*

Define an n-admissible number k as one such that an n-dimensional cube may be subdivided into k cubes. Prove that for each n there exists an integer A_n such that all integers exceeding A_n are n-admissible.

E 725. *Proposed by Henry Scheffé, Princeton, N. J.*

If two events have probabilities p_1 and p_2 $(0<p_i<1)$, and correlation coefficient ρ, show that the range of possible ρ is the interval $-f(a_1a_2) \leqq \rho \leqq f(a_1/a_2)$, where $a_i=[p_i/(1-p_i)]^{1/2}$ and $f(x)=\min(x, 1/x)$. The correlation coefficient between two events E_1 and E_2 may be defined as that between x_1 and x_2, where $x_i=1$ if E_i happens and $x_i=0$ if not $-E_i$ happens.

SOLUTIONS

Iterated Factorials

E 684 [1945, 395]. *Proposed by Paul Erdös, Stanford University*

Prove that $n!^{(n-1)!}$ divides $(n!)!$; and that if n is not a power of a prime, $n!^{n(n-2)!}$ divides $(n!)!$.

Solution by Leo Moser, University of Toronto. The number of times that a prime p divides $m!$ is given by $\sum_{i=1}^{\infty}[m/p^i]$.

For the first part all we need show is that $\sum[n!/p^i] \geqq (n-1)!\sum[n/p^i]$, which is clearly true.

In the second part, since n is not a power of a prime, the number of times that p divides $n!$ is the same as the number of times that p divides $(n-1)!$. Hence $\sum[n/p^i]=\sum[(n-1)/p^i]$. We therefore have

$$\sum[n!/p^i] \geqq n(n-2)!\sum[(n-1)/p^i] = n(n-2)!\sum[n/p^i],$$

which proves the theorem.

The first part may be generalized as follows: *If* $ab \leqq n!$, *then* $a!^b$ *divides* $(n!)!$. The second part may be generalized in this way: *If none of* $n, n-1, n-2, \cdots, n-k+1$ *is a power of a prime, then* $n!^{n!/(n-k)}$ *divides* $(n!)!$. The proofs are similar to those given above.

Also solved by the proposer.

The Careless Professor

E 686 [1945, 457]. *Proposed by E. D. Schell, Arlington, Virginia*

The professor writes out a series of positive terms, $\sum a_n$, on the blackboard. Because of his carelessly written addition signs, his students consider instead the product $\prod a_n$. Unfortunately the misunderstanding cannot be discovered even though he calls out the partial sum as he writes each term. Given that a_1 and a_2 are integers, find the first five terms.

Solution by Roy Dubisch, Montana State University. Since a_1 and a_2 are integers, the relation $a_1+a_2=a_1a_2$ implies that $a_1=a_2=2$. Then $a_1+a_2+a_3=a_1a_2a_3$ implies that $a_3=4/3$. Similarly, we obtain $a_4=16/13$ and $a_5=256/217$.

While it is easy to show that the numerator of a_n $(n>1)$ is given by 2^t, $t=2^{n-2}$, it appears more difficult to construct a formula for the denominator of a_n. Even an iterative formula would seem to involve the partial sums (or products).

Also solved by D. W. Alling, Murray Barbour, W. G. Brady, R. E. Crane, J. H. Cross, Monte Dernham, M. P. de Regt, George Grossman, V. L. Klee, Jr., Margaret Olmstead, P. A. Piza, A. P. Rhodes, E. P. Starke, W. R. Talbot, F. Underwood, R. H. Urbano, Maud Willey, L. B. Zeropa, and the proposer.

Dernham and Underwood observed that for $n>1$, $\{a_n\}$ is a monotone decreasing sequence with limit 1.

A Fermagoric Triangle

E 688 [1945, 457]. *Proposed by P. A. Pizá, San Juan, Puerto Rico*

Consider the right triangle ABC with sides $a=12$, $b=5$, $c=13$. $(a^2+b^2=c^2.)$ Keeping the base BC fixed, displace the vertex A without altering the perimeter (*i.e.*, along an ellipse with foci B and C) until the value of $\cos A$ is reduced from 5/13 to 7/38. Show that the sides of the new triangle satisfy the relation

$$a^3 = b^3 + c^3.$$

I. *Solution by F. L. Celauro, Lehigh University.* The perimeter of the new triangle is $a+b+c$, or 30. Since $a=12$, then $c=18-b$. Using the law of cosines, $\cos A=(b^2+c^2-a^2)/2bc$, with $\cos A=7/38$, we have

$$7/38 = [b^2 + (18 - b)^2 - 12^2]/2b(18 - b).$$

There are two solutions to this equation, $b=9\pm\sqrt{5}$. The corresponding values of c are $9\mp\sqrt{5}$. Since

$$12^3 = (9 \pm \sqrt{5})^3 + (9 \mp \sqrt{5})^3$$

the sides of the new triangle satisfy the required relation.

II. *Solution by Maud Willey, Denton, Texas.* In the new triangle $a=12$, $b+c=18$, $\cos A=7/38$, and (by the law of cosines)

$$a^2 = (b + c)^2 - 2bc(\cos A + 1).$$

Substituting the known values and solving for bc we find $bc=76$. Now

$$b^3 + c^3 = (b + c)^3 - 3bc(b + c)$$
$$= 18^3 - 3\cdot 76\cdot 18 = 12^3 = a^3.$$

Also solved by Frank Arena, LeRoy Babcock, Murray Barbour, James Carlisle, W. B. Clarke, M. P. de Regt, Archibald Henderson, Sam Kravitz, Albert Medwid, C. T. Oergel, Margaret Olmstead, A. P. Rhodes, J. T. Rodriguez, F. W. Saunders, E. D. Schell, E. S. Smith, Mary Sultzer, W. R. Talbot, J. A. Tierney, F. Underwood, R. H. Urbano, Jeanette Van Os, Hazel Wilson, B. F. Yanney, and the proposer.

Henderson observed that the new triangle is "Glenie's Triangle," as treated by J. Glenie in *The Antecedental Calculus* (London, 1793). As further references he gave L. E. Dickson's *History of the Theory of Numbers*, II, p. 546, and p. 33 of the proposer's recent book *Fermagoric Triangles*.

The proposer pointed out that in his book *Fermagoric Triangles* he establishes an identity equivalent to

$$(2rs)^3 = (r^2 + \sqrt{r(4s^3 - r^3)/3})^3 + (r^2 - \sqrt{r(4s^3 - r^3)/3})^3.$$

For $r=3$, $s=2$ we obtain the triangle considered in the problem.

The proposer also pointed out that if BC is kept fixed and the vertex A moved along the ellipse so that $\cos A$ is successively reduced from 5/13 to $(\sqrt{200340}-360)/(810-\sqrt{200340})$, $(\sqrt{62856}-234)/(324-\sqrt{62856})$, 7/38 we obtain three triangles in which, respectively, $a^5=b^5+c^5$, $a^4=b^4+c^4$, $a^3=b^3+c^3$. It is easily shown that these triangles are constructible by euclidean tools. We thus have here, along with the given right triangle, a set of isoperimetric "fermagoric" triangles, of orders 2, 3, 4, 5, having a common base BC, and all of them constructible by euclidean tools.

Four Collinear Points

E 689 [1945, 457]. *Proposed by Morgan Ward, California Institute of Technology*

Let A, B, C, D be four collinear points in the order written, and let P be any other point in space. Prove that the inequality

$$PA + PD \geqq PB + PC$$

holds for all positions of P if and only if $AB=CD$.

Solution by E. J. Stulken, Geophysical Service Inc., Dallas, Texas. Consider $AB=CD$. Designate the midpoint of AD by O and extend PO its own length to the point P_1. Then $PA=P_1D$, $PB=P_1C$, $PC=P_1B$, $PD=P_1A$. If P is on A, D, or on AD extended, $PA+P_1A=PB+P_1B$. If P is elsewhere, $PA+P_1A>PB+P_1B$. Therefore $PA+PD\geqq PB+PC$.

With $AB\neq CD$ and P beyond the longer segment on AD extended, $PA+PD<PB+PC$.

Also solved by Murray Barbour, J. B. Kelly, V. L. Klee, Jr., and the proposer.

An Extension of E 614

E 690 [1945, 457]. *Proposed by F. J. Duarte, Caracas, Venezuela*

If p is a prime number greater than 3, prove that the polynomial

$$(x + y)^p - x^p - y^p$$

is divisible by $x^3+2x^2y+2xy^2+y^3$. (*Cf.* E 614 [1944, 591].)

Solution by F. W. Saunders, student, University of North Carolina. It was proved by C. D. Olds (*cf.* E 614 [1944, 591]) that $(x+y)^p-x^p-y^p$ is divisible by x^2+xy+y^2 for $p\equiv 1$ or 5 (mod 6).

Now, for every prime $p>3$, we have $p\equiv 1$ or 5 (mod 6). Also

$$x^3 + 2x^2y + 2xy^2 + y^3 = (x + y)(x^2 + xy + y^2).$$

Moreover, $(x+y)^p-x^p-y^p$ is always divisible by $x+y$ for odd p. Therefore, since $x+y$ is prime to x^2+xy+y^2, the theorem follows.

Also solved by E. D. Schell, E. P. Starke, and the proposer. P. A. Piza gave the following reference from Vol. II of Dickson's *History of the Theory of Numbers:* "G. Heppel [Math. Quest. Educ. Times, 40, 1884, 124] proved that, if n is a prime >3, $(x+y)^n-x^n-y^n$ is divisible by $nxy(x+y)(x^2+xy+y^2)$ and found the coefficients of the general term of the quotient."

Arrow over Bridge

E 691 [1945, 516]. *Proposed by E. K. Paxton, Washington and Lee University*

The top of a certain bridge is 215 feet above the ground and 100 feet wide. Neglecting air resistance, find the minimum initial velocity with which an arrow is shot in order to go over the bridge. At what distance from the base must the archer stand? What is the angle of elevation? How high does the arrow go? (Assume the arrow leaves the bow at a height of 5 feet above the ground.)

I. *Solution by G. A. Williams, Oregon State College.* Assuming the plane of the trajectory to be perpendicular to the bridge and taking as axes the horizontal line in this plane 5 feet above the ground and the vertical line at the center of the bridge, the equation of the trajectory is

$$y = v_0^2 (\sin^2 \alpha)/2g - (x^2 g \sec^2 \alpha)/2v_0^2. \tag{1}$$

For $x=50$ we have $y=210$. Substituting these values in (1) and rearranging terms gives v_0 as a function of α:

$$v_0^4 \sin^2 \alpha - 420gv_0^2 - 2500g^2 \sec^2 \alpha = 0. \tag{2}$$

Differentiating with respect to α and setting $dv_0/d\alpha=0$, gives

$$2v_0^4 \sin \alpha \cos \alpha - 5000g^2 \sec^2 \alpha \tan \alpha = 0,$$

or

(3) $$v_0 = 5\sqrt{2g}\sec\alpha.$$

Substituting this in (2) and solving for $\tan\alpha$ gives

(4) $$\tan\alpha = \sqrt{47/5}, \quad \text{or} \quad \alpha = 71°56.1'.$$

Substituting from (4) in (3) we find

(5) $$v_0 = 2\sqrt{130g}.$$

Substituting (4), (5), and $x=0$ in (1) we obtain $y=235$ feet. Substituting (4), (5), and $y=0$ in (1) we obtain $x=10\sqrt{235}$. Hence the archer must stand $(10\sqrt{235}-50)$ feet from the base of the bridge.

II. *Solution by Frank Hawthorne, Columbia University.* As a consequence of the law of conservation of energy it can easily be shown that the velocity of a projectile at any point is that due to a fall from the directrix of the parabolic path. (See, *e.g.*, p. 206 of Jeans' Theoretical Mechanics.) Consider the plane of the projectile and let O_1 and O_2 be the edges of the bridge in this plane. The focus F of the parabola is somewhere on the vertical bisector of O_1O_2. The distance of the directrix above O_1 and O_2 is equal to the distance O_1F (or O_2F). By the opening sentence, in order to minimize the initial velocity, we must take the directrix as low as possible. This will occur when F is the midpoint of O_1O_2. We thus have the following construction:

With centers at O_1 and O_2 describe circles of 50 ft. radius, tangent to each other at F. F is the midpoint of O_1O_2. Draw the common tangent AB above the two circles. Draw a horizontal line DE parallel to O_1O_2 and 210 ft. below it. With center F and radius 260 ft. strike an arc intersecting DE at P. Draw PC perpendicular to AB. P is the point at which the arrow leaves the bow; F and AB are the focus and directrix of the parabolic path.

(1) The minimum initial velocity is the velocity of free fall from 260 ft., *i.e.*, $\sqrt{520g}=129$ ft./sec. approximately. ($g=32$ ft./sec^2.)

(2) The horizontal distance from P to a point directly beneath the center of the bridge is $\sqrt{260^2-210^2}=153$ ft. approximately.

(3) The initial direction of the arrow is along the bisector of angle FPC. The angle of elevation is arc sin $210/260+\frac{1}{2}$ arc cos $210/260=71°\ 56'$ approximately.

(4) The highest point on the trajectory is half way between F and the line AB, or 25 ft. above the bridge, or 240 ft. above the ground under the bridge.

Also solved by D. W. Alling, G. W. Grotts, Norman Miller, S. T. Parker, Herbert Reisman, A. Sisk, E. S. Smith, R. H. Urbano, and the proposer.

As a corollary to his method Hawthorne gave the following interesting construction. To find the path, for minimum initial velocity, which passes through two given points, simply divide the line joining the two points into two parts such that the lower part is h longer than the upper part, where h is the difference in height of the two points. This gives the focus F. The directrix is horizontal above the points at a distance from each equal to their respective distances from F.

The proposer furnished the following background for this problem. "The dimensions of the bridge are those of the Natural Bridge of Virginia. Some months ago Mr. Earl Chapin May, New York City author, on a visit to the Bridge related an incident recorded in his father's diary in the year 1858. According to Mr. May, his father was traveling with a road circus through this section in that year and states that one Charlie Tubbs, a strong man of the show, threw a stone over Natural Bridge. This statement immediately brought up the question of the physical possibility of such a feat."

A Property of an Odd Number of Consecutive Integers

E 692 [1945, 516]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider $2p+1$ consecutive integers. Show that: (1) the product of the first p of them diminished by the product of the last p gives an integer divisible by $p(p+1)$; (2) the product of the first p increased by the product of the last p gives an integer divisible by $p(p+1)$, unless $p+1$ is a prime factor of the remaining integer of the set.

(This is a generalization of problem 203 by Fitz-Patrick, p. 32 of G. de Longchamps' *Exercises d'arithmétique.*)

Solution by E. M. Beesley, University of Nevada. For $p \geqq 1$ and any integer k, let

$$A_k = \prod_{j=1}^{p} (k + j), \qquad B_k = \prod_{j=1}^{p} (k + p + 1 + j).$$

The following relations will be established:

(1) $A_k \pm B_k \equiv 0 \pmod{p}$

(2) $A_k - B_k \equiv 0 \pmod{p+1}$

(3) $A_k + B_k \equiv 0 \pmod{p+1}$, unless $p+1$ is an odd prime and $k \equiv 0 \pmod{p+1}$.

Since any collection of p consecutive integers must have a multiple of p as a factor, (1) is evident. If $k \not\equiv 0 \pmod{p+1}$, both A_k and B_k have a multiple of $p+1$ as a factor. Hence (2) and (3) are valid for such k. If $k \equiv 0 \pmod{p+1}$, then

$$k + j \equiv k + p + 1 + j \equiv j \pmod{p + 1},$$

and consequently $A_k + B_k \equiv p! \pmod{p+1}$. This completes the verification of (2). Moreover

$$A_k + B_k \equiv 2(p!) \pmod{p + 1},$$

and since $2(p!) \equiv 0 \pmod{p+1}$ if and only if $p+1$ is not an odd prime, this establishes (3).

Also solved by D. W. Alling, Murray Barbour, P. T. Bateman, B. A. Hausmann, Z. I. Mosesson, P. A. Pizá, E. P. Starke, and the proposer.

An Anharmonic Ratio in Space

E 693 [1945, 516]. *Proposed by N. A. Court, University of Oklahoma*

Through a given point to draw a line meeting three planes in three points so that the anharmonic ratio of the four points shall have a given value.

Solution by the Proposer. Let a line m through the given point P meet the two given planes α, β in the points Q, R, and let S be the point on m such that $(PQRS)=k$, where k is the given value of the anharmonic ratio.

The plane δ determined by the point S and the line $\alpha\beta$ cuts the third given plane γ along a line u. A line joining P to any point of u satisfies the conditions of the problem. The required lines thus form a flat pencil in the plane P-u and having P for center.

Limit of a Recursive Sequence

E 694 [1945, 516]. *Proposed by J. D. Bankier, McMaster University*

A sequence $\{x_n\}$ is defined recursively, in terms of two numbers x_0 and x_1, by the formula

$$x_n = \frac{n-1}{n} x_{n-1} + \frac{1}{n} x_{n-2}.$$

To what value does the sequence converge?

Solution by G. T. Williams, student, Harvard University. We write the equation as

$$x_n - x_{n-1} = -(x_{n-1} - x_{n-2})/n.$$

Then, by iteration,

$$x_n - x_{n-1} = (-1)^n(x_0 - x_1)/n!.$$

Summing both sides,

$$x_n - x_0 = (x_0 - x_1)\sum_1^n (-1)^n/n!$$

and

$$x_n \to x_1 + (x_0 - x_1)/e.$$

Also solved by D. W. Alling, Murray Barbour, P. T. Bateman, E. M. Beesley, Barney Bissinger, Harley Flanders, N. G. Gunderson, J. E. Hanson, J. B. Kelly, H. A. Luther, Norman Miller, Leo Moser, Z. I. Mosesson and M. J. Singer (jointly), S. T. Parker, A. S. Peters, E. P. Starke, Morgan Ward, J. G. Wendel, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4202. *Proposed by Vladimir Karapetoff, New York, N. Y.*

In a certain game of chance, consecutive numbers from 1 to n are written on a table. The same number n of discs are provided with consecutive numbers written on them. The discs are turned with the numbers down so that the players cannot see the numbers written on them. A player covers all the numbers on the table with the discs at random, because he does not see the numbers on them. The discs are then turned over and the score is made on the basis of the number of discs whose numbers agree with the numbers on the table which they are covering. It is required to deduce an expression for the chance that k of the n discs covered the correct numbers.

4203. *Proposed by N. J. Fine, Indianapolis, Ind.*

This is a generalization of E 651 [1945, 42]. If one is allowed n weighings on a beam balance, what is the maximum number A_n of coins, exactly one of which is bad, from which one can isolate the bad coin and determine whether it is heavy or light? (Cf. E 712 [1946, 156].)

4204. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The four products of the three sides of faces of a tetrahedron $ABCD$ are proportional to the tangents of the half angles of the corresponding cones of revolution inscribed in the trihedral angles at the vertices.

4205. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If a right triangle has sides of integral lengths and the sum of the sides forming the right angle is a square, then the sum of the cubes of these two sides is the sum of two squares. Can the hypotenuse be a square?

SOLUTIONS

Feuerbach Points

3945 [1940, 115]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If in a triangle the distances d_1, d_2, d_3 of the midpoints of the sides of the triangle to a tangent to the nine-point circle satisfy a relation $\sqrt{d_1} \pm \sqrt{d_2} \pm \sqrt{d_3} = 0$, the point of contact of that tangent is one of the Feuerbach points.

Solution by R. Goormaghtigh, Bruges, Belgium. According to Godt's theorem (*Münchener Berichte*, 1896), the Feuerbach points of a triangle are the four points of the nine-point circle such that the distance of any of them to the midpoint of one of the sides equals the sum of its distances to the midpoints of the two other sides.

But, if d_1 is the distance of the midpoint A_m of the side BC of a triangle ABC to the tangent at a point T to the nine-point circle, then

$$d_1 = \overline{TA}_m^2/R$$

R being the circumradius of ABC. The proposed theorem is therefore another form of Godt's theorem.

Infinite Series and Products Deranged

4138 [1944, 533]. *Proposed by G. Pólya, Stanford University*

Given two positive integers p and q define

$$S_{p,q} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2p} - \frac{1}{3} - \frac{1}{5} - \cdots - \frac{1}{2q+1} + \frac{1}{2p+2} + \cdots + \frac{1}{4p} - \frac{1}{2q+3} - \cdots - \frac{1}{4q+1} + \cdots,$$

$$P_{p,q} = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{2p}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots \cdot\left(1-\frac{1}{2q+1}\right)\left(1+\frac{1}{2p+2}\right)\cdots.$$

We obtain the series in which blocks of p positive terms alternate with blocks of q negative terms by rearranging the terms of the well known series

$$\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \cdots = S_{1,1} = 1 - \log 2,$$

and the product by rearranging correspondingly

$$(1 + \tfrac{1}{2})(1 - \tfrac{1}{3})(1 + \tfrac{1}{4})(1 - \tfrac{1}{5}) \cdots = P_{1,1}.$$

Show directly that

$$P_{p,q} = (p/q)^{1/2}$$

and hence derive the well known result

$$S_{p,q} - S_{1,1} = \log (p/q)^{1/2}.$$

Solution by the Proposer. 1. Define

$$F_n = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{6}\right)\cdots\left(1+\frac{1}{2n}\right).$$

Then, by Wallis' formula, or Stirling's formula,

$$F_n = \frac{3}{2}\,\frac{5}{4}\,\frac{7}{6}\cdots\frac{2n+1}{2n} = \frac{(2n+1)!}{(n!2^n)^2} = \frac{2n+1}{2^{2n}}\binom{2n}{n} \sim \frac{2n^{1/2}}{\pi^{1/2}}$$

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\cdots\left(1-\frac{1}{2n+1}\right)$$

$$= \frac{2}{3}\,\frac{4}{5}\,\frac{6}{7}\cdots\frac{2n}{2n+1} = \frac{1}{F_n}$$

and the partial product of $P_{p,q}$ with subscript $(p+q)n$ is

$$\frac{F_{pn}}{F_{qn}} \sim \left(\frac{pn}{qn}\right)^{1/2}.$$

This last limit can also be found by more elementary methods, without using Wallis' or Stirling's formula.

2. The series

$$\log P_{1,1} - S_{1,1} = [\log(1+\tfrac{1}{2}) - \tfrac{1}{2}] + [\log(1-\tfrac{1}{3}) + \tfrac{1}{3}] + [\log(1+\tfrac{1}{4}) - \tfrac{1}{4}] + \cdots$$

is absolutely convergent since

$$|\log(1-x) + x| = \left|-\frac{x^2}{2} - \frac{x^3}{3} - \cdots\right| < \frac{|x|^2}{2(1-|x|)} \leqq |x|^2$$

for $|x| \leqq 1/2$. The value of an absolutely convergent series is not altered by a rearrangement of the terms, and so

$$\log P_{p,q} - S_{p,q} = \log P_{1,1} - S_{1,1}.$$

Editorial Note. The functions F_n and $1/F_n$ are related to the definite integrals

$$I_m = \int_0^{\pi/2} \sin^m x\,dx, \qquad I_0 = \pi/2, \qquad I_1 = 1,$$

where m is an integer greater than unity. Integration by parts gives

$$I_m = \frac{m-1}{m}\,\frac{m-3}{m-2}\cdots\frac{m-(2k-1)}{m-2(k-1)}\,I_{m-2k}.$$

From the relations $I_{2m+1} < I_{2m} < I_{2m-1}$, we find that

$$\lim_{m\to\infty} I_{2m}/I_{2m-1} = 1.$$

We now have

$$I_{2np} = F_{np}\frac{I_0}{2np+1}, \qquad I_{2np-1} = \frac{1}{F_{np}}\frac{2np+1}{2np},$$

$$I_{2np}/I_{2np-1} = (F_{np})^2 I_0\, 2np/(2np+1)^2,$$

$$\frac{I_{2np}/I_{2np-1}}{I_{2nq}/I_{2nq-1}} = \left(\frac{F_{np}}{F_{nq}}\right)^2 \frac{p}{q}\left(\frac{2nq+1}{2np+1}\right)^2,$$

$$P_{pq} = \lim_{n\to\infty}\frac{F_{np}}{F_{nq}} = \sqrt{\frac{p}{q}}. \tag{1}$$

In order to derive Wallis' formula we consider

$$I_{2m}/I_{2m-1} = I_0/(I_{2m-1})^2 2m$$

$$= I_0\frac{1\cdot 3}{2\cdot 2}\frac{3\cdot 5}{4\cdot 4}\cdots\frac{(2m-3)\cdot(2m-1)}{2(m-1)\cdot 2(m-1)}\frac{2m-1}{2m}.$$

Hence

$$\frac{\pi}{2} = \prod_{m=1}^{\infty}\frac{2m\cdot 2m}{(2m-1)\cdot(2m+1)}.$$

See Goursat-Hedrick, *Mathematical Analysis*, vol. 1, p. 240.

An elementary proof of (1) not involving I_m is as follows. It will suffice to consider $p>q$, then by reduction we have

$$F_{np}/F_{nq} = \prod_{j=1}^{n(p-q)}\left[1+\frac{1}{2(j+nq)}\right],$$

$$\log(F_{np}/F_{nq}) - \sum_{j=1}^{n(p-q)} 1/2(j+nq)$$

$$= \sum_{j=1}^{n(p-q)}\left\{\log\left[1+\frac{1}{2(j+nq)}\right] - \frac{1}{2(j+nq)}\right\}.$$

The absolute value of the summand $\{\cdots\}$ is less than $1/8(1+nq)^2$, and hence the absolute value of the last sum is less than $n(p-q)/8(1+nq)^2$. Thus

$$\lim_{n\to\infty}\log\left(\frac{F_{np}}{F_{nq}}\right) = \frac{1}{2}\lim_{n\to\infty}\sum_{j=1}^{n(p-q)}\frac{1}{j+nq}.$$

In order to find this limit, divide the interval on the x axis from the origin to $(p-q)/q$ into $n(p-q)$ equal subintervals of length $1/qn$, and at the end of the jth interval erect the ordinate $1/(j+nq)$. We now have

$$\lim_{n\to\infty}\sum_{j=1}^{n(p-q)} 1/(j+nq) = \int_0^{(p-q)/q}\frac{dx}{1+x} = \log\left(\frac{p}{q}\right).$$

Hence

$$P_{pq} = \lim_{n\to\infty}\frac{F_{np}}{F_{nq}} = \sqrt{\frac{p}{q}}.$$

A direct proof of the formula for S_{pq} is as follows. Again taking $p>q$ set

$$T_{np} = \sum_{j=1}^{np} \frac{1}{2j}, \qquad T_{nq} = \sum_{j=1}^{nq} \frac{1}{2j+1},$$

Then

$$T_{np} - T_{nq} = \sum_{j=1}^{nq} \left(\frac{1}{2j} - \frac{1}{2j+1}\right) + \frac{1}{2} \sum_{j=1}^{n(p-q)} \frac{1}{j+nq},$$

$$\lim_{n\to\infty} (T_{np} - T_{nq}) = S_{11} + \log \sqrt{p/q},$$

$$S_{pq} = S_{11} + \log \sqrt{p/q}, \tag{2}$$

the log term being derived as before, and the terms of $T_{np}-T_{nq}$ may be rearranged so that we have first the first p positive terms of S_{11}, then next the first q negative terms, the next set of p positive terms, and so on exhausting all of the $n(p+q)$ terms.

Two Cubes and a Square

4140 [1944, 534]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find an integer of two digits such that the difference between its cube and that of the integer whose digits are those of the first in reversed order is a perfect square.

Solution by Monte Dernham, San Francisco. We have

$$(10t + u)^3 - (10u + t)^3 = A^2. \tag{1}$$

This may be written

$$3(t - u)[37(t + u)^2 - 27tu] = (A/3)^2. \tag{2}$$

Hence either $t+u$ or $t-u$ is a multiple of 3, which serves to eliminate from consideration all but 16 pairs of two digit integers. Further discarding those the difference of whose cubes terminates in 2, 3, 7 or 8, we are restricted to one or more of the following integers

$$87, 82, 81, 74, 72, 63, 54, 51, 42, \tag{3}$$

an examination of which discloses a single solution,

$$74^3 - 47^3 = 549^2.$$

Solved also by D. H. Browne, M. L. Constable, B. A. Hausmann, Irving Kaplansky, J. B. Kelly, A. D. Maxwell, W. C. Rufus, E. P. Starke, R. H. Wilson, Jr. and the proposer.

Editorial Note. It will be seen that the case $u=0$ must be rejected since 37 is a prime, and this is also easily seen without using this primality. Tests of the numbers in (3) can be easily applied without use of tables for cubes or squares as follows. We take outside the $[\cdots]$ in (2) the common factors of the two terms inside leaving the remaining factor c. Then we have for the two factors:

87, 3^3, $c \equiv 1 \bmod 3$; 82, $3^2 2^3$, $c \equiv 1 \bmod 2$; 81, $3^4 7$, $c \equiv 5 \bmod 7$;
74, 3^2, $c = 61^2$; 72, $3^4 5$, $c \equiv -3 \bmod 5$; 63, 3^6, $c = 31$;
54, 3^4, $c = 13 \cdot 7$; 51, $3^3 4$, $c \equiv 1 \bmod 3$; 42, $3^3 2^3$, $c = 31$.

Hence the only solution is for 74 as stated above. A later note from Dernham eliminated from (3) the cases 87, 51, 42.

Classes of Infinite Series

4142 [1944, 593]. *Proposed by G. Pólya, Stanford University*

Find a sequence of real numbers $a_1, a_2, a_3, \cdots$ so that $\sum_1^\infty a_n$ converges, $\sum_1^\infty a_n^3$ diverges, $\sum_1^\infty a_n^5$ converges. More generally, let C be an arbitrarily given (finite or infinite) class of positive integers. There exists a sequence of real numbers $a_1, a_2, \cdots, a_n, \cdots$ adapted to C so that, for $l = 1, 2, 3, \cdots$, the series

$$a_1^{2l-1} + a_2^{2l-1} + \cdots + a_n^{2l-1} + \cdots$$

converges or diverges according as l does or does not belong to C.

Solution by N. J. Fine, Indianapolis, Ind. For a given positive odd integer w we shall construct a sequence of real numbers c_i with the following properties:

(1) $$S_w(k) \equiv \sum_{i=1}^{\infty} c_i^k \quad \text{converges for all positive odd } k \neq w,$$

(2) $$S_w(w) \equiv \sum_{i=1}^{\infty} c_i^w \quad \text{diverges.}$$

Throughout this discussion k will be restricted to be a positive odd integer.

The c_i will be constructed in blocks $B_n(w)$ of the form

$$\begin{aligned} B_n(\omega) = {} & A_1 n^{-1/w}, \cdots, A_1 n^{-1/w} && (k_1 \text{ terms}) \\ & A_2 n^{-1/w}, \cdots, A_2 n^{-1/w} && (k_2 \text{ terms}) \\ & \cdots\cdots\cdots\cdots\cdots\cdots && \cdots\cdots \\ & A_r n^{-1/w}, \cdots, A_r n^{-1/w} && (k_r \text{ terms}), \end{aligned}$$

where $r = (w+1)/2$ and the A_i and k_i are to be determined subject to the conditions that

(1) k_i are positive integers

(2) $$|A_i| \leqq 1, \qquad A_i \neq A_j \quad \text{if} \quad i \neq j$$

(3) $$\sum_{i=1}^{r} k_i A_i^k = 0 \quad \text{if} \quad k < w$$

(4) $$\sum_{i=1}^{r} k_i A_i^w = \Delta \neq 0.$$

These conditions may be satisfied by first choosing the A_i as any set of distinct rationals not greater than unity, solving the system of equations (3) and (4) for k_i in the form $k_i=\Delta D_i/D$, where D is the determinant of the system, choosing Δ as a common multiple of all the denominators of D_i/D expressed as fractions, and finally changing the signs of all the negative k_i by changing the signs of the corresponding A_i. These steps are all possible since the D_i and D are Vandermonde determinants and hence are non-zero rationals. With conditions (1) $\cdots$ (4) satisfied it is easy to verify that the required properties 1 and 2 belong to the sequence c_i.

Now, given an arbitrary class M of positive odd integers $\{w_j\}$, we construct a series $S(k)$ which diverges if $k\in M$ and converges otherwise. Let C_j be a bound for the partial sums of $S_{w_j}(k)$ for all $k\neq w_j$; this bound exists since the $|A_i|$ were chosen $\leqq 1$, and $S_{w_j}(k)$ converges absolutely and uniformly for $k>w_j$. Now write down the double series in which the terms of the jth row are $2^{-j}C_j^{-1}$ multiplied by the terms of $S_{w_j}(k)$, and sum this series by diagonals to give the required series. As an example we construct the series S_3:

$$1^{-1/3}-\tfrac{1}{2}\cdot 1^{-1/3}-\tfrac{1}{2}\cdot 1^{-1/3}+2^{-1/3}-\tfrac{1}{2}\cdot 2^{-1/3}-\tfrac{1}{2}\cdot 2^{-1/3}+\cdots$$
$$+n^{-1/3}-\tfrac{1}{2}n^{-1/3}-\tfrac{1}{2}n^{-1/3}+\cdots.$$

Eight-Digit Squares

4144 [1944, 593]. *Proposed by V. Thébault, Tennie, Sarthe, France*

For what base is a number of eight digits of the form *ababcdcd* the square of a four digit number *mnmn*, given that *a b c d m n* is a permutation of six consecutive digits?

Solution by E. P. Starke, Rutgers University. Let R be the base and, for brevity, put $A=aR+b$, $C=cR+d$, $M=mR+n$. We have the

$$AR^6+AR^4+CR^2+C=M^2(R^2+1)^2$$

or

$$AR^2-A+(A+C)/(R^2+1)=M^2.$$

Now the fact that $0<A+C<2R^2$ implies that the integer $(A+C)/(R^2+1)$ is unity. Thus we have

$$AR^2-A+1=M^2, \qquad A+C=R^2+1.$$

Adopting the following procedure, we may easily test any proposed value of R: determine the values of M for which

$$A=(M^2-1)/(R^2-1) \tag{1}$$

is an integer, find $C=R^2+1-A$, and compare the resulting digits.*

For $M<R^2$, R^2-1 divides M^2-1 for $M=1$, R, R^2-R-1, R^2-2. The corre-

* There must be no duplicate digits, and the difference between the greatest and smallest of the six digits must be 5.

sponding values of A are 0, 1, R^2-2R, R^2-3; and $C=R^2+1$, R^2, $2R+1$, 4. The first two sets must be discarded because $C \geqq R^2$. For the last two we easily compute $m=R-2$, $R-1$; $n=R-1$, $R-2$; $a=R-2$, $R-1$; so that in each case there are duplicate digits. Furthermore, $M^2-1 \equiv 0 \pmod{R^2-1}$ has no other solutions than the values of M listed above, when $R-1$ and $R+1$ are primes or powers of primes (thus there is no solution for $R=10$.)

If R is odd we have, besides the above, the following values:

$$
\begin{aligned}
M &= \tfrac{1}{2}(R^2-2R-1), \quad \tfrac{1}{2}(R^2+2R-1), \quad \tfrac{1}{2}(R^2-3), \quad \tfrac{1}{2}(R^2+1);\\
A &= \tfrac{1}{4}(R^2-4R+3), \quad \tfrac{1}{4}(R^2+4R+3), \quad \tfrac{1}{4}(R^2-5), \quad \tfrac{1}{4}(R^2+3);\\
C &= \tfrac{1}{4}(3R^2+4R+1), \quad \tfrac{1}{4}(3R^2-4R+1), \quad \tfrac{3}{4}(R^2+3), \quad \tfrac{1}{4}(3R^2+1).
\end{aligned} \tag{2}
$$

When reduced to base R, the digits of the numbers in the first column above are

$$m=\tfrac{1}{2}(R-3),\ n=\tfrac{1}{2}(R-1),\ a=\tfrac{1}{4}(R-5),\ b=\tfrac{1}{4}(R+3),\ c=\tfrac{1}{4}(3R+1),\ d=\tfrac{1}{4}(3R+1);$$
$$m=\tfrac{1}{2}(R-3),\ n=\tfrac{1}{2}(R-1),\ a=\tfrac{1}{4}(R-7),\ b=\tfrac{3}{4}(R+1),\ c=\tfrac{3}{4}(R+1),\ d=\tfrac{1}{4}(R+1);$$

according as $R=4x+1$ or $4x-1$. When a similar break-down is made for the other values of M, A, C, it is easily seen that the difference between the smallest and greatest digits exceeds 5 whenever $R>9$. Obviously $R>5$ if there exist six consecutive digits. Trying $R=7$, 9 we obtain just two solutions with no duplicate digits, both of which come from the fourth column of (2) and in which A, M, C form an arithmetic progression. The solutions are

$$\text{base } 7,\ 16165252 = (3434)^2,$$
$$\text{base } 9,\ 23236767 = (4545)^2.$$

For other solutions, if they exist, R^2-1 must admit further factorization than appears above, but with larger R and increasing range of digits the chance diminishes that all six digits be consecutive. Further trials have yielded no more solutions, and a general treatment seems out of the question.

Malfatti Problem

4145 [1945, 47]. *Proposed by Howard Eves, College of Puget Sound*

Find the positions of three non-overlapping circles in a triangle which have a maximum combined area.

Remarks by the Proposer. In 1803 Malfatti proposed the problem of cutting three circular cylindrical holes out of a solid right triangular prism in such a way that the cylinders and the prism have the same altitude and that the combined volumes of the cylinders be a maximum. Malfatti intuitively reduced this problem to another, now commonly referred to as "Malfatti's problem": To inscribe within a triangle three circles each of which shall be tangent to the other two and to two sides of the triangle. This reduced problem has enjoyed a long history and has been solved and generalized in many ways. Apparently

Malfatti considered this reduced problem as equivalent to the original problem. That such is not the case was indicated in 1929, when Lob and Richmond pointed out that in the case of the equilateral triangle the inscribed circle and two of the smaller circles that can be fitted into the corners have a combined area which is greater than that given by the Malfatti arrangement, the ratio of the areas of the two respective sets of circles being $22(2+\sqrt{3})/81>1$.

That the Malfatti arrangement certainly cannot always lead to a maximum combined area is more easily and more strikingly seen by considering an isosceles triangle with unit base and very long altitude, h, on this base. Then, letting A be the combined area of the Malfatti circles, we have

$$\lim_{h\to\infty} A = 3\pi/8.$$

On the other hand, if A' is the combined area of three circles fitted one above the other and tangent to the legs of the triangle, then

$$\lim_{h\to\infty} A' = 3\pi/4.$$

Clearly, then, the original and the reduced Malfatti problems are not equivalent, and the problem stated at the head of these remarks is raised. The analogous problem of finding the positions of four non-overlapping spheres inside a tetrahedron and having a maximum combined volume is also raised.

Note. Solutions of 4145 are still invited.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

A conference on algebra will be held at the University of Chicago during the week of July 15–19, 1946. Some of the sessions will emphasize the recent contacts of algebra with other branches of mathematics such as topology, function theory, and geometry. It is hoped that the meetings will be held in Eckhart Hall. Reservations should be made directly with Chicago hotels. Hotels in the vicinity of the University of Chicago are the Broadview, 5540 South Hyde Park Blvd.; Del Prado, 5307 South Hyde Park Blvd.; Shoreland, 5454 South Shore Drive, Windermere, 1642 East 56th Street; Mira-Mar, 6218 South Woodlawn. Because of the acute shortage of hotel space, reservations should be made as soon as possible.

Dr. Theodore Hailperin has been appointed to an assistant professorship at Lehigh University.

W. I. Layton of Amarillo College has been appointed Professor of Mathematics and Head of the Department at Austin Peay State College, Clarksville, Tennessee.

Dr. P. V. Reichelderfer of Ohio State University and Assistant Professor George Whaples of the University of Pennsylvania have been appointed to assistant professorships at the University of Wisconsin.

Dr. M. F. Smiley has returned to Lehigh University as an associate professor.

The following appointments to instructorships are announced:
Michigan State College: Stanley Johnson, Mrs. Madelyn Kintner, Miss J. E. Master, O. T. McMillan, Nicholas Musselman, Frank Saidel.
University of Houston: Dr. Albert Newhouse

Associate Professor Emeritus Joseph Allen of the College of the City of New York died March 4, 1946. He was a charter member of the Association.

Professor Emeritus Arthur Korn of Stevens Institute of Technology died December 21, 1945.

Professor Emeritus T. A. Pierce of the University of Nebraska died August 18, 1945.

R. G. M. Sabel of Bristol High School, Bristol, Connecticut, died February 24, 1946.

Professor Althéod Tremblay of Laval University, Quebec, died March 1, 1946.

SUMMER COURSES

The following institutions announce courses in mathematics for the summer of 1946:

The Catholic University of America. From July 1 to August 8 the following advanced courses will be offered: By Professor Finan: theory of equations, fundamentals of mathematics. By Professor Ramler: advanced Euclidean geometry, synthetic projective geometry, differential equations. By Professor Rice: solid analytic geometry, calculus of observations.

Teachers College, Columbia University. From July 8 to August 16 the following courses in the teaching of mathematics will be offered: By Professor Fawcett: a one-week course on the teaching of geometry. By Professor Fehr: teaching algebra in secondary schools, professionalized subject matter in advanced secondary mathematics. By Dr. Lazar: teaching geometry in secondary schools, history of mathematics. By Mr. Mirick: elementary mechanics (statics), observation and participation in the teaching of geometry. By Professor Reeve: teaching and supervision of mathematics—junior high school, teaching and supervision of mathematics—senior high school. By Professor Schlauch: modern business arithmetic. By Professor Shuster: teaching arithmetic in elementary school,

field work in mathematics. In addition, on consecutive Thursdays beginning on July 11, there will be five special lectures and discussions relating to the reorganization and teaching of mathematics in the post-war world.

The University of Southern California. From June 24 to August 2 the following advanced courses will be offered: By Dr. Hammer: calculus of observations. By Professor Hyers: theory of functions of a complex variable. By Professor Snapper: theory of equations and determinants, matrix theory. By Professor Steed: differential equations. By Professor White: introduction to higher geometry. By Professor Ulam: theory of probability and statistics.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

IMPORTANT LEGISLATION BEFORE CONGRESS

For over a year Congress has been debating various bills proposing the creation of a National Science Foundation or a National Research Board for the encouragement of a more extensive, national program of scientific research. The first bill to appear was S. 825, introduced by Senator Byrd on April 4, 1945. It proposed the establishment of a Research Board for National Security as an independent government agency. A somewhat similar bill was introduced into the House by Representative May on June 11, 1945, but in that measure it was proposed that the Research Board be set up by the National Academy of Sciences in cooperation with the War and Navy Departments. On July 9, 1945, Senator Fulbright proposed S. 1248, which would set up within the Department of Commerce a Bureau of Scientific Research. The proposed Bureau would absorb the Office of Production Research and Development of the War Production Board and the National Inventors Council. On July 19, Senator Magnuson introduced S. 1285, which provided for the creation of a National Science Foundation, and generally followed the major recommendations of Vannevar Bush's Report to the President. Four days later Senator Kilgore introduced S. 1297, which differed from S. 1285 in proposing that a National Science Foundation be administered by a Director rather than by a Board of unpaid, part-time scientists. In addition, S. 1297 included the social sciences within the proposed program and provided for some new patent regulations.

Approximately 100 scientists and a few laymen participated in hearings upon the various bills during August, September, and October, 1945. Opinions were obtained from university, industrial, and government scientists, as well as from agencies and businesses whose chief concern is the application of science. All

but one witness favored the establishment of a National Science Foundation.

The president of the National Academy of Sciences opposed the creation of a National Science Foundation, as proposed, partly because of his belief that politics could not be eliminated from the distribution of funds set aside for scientific research. A majority of witnesses favored the inclusion of the social sciences in the Foundation. A majority also seemed to favor the type of administration previously advocated in S. 1285. These hearings and later discussions led to the introduction of S. 1720 on December 21. This bill, introduced by Senator Kilgore, with Senators Johnson, Pepper, Fulbright, and Saltonstall as co-sponsors, embodied many improvements. However, it failed to gain the support of many influential scientists who continued to back the original Magnuson Bill. Representatives of the latter group were interviewed in January by a group from the Senate headed by Senators Kilgore and Saltonstall, and several modifications in the provisions of S. 1720 were considered.

As a result, another bill, S. 1850, was introduced into the Senate on February 21; the sponsors of this compromise measure were Senators Kilgore, Magnuson, Pepper, Johnson, Fulbright, Saltonstall, Ferguson, and Thomas. Some of the major provisions of the bill are briefly summarized as follows:

1. Establishes a National Science Foundation administered by an Administrator appointed by the President with the consent of the Senate and after consultation with the National Science Board.

2. Provides for the creation of a National Science Board to be made up of nine persons appointed by the President after obtaining the advice and consent of the Senate and the Chairmen of the several Divisional Scientific Committees.

3. Designates the following Divisions to be established within the Foundation: (a) Mathematical and Physical Sciences; (b) Biological Sciences; (c) Social Sciences; (d) Health and Medical Sciences; (e) National Defense; (f) Engineering and Technology; (g) Scientific Personnel and Education; (h) Publications and Information; and (i) additional divisions not to exceed three in number. Each Divisional Committee, except National Defense, shall consist of 5 to 15 members appointed by the Administrator with the advice and approval of the Board. The term of service of each member shall be three years.

4. Provides that "The Board shall continuously survey the activities and management of the Foundation, and shall periodically evaluate the achievements of the Foundation in accomplishing the objectives of this Act. Each divisional scientific committee shall survey continuously the scientific field which it encompasses, shall undertake to determine the specific scientific needs of such field, and shall evaluate proposed programs and projects." "Until the Administrator and the Board have received general recommendations from the Division of Social Sciences regarding the support of research through that Division, support of social science research shall be limited to studies of the impact of scientific discovery on the general welfare and studies required in connection with other projects supported by the Foundation."

5. Authorizes the awarding of scholarships and fellowships "in any field of science, including but not limited to the mathematical, physical, biological, medical, and social sciences at nonprofit institutions of higher education, or other institutions, selected by the recipient of such aid, for such periods as the Administrator may determine, in the United States or in foreign countries."

6. Establishes a national register of scientific personnel.

7. Provides for the use and dissemination of research findings.

8. Authorizes the head of any Government agency, "with the approval of the President and through the Department of State, to conclude reciprocal agreements with foreign governments or agencies thereof, relating to the interchange of scientific and technological information, and the use and availability of patents and patent rights owned or controlled by the respective governments." "The Administrator may defray the expenses of representatives of Government agencies and other organizations and of individual scientists to accredited international scientific congresses and meetings whenever he deems it necessary in the promotion of the objectives of this Act.

Just as it appeared that an acceptable bill had finally been written, two new science bills were introduced. One by Congresswoman Luce (H.R. 5332) would create a Department of Science and Health with a Secretary in the Cabinet. The Willis Bill, S. 1777, proposes that the President appoint a self-perpetuating committee of 50 from a list of nominees submitted by the National Academy of Science for the purpose of administering a program of scientific activity. This latter bill is receiving support from the minority of scientists who have opposed the creation of a National Science Foundation because of its possible political implications.

EDUCATION FOR ALL AMERICAN YOUTH: A REVIEW

H. E. WAHLERT, New York University

Teachers of college mathematics become aware of the results of secondary education, sometimes painfully, when they meet these results in their freshman classes. It may then be too late to suggest changes that would lead to better results. Thus it seems desirable for college mathematics teachers to be aware of the opinions, plans, and ideas, as expressed by leading educators in the field of secondary education, before they are put into effect. They can then be evaluated from the point of view of their probable effect on our future students; we can give support to those we find desirable and oppose those we find undesirable. The Mathematical Association of America has long been interested in educational questions and has recently expressed this interest, even more definitely, through the appointment of a Coordinating Committee "to keep in close touch with all educational movements in the United States and Canada.* Thus, following the aim of this committee, it is the purpose of this article to discuss the attitude toward mathematics of one of the many recent publications professing

* This MONTHLY, vol. 52, May, 1945, p. 294.

to describe a desirable secondary education, *Education for All American Youth.**

Two hypothetical schools, one in "Farmville" and one in "American City", are described in Chapters 2–4 as if they had been in existence for five years following the end of the war. These schools are planned for all youth in grades 10 to 14. In junior high school, grades 7–9, these youth will have had a curriculum which, "in its broad outlines," is "the same for all pupils." An analysis of the common qualities and differences of youth leads to a broad point of view toward education. From this point of view evolve the curricula of the two schools. Since they are basically similar, the curriculum of American City can serve as an example. The program is divided into two areas: "Common Studies" and "Differential Studies." All students, in heterogeneous groups, take the work in Common Studies. The differences in the interests of students, for example, occupational, recreational, and so on, are determined, and, by suitable guidance and planning, the student and his advisers map out his program in the field of Differential Studies. The material studied in the area of Common Studies includes health and physical education, a year course whose primary aim is to "understand the social significance of science," and "a continuous course for all, planned to help students grow in competence as citizens of the community and the nation; in understanding of economic processes and of their roles as producers and consumers; in cooperative living in family, school, and community; in appreciation of literature and the arts; and in the use of the English language."

The average number of periods per day allotted to these divisions in the grades 10–14 is given in the table below:

	10	11	12	13	14
Common Learnings	2	2	2	1	1
Science	1	0	0	0	0
Health and Physical Education	1	1	1	1	1
Vocational Preparation	1	2	2	3	3
Individual Interests	1	1	1	1	1

Since this report deals with education in grades 10–14, mathematics, as such, is not required of all students.† It is assumed that, in the course of study through junior high school, "mastery over the processes and principles" of mathematics "which everyone needs to know" has been developed. However, it is recognized

* National Education Association and American Association of School Administrators, Educational Policies Commission, Education for All American Youth, Washington, D. C., the Commission, 1944. Quotations in this article, not otherwise footnoted, are from this book.

† This is not inconsistent with the opinion expressed in the final report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, The Place of Mathematics in Secondary Education, Bureau of Publications, Teachers College, Columbia University, New York 1940, p. 36.

that some students may not have developed this mastery, and so, "if a student lacks that mastery, remedial instruction is given in the tenth grade, longer if necessary" in American City. In Farmville the school operates "a mathematics workshop with a teacher in charge at all times, where remedial instruction is given and where any student may go at any time for help with the mathematical operations which he needs to use."

In the high schools, mathematics proper follows the line of vocational or occupational interests and "the amount of formal instruction may vary from nothing to three full years of systematic study." Students who plan to enter trades in which some mathematics is used would learn their mathematics in the trade course at the time that its use arises. Students who plan to enter such an occupation as engineering, in which much formal mathematics is needed, would take the necessary mathematics, probably in sequential courses, in part of the time set aside in their program for "Vocational Education."

Throughout the report primary emphasis is placed on "education in the responsibilities and privileges of citizenship." However, the ability to think rationally is also one of the main goals of the proposed plan. It is of interest to note that although mathematicians have often felt that "the chief end of mathematical study must be to make the pupil think,"* nowhere is it proposed that any use be made of mathematics in contributing to the general student's ability to think rationally.

Teachers of college mathematics are especially interested in the "boy who enjoys mathematics for its own sake" and in "Philip Scott, who solves problems in calculus for the fun of it." What do the secondary schools of Farmville and American City offer to these boys? First, they would take, along with all their fellow students, including Johnnie Lowique, the courses in the area of Common Studies. In these classes, with heterogeneous grouping, students with differing talents could offer their different abilities to the learning of the group. Thus Philip Scott and Johnnie Lowique would come to appreciate the talents of each other. In the area of Differential Studies, Philip Scott would probably choose sequential courses in mathematics for his work in the division of Vocational Preparation, and might pursue further studies in mathematics in the division of Individual Interests. He would "be encouraged to work well beyond the average of the class"; he would be able to pursue his interest within the regular classes or as an individual or in a small group, going as rapidly and as far as possible, and he would be "allowed extra time to pursue his own interests." He could get additional assistance from correspondence courses from the extension divisions of universities and colleges. In his study of mathematics the teachers would have acquainted him "with the nature and requirements of scientific and engineering occupations, as well as—the uses of mathematics and science in other occupations." In the course in science, required of all students, he will have learned "that most scientific advances have depended upon precise measurement and accurate calculation and that mathematics is indispensable to scientific inquiry."

* Young, J. W., Lectures on Fundamental Concepts of Algebra and Geometry, New York, The Macmillan Company, 1911, p. 4.

His interest in mathematics might lead him to go on the field trip in which dams and power projects are studied at firsthand and thus he will see further "that knowledge of mathematics and physics is essential for the engineer." He will learn, perhaps from the other students with related interests, (he may not look into it himself, since he does mathematics "for the fun of it"), that mathematics and science "must *stand up to practical testing*" by applying them "to gasoline motors, electrical equipment, crop production, food preparation and health."

What would happen in the high schools described in *Education for All American Youth* to "the stupid boy, the one who did not like school, the one who liked to play with little wind wheels, who liked to fight, who actually did run away from school, and who stood near the bottom of the class in mathematics?"* It is of course the hope of the authors of the report that the potential abilities of a Newton would be recognized with the attention these schools give to guidance, and that the schools themselves, with their opportunities for the pursuit of individual interests, would so appeal to a Newton that he would not run away from school. It is recognized, however, that a pupil may be "working far below his ability." In this event "the problem is one to be approached through counseling"; however, "neither skilful guidance nor good teaching—is a panacea. When counselors and teachers have done their best, a student may yet persist in doing far less than he is able to do. When this occurs, the school must enter the fact on the student's record, while it continues its search for causes and cures."

The report is well aware of the possible effects which the poverty of the environment may have on the mind of a child. It is pointed out that "it is possible for a boy or girl to grow up in this community or any other, and have his personality dwarfed or distorted because of the poverty of his environment. Not only is that possible, but it happens, year in and year out, to thousands of children." It is of course the aim of the authors of the report to have the school environment so rich and interesting that, even though the curriculum is organized for *all* American youth, Philip Scott or even an Isaac Newton would be challenged and would realize his potentialities. In the attempt to develop an education for *all* American youth, the question arises as to whether the education proposed has "been largely developed in the interests of the weaker students" and whether difficulties have "been placed in the path of the superior student which—make it difficult for him to obtain the education of which he is capable?" Will the gifted student find these schools challenging, or will they be "wasteful of a talented student's precious years of youth"?†

The report closes with the question: "Would you like your children to attend schools like those of Farmville and American City?" The question of interest to college teachers of mathematics is whether we would like to have our children and the children who grow up to be our students attend such schools?

* Sir Isaac Newton as described in Smith, D. E., Why should not all mathematics be elective?, Mathematics Teacher, vol. 38, December, 1945, p. 360.

† The last three quotations are from: MacDuffee, C. C., An objective in education, This MONTHLY, vol. 52, August-September, 1945, pp. 359, 361.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE ANNUAL MEETING OF THE MICHIGAN SECTION

The twenty-second annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan in Ann Arbor on Saturday, March 18, 1944. This meeting also constituted the meeting of the Mathematics Section of the Michigan Academy of Science, Arts, and Letters. Morning and afternoon program sessions were held in addition to a luncheon-business meeting. The Chairman of the Section, Professor G. G. Speeker, presided at all sessions.

There were fifty-one people in attendance, including the following thirty-six members of the Association: J. W. Bradshaw, J. B. Brandeberry, Father Richard Cebula, C. J. Coe, A. H. Copeland, C. C. Craig, Wayne Dancer, P. S. Dwyer, C. M. Erikson, F. D. Faulkner, Peter Field, K. W. Folley, J. S. Frame, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, E. E. Ingalls, L. S. Johnston, P. S. Jones, Wilfred Kaplan, L. C. Karpinski, C. E. Love, Sister M. Mercedes, A. L. Nelson, E. A. Nordhaus, J. E. Powell, G. Y. Rainich, C. C. Richtmeyer, L. J. Rouse, T. R. Running, E. R. Sleight, G. G. Speeker, R. M. Thrall, Fern Welker, E. T. Welmers, R. L. Wilder.

Professor J. W. Bradshaw of the University of Michigan was elected Chairman for the following year, and Professor C. J. Coe of the University of Michigan was elected Secretary-Treasurer.

The following program of eight papers was presented:

1. *Prediction of the spacings in a certain molecule*, by Dr. Wilfred Kaplan, University of Michigan.

Professor K. Fajans has raised the question as to how much information concerning the spacings of the atoms in a molecule can be obtained from consideration of the Coulomb forces alone. In this paper Dr. Kaplan answered the question for the molecule P_4O_6 in which the Coulomb potential energy V is regarded as a function of a single parameter t determining the spacings of the oxygen atoms. He computed the value of t for a minimum value of V, obtaining a value in fair agreement with the experimental result, and offered an explanation of the discrepancy.

2. *A type of transformation of plane curves*, by Professor V. G. Grove, Michigan State College.

With a given source of light, each plane curve as a reflector is associated with another plane curve as its caustic. Professor Grove discussed this transformation, employing line and point coördinates, and showed how one curve can be obtained from the other. He discussed, in particular, the case in which the caustic is a circle with its center at the source of light, and established a geometric characteristic of the reflecting curves for this case.

3. *Sequences of Kummer bounds and modified continued fractions*, by Professor J. W. Bradshaw, University of Michigan.

Employing Kummer's method, Professor Bradshaw calculated a series of bounds for $\sum_{n+1}^{\infty} k^{-2}$ and $\sum_{n+1}^{\infty} k^{-3}$. He showed that they constitute "modified continued fractions" as defined in his article in this MONTHLY, vol. 49, 1942, pp. 513–519.

4. *A Cantor function constructed with continued fractions*, by Fritz Herzog and B. H. Bissinger, Michigan State College, introduced by the Secretary.

The authors define a function $w(x)$ on the interval $0<x<1$ as follows. If x has a simple continued fraction expansion $x=\{c_1+1, c_2+1, \cdots\}$ in which no c_i is zero, then $w(x)=\{c_1, c_2, \cdots\}$. Otherwise $w(x)$ has the same value as for the next such number less than x. Then $w(x)$ is continuous, non-decreasing, and constant almost everywhere, but is not absolutely continuous. Its graph is rectifiable with length 2. At each point of a set of the power of the continuum the two upper derivatives are equal, positive, and finite, while the two lower derivatives are zero. Thus the derivative fails to exist.

5. *Mathematics in Scotland before 1700*, by Professor E. R. Sleight, Albion College.

A considerable time spent in careful investigation on the site has enabled Professor Sleight to present a clear picture of the early development of mathematics in Scotland. He finds two periods, the first in which the study of mathematics was confined to the monasteries and restricted to the elements of arithmetic, algebra, and geometry; and the second in which, after the founding of the universities, a gradual development took place. This was, however, rather sporadic until the latter half of the seventeenth century.

6. *The summation of* $\sum 1/n$, by Professor C. J. Coe, University of Michigan.

Let a row of rectangles of unit width and with altitudes 1, 1/2, 1/3, $\cdots$ be placed in contact on the x-axis. We seek a continuous curve $y=f(x)$ such that the area under the curve between the sides of each rectangle shall be equal to the area of the rectangle. In other words, it is required that

$$\int_{x-1}^{x} f(t)dt = \frac{1}{x}.$$

It was shown that a particular determination of $f(x)$ is the trigamma function

$$f(x) = \sum_{i=1}^{\infty} (x+i)^{-2} = \frac{d^2}{dx^2} \log (x!).$$

Various properties of the function were developed, and it was shown that it could be represented by any one of a sequence of more and more rapidly convergent series and also by certain asymptotic series.

7. *A cubic surface with rational lines*, by Professor J. S. Frame, Michigan State College.

Professor Frame discussed the determination of cubic surfaces having 27 distinct real lines with coördinates rational in some projective system. By choosing the coördinate planes and the unit plane as the tritangent planes to the surfaces, the equation in projective coördinates (x_1, x_2, x_3, x_4) becomes $x_0x_2x_4 = x_1x_3x_5$ where

$$x_0 + x_1 + x_2 + x_3 + x_4 = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = 0,$$

and in which $a_1, \cdots, a_5$ are homogeneous parameters. A plane $\rho x_0 = a_5x_5$ will be tritangent only when ρ is one of the roots λ, μ, ν of the cubic

$$\rho(\rho - a_2)(\rho - a_4) = a_5(\rho - a_1)(\rho - a_3).$$

If we let

$$\xi_0 = \rho x_0, \quad \xi_5 = -a_5x_5, \quad \xi_i = (\rho - a_i)x_i, \quad i = 1, 2, 3, 4$$

we may write the equation of the surface in the form

$$\xi_0\xi_2\xi_4 + \xi_1\xi_3\xi_5 = 0$$

or in any one of the 27 forms obtained from

$$\xi_0(\xi_2 + \xi_1)(\xi_4 + \xi_1) + \xi_1(\xi_3 + \xi_0)(\xi_5 + \xi_0) = 0$$

by letting ρ be λ, μ, ν, and permuting the subscripts 0, 2, 4 and 1, 3, 5 respectively. The first equation displays the nine lines of the form $\xi_0 = \xi_1$, and the second displays the eighteen lines of the form

$$\xi_0 + \xi_1 = \xi_2 + \xi_3 = \xi_4 + \xi_5 = 0.$$

For these lines to be rational we must have the a_i and λ, μ, ν all rational. To accomplish this, assign integral values to a_1, λ, μ, ν, compute the number

$$F = \frac{1}{a_1}(a_1 - \lambda)(a_1 - \mu)(a_1 - \nu) = (a_1 - a_2)(a_1 - a_4)$$

and let $a_2 - a_1$ be any factor R of F. Then

$$a_2 = a_1 + R, \quad a_4 = a_1 + \frac{F}{R}, \quad a_5 = \lambda + \mu + \nu - a_2 - a_4, \quad a_3 = \frac{\lambda\mu\nu}{a_1a_5}.$$

For the 27 lines to be distinct we exclude cases in which a_2 or a_4 is equal to λ, μ, ν. An example of a surface with 27 distinct rational lines is

$$(x_1 + x_2 + x_3 + x_4)x_2x_4 - x_1x_3(2x_1 + 3x_2 + 15x_3 + 8x_4).$$

8. *Sum of the distances to sides of a triangle*, by D. K. Kazarinoff, University of Michigan, introduced by the Secretary.

C. J. Coe, *Secretary*

THE FALL MEETING OF THE INDIANA SECTION

The twenty-third annual meeting of the Indiana Section of the Mathematical Association of America was held at Butler University, Indianapolis, Indiana, on Friday, October 19, 1945, in conjunction with the fall meeting of the Indiana Academy of Science. Professor Juna L. Beal presided.

Thirty-three persons registered at the meeting, including the following twenty-one members of the Association: G. E. Albert, W. L. Ayres, Juna L. Beal, Stanley Bolks, I. W. Burr, W. E. Edington, G. H. Graves, Cora B. Hennel, H. K. Hughes, M. W. Keller, E. L. Klinger, H. A. Meyer, A. N. Milgram, G. T. Miller, Ivan Niven, P. M. Pepper, C. K. Robbins, T. Y. Thomas, M. S. Webster, H. E. Wolfe.

At the business meeting the following officers were elected for the next year: Chairman, W. L. Ayres, Purdue University; Vice-Chairman, G. H. Graves, Purdue University; Secretary-Treasurer, M. W. Keller, Purdue University. It was decided to hold the next annual meeting again in conjunction with the Indiana Academy of Science.

The following papers were presented:

1. *A practical form of the comparison test for series of positive terms*, by Professor H. K. Hughes, Purdue University.

Let $\sum u_n$ and $\sum v_n$ be two infinite series of positive terms. If $\lim_{n\to\infty}(u_n/v_n)=L$, where L is a positive number, then it is well known that the two series are both convergent or both divergent. This modified form of the usual comparison test has not been much used in classes studying series for the first time, but it actually is very practical. The speaker cited examples of series for which a young student might have difficulty in setting up a "comparison series" but which could be handled easily by the test in the form here described.

2. *Statistical methods for controlling the quality of industrial products*, by Professor I. W. Burr, Purdue University.

Since industrial data are statistical in nature, it is only to be expected that they may best be analyzed by statistical methods. In this connection the concepts of frequency distribution, control charts, correlation, and probability are especially useful. It was the purpose of this paper to show how these tools are used in the practical applications, to suggest this field as a new and attractive career, and to point out that there are many unsolved problems.

3. *A program for increasing interest in mathematics in Indiana high schools*, by Professor W. H. Carnahan, Purdue University, introduced by M. W. Keller.

The speaker illustrated by various devices the manner in which he attempts to interest high school students in mathematics by showing them the part it plays in their daily lives.

4. *On length of curves*, by Professor A. N. Milgram, University of Notre Dame.

Let R be a plane region bounded by a simple closed curve J. We say that R bends toward the region at the point $P \subset J$ if there exist arbitrarily small segments in R with endpoints A and B on J in the order APB on a "small" subarc of J. *Every simple closed curve has at least three points at which the curve bends toward the region.* This may be used to prove that in any closed region which is simply connected two interior points have a unique geodesic joining them. A

simple closed curve J of finite length L has in any ϵ neighborhood a curve interior to J of length $L^* < L$, and exterior to J a curve of length $L^* < L + \epsilon$.

5. *Absolute scalar invariants and the isometric correspondence of Riemann spaces*, by Professor T. Y. Thomas, Indiana University.

Necessary and sufficient conditions for the isometric correspondence of Riemann spaces R_n and $\overline{R}_n$ are given in terms of the equality of absolute scalar invariants of the spaces. In the general case for which the spaces admit a complete set of n functionally independent scalars, it is proved that these and a certain derived set of scalars suffice for the solution of the problem. The solution of the corresponding problem is given for spaces of two dimensions which do not admit two functionally independent scalars.

6. *Symmetry in metric spaces*, by Professor P. M. Pepper, University of Notre Dame.

In an abstract metric space S, a point c is called a *center of pointwise symmetry* if for each x in S there exists a point $y(x)$ such that the distance xc is equal to the distance $cy(x)$ and one half the distance $xy(x)$. If S has at least two centers of pointwise symmetry, then S is unbounded. A point c of S is called a *center of η-symmetry* (fractional symmetry) if $0 < \eta \leqq 1$, and for each x in S there exists a point $y(x)$ for which $xc = cy(x)$ and $xy(x) \geqq 2\eta xc$. For each positive η less than 1 there exist bounded metric spaces of arbitrarily small diameter with two centers of η-symmetry. (Examples related to the Chebychef polynomials of the second kind are shown for each η less than 1.) A point c of S is called a *center of pointwise open symmetry* if for each number $\eta > 0$ and each x in S there exists a point $y(\eta, x)$ such that $xy(\eta, x) \geq 2xc - \eta$ and $|cy(\eta, x) - xc| \leqq \eta$.

M. W. KELLER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-eighth Summer Meeting, Ithaca, New York, August 19–20, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., October, 1946
ILLINOIS, Peoria, May 9–10, 1947
INDIANA, Terre Haute, October 18, 1946
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Claremont, March 8, 1947
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SYMBOLIC RELATIONS AMONG CLASSICAL POLYNOMIALS

E. D. RAINVILLE, University of Michigan

1. Introduction. We shall obtain several algebraic relations involving the polynomials of Legendre, Laguerre, and Hermite. Authors are not uniform in their notations for these polynomials. The notation used here is that implied by the following generating function definitions: For Legendre polynomials,

$$(1) \qquad (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n;$$

For Hermite polynomials,

$$(2) \qquad e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!};$$

For Laguerre polynomials,

$$(3) \qquad e^t J_0(2\sqrt{xt}) = \sum_{n=0}^{\infty} L_n(x)\frac{t^n}{n!},$$

where $J_0(x)$ is the Bessel function of the first kind and of index zero.

Explicit expressions for these polynomials are included for convenience. They are as follows:

$$(4) \qquad P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!x^{n-2k}}{2^n k!(n-k)!(n-2k)!},$$

where [ ] is the greatest integer symbol,

$$(5) \qquad H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!},$$

$$(6) \qquad L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n!x^k}{(k!)^2(n-k)!}.$$

The relations to be obtained here are particularly attractive in symbolic form. Whenever $\doteq$ is used to replace $=$, it is to be understood that exponents shall be lowered to subscripts on any symbol, such as P, H, L, which is undefined here except with subscripts. For example, since

$$L_2(x) = 1 - 2x + \tfrac{1}{2}x^2,$$

then

$$L_2(P(x)) \doteq P_0(x) - 2P_1(x) + \tfrac{1}{2}P_2(x).$$

In the same way

$$\{H(x) - P(x)\}^2 \doteq H_2(x)P_0(x) - 2H_1(x)P_1(x) + H_0(x)P_2(x).$$

2. Examples of known relations. Several of the known [1] algebraic relations involving the above polynomials may be expressed neatly in symbolic form. A few of them are:

$$H_n(x+y) \doteq \{H(x)+2y\}^n, \tag{7}$$

$$\{H(x)+H(y)\}^n \doteq 2^{n/2}H_n(2^{-1/2}(x+y)), \tag{8}$$

$$\{1-L(x)\}^n \doteq x^n/n!, \tag{9}$$

$$L_n(xy) \doteq \{1-y+yL(x)\}^n, \tag{10}$$

$$P_n(x) \doteq \{2x-P(x)\}^n, \tag{11}$$

$$P_n(1-2x^2) \doteq \{1-2xP(x)\}^n, \tag{12}$$

$$P_n(x) \doteq n!\left\{L\left(\frac{1-x}{2}\right)-L\left(\frac{1+x}{2}\right)\right\}^n. \tag{13}$$

3. Methods and results. We use three methods to obtain additional formulas. One method is the combination in an appropriate manner of equation (7) with itself or with other known relations. The second method is the application of a linear differential operator to a Hermite polynomial, followed by two distinct evaluations of the result. The third method is the manipulation of generating functions.

Each of the three methods will be exhibited in a representative derivation. For most of the results the proof will be omitted. In its place we shall give sufficient hints to enable one to supply the proof by one or more of the methods of this paper.

It is convenient to group our results here. The relations involving both the Hermite and the Legendre polynomials are:

$$H_n(P(x)) \doteq \{H(2x)-2P(x)\}^n, \tag{14}$$

$$H_n(\tfrac{1}{2}P(x)) \doteq \{H(x)-P(x)\}^n, \tag{15}$$

$$\{H(x)-2P(x)\}^{2n+1} \doteq 0, \tag{16}$$

$$\{H(x)-P(2x)\}^{2n+1} \doteq 0. \tag{17}$$

A relation involving the Legendre and the Laguerre polynomials is

$$2^nL_n(P(x)) \doteq \{L(x-1)+L(x+1)\}^n. \tag{18}$$

A relation involving the Hermite and the Laguerre polynomials is

$$H_n\left((1+x)L\left(\frac{x}{1+x}\right)\right) \doteq \{H(x)+2L(x)\}^n. \tag{19}$$

Relations in which all three, the Hermite, Legendre, and Laguerre polynomials enter are

$$n!\{H(x) - 2P(x)\}^{2n} \doteq (-1)^n(2n)!L_n(x^2 - 1), \tag{20}$$

$$n!\{H(x) - P(2x)\}^{2n} \doteq (-1)^n(2n!)L_n(x^2 - \tfrac{1}{4}). \tag{21}$$

Relations involving only the Hermite polynomials are:

$$H_n(\tfrac{1}{2}H(x + y)) \doteq \{H(x) + H(y)\}^n, \tag{22}$$

$$H_n(\tfrac{1}{2}H(x)) \doteq 2^{n/2}H_n(2^{-1/2}x), \tag{23}$$

$$H_n\left(\frac{w}{2}H(x)\right) \doteq w^nH_n\left(\frac{H(wx)}{2w}\right), \tag{24}$$

$$y^nH_n(x/y) \doteq (y^2 - 1)^{n/2}H_n\left(\frac{H(x)}{2\sqrt{y^2 - 1}}\right), \tag{25}$$

$$H_n(H(x)) \doteq 5^{n/2}H_n(2x/\sqrt{5}), \tag{25a}$$

$$H_n(xy) \doteq \{H(x) + 2x(y - 1)\}^n. \tag{26}$$

The Hermite polynomials and polynomials $g_n(x)$ studied earlier [2] are related by

$$H_n\left((1 + x)g\left(\frac{x}{1 + x}\right)\right) \doteq \{H(x) + 2g(x)\}^n. \tag{27}$$

A relation involving the Hermite polynomial and the hypergeometric polynomial $\phi_n(\alpha, \gamma, x) = F(\alpha, -n; \gamma; -x)$ of [2] is

$$H_n(\phi(\alpha, \gamma, x)) \doteq \{H(x) + 2\phi(\gamma - \alpha, \gamma, -x)\}^n. \tag{28}$$

It should be kept in mind that these relations are identities involving finite series in which the pertinent polynomials enter. The expanded form of equations (14) and (15), for example, is

$$\sum_{k=0}^{[n/2]} \frac{(-1)^k n!2^{n-2k}P_{n-2k}(x)}{k!(n - 2k)!} = \sum_{k=0}^{n} (-1)^k C_{n,k}2^kP_k(x)H_{n-k}(2x), \tag{14}$$

and

$$\sum_{k=0}^{[n/2]} \frac{(-1)^k n!P_{n-2k}(x)}{k!(n - 2k)!} = \sum_{k=0}^{n} (-1)^k C_{n,k}P_k(x)H_{n-k}(x), \tag{15}$$

in which $C_{n,k}$ is the binomial coefficient.

4. Derivation of the relation (14). Since equation (7) is an identity in y it follows for any set of polynomials $T_n(x)$ we have

$$H_n(x + T(x)) \doteq \{H(x) + 2T(x)\}^n. \tag{29}$$

Therefore

$$\{H(x) - 2P(x/2)\}^n \doteq H_n(x - P(x/2)),$$

in the right member of which various powers of $x-P(x/2)$ occur. Equation (11) with x replaced by $x/2$ shows that

$$\{x - P(x/2)\}^n \doteq P_n(x/2).$$

Hence

$$\{H(x) - 2P(x/2)\}^n \doteq H_n(P(x/2)),$$

from which the relation (14) follows upon our replacing x by $2x$.

Equation (14) may also be derived by applying the linear differential operator

$$B_1 = \sum_{k=0}^{\infty} \frac{(-1)^k P_k(x) D^k}{2^k k!},$$

where $D=d/dx$, to the polynomial $H_n(2x)$ in the manner of Section 5 below.

It is known [3] that the Legendre polynomials have the generating function indicated in

$$e^{tx} J_0(t\sqrt{1-x^2}) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}. \tag{30}$$

With equation (30) at hand it is easy to show by multiplication of series that

$$e^{2xt-t^2} J_0(2t\sqrt{1-x^2}) \doteq \sum_{n=0}^{\infty} H_n(P(x)) \frac{t^n}{n!}.$$

The left member of the above equation may also be obtained from the product of the series

$$e^{4xt-t^2} = \sum_{n=0}^{\infty} H_n(2x) \frac{t^n}{n!}$$

and

$$e^{-2xt} J_0(2t\sqrt{1-x^2}) = \sum_{n=0}^{\infty} (-1)^n 2^n P_n(x) \frac{t^n}{n}.$$

Thus the relation (14) may be obtained by the third method.

5. Derivation of the relation (24). Let us now consider the operator

$$B_2 = \sum_{k=0}^{\infty} \frac{(-1)^k D^{2k}}{2^{2k} k!},$$

where $D=d/dx$. First we find that

$$B_2 x^n = \sum_{k=0}^{[n/2]} \frac{(-1)^k n! x^{n-2k}}{2^{2k} k! (n-2k)!} = 2^{-n} H_n(x).$$

Next we apply B_2 to the polynomial $H_n(wx)$ where w is independent of x. Of course,

$$B_2H_n(wx) \doteq H_n\left(\frac{w}{2}H(x)\right),$$

but also

$$B_2H_n(wx) = \sum_{k=0}^{[n/2]} \frac{(-1)^k D^{2k} H_n(wx)}{2^{2k}k!}.$$

A familiar result on the Hermite polynomials is that

$$D^s H_n(x) = \frac{2^s n! H_{n-s}(x)}{(n-s)!}.$$

Therefore,

$$B_2H_n(wx) = \sum_{k=0}^{[n/2]} \frac{(-1)^k w^{2k} n! H_{n-2k}(wx)}{k!(n-2k)!}$$

$$\doteq w^n H_n\left(\frac{H(wx)}{2w}\right),$$

and equation (24) follows from the two formulas for $B_2H_n(wx)$.

The relation (24) may also be obtained by showing that each member of it is the coefficient of $t^n/n!$ in the expansion of the generating function $\exp[2xwt-(w^2+1)t^2]$.

6. Derivation of the relation (18). If the polynomials $\psi_n(x)$ are generated as indicated in

$$F(x, t) = \sum_{n=0}^{\infty} \psi_n(x) \frac{t^n}{(n!)^2},$$

then

$$e^t F(x, -t) = \left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n \psi_n(x) t^n}{(n!)^2}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k \psi_k(x) n!}{(k!)^2(n-k)!} \frac{t^n}{n!}$$

$$\doteq \sum_{n=0}^{\infty} L_n(\psi(x)) \frac{t^n}{n!}.$$

From Bateman [4] we borrow the fact that

$$I_0(\sqrt{2t(x-1)})I_0(\sqrt{2t(x+1)}) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{(n!)^2},$$

where $I_0(x)$ is the modified Bessel function, $I_0(iy) = J_0(y)$. Letting $P_n(x)$ play the role of the $\psi_n(x)$ above, we may write

$$e^t J_0(\sqrt{2t(x-1)})J_0(\sqrt{2t(x+1)}) \doteq \sum_{n=0}^{\infty} L_n(P(x)) \frac{t^n}{n!}. \tag{31}$$

Let us replace t by $2t$ in (31), thus arriving at

$$e^{2t}J_0(2\sqrt{t(x-1)})J_0(2\sqrt{t(x+1)}) \doteq \sum_{n=0}^{\infty} 2^n L_n(P(x))\frac{t^n}{n!}.$$

This reminds us of equation (3), from which we get

$$e^t J_0(2\sqrt{t(x-1)}) = \sum_{n=0}^{\infty} L_n(x-1)\frac{t^n}{n!}$$

and

$$e^t J_0(2\sqrt{t(x+1)}) = \sum_{n=0}^{\infty} L_n(x+1)\frac{t^n}{n!}.$$

Now we can conclude that

$$\sum_{n=0}^{\infty} 2^n L_n(P(x))\frac{t^n}{n!} \doteq \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{L_k(x-1)L_{n-k}(x+1)n!}{k!(n-k)!}\frac{t^n}{n!},$$

so that we have the desired relation

$$2^n L_n(P(x)) \doteq \{L(x-1)+L(x+1)\}^n. \tag{18}$$

7. Suggestions for proofs of the remaining relations. The relation (15) can be obtained from equations (11) and (29). It can be obtained by applying the operation B_1 above to $H_n(x)$. Finally, the generating function $\exp(xt-t^2)J_0(t\sqrt{1-x^2})$ may be expanded in powers of t in two ways, thus leading to equation (15).

Relations (16) and (20) can be derived simultaneously by proper use of the generating function $\exp(-t^2)J_0(2t\sqrt{1-x^2})$. In the same way (17) and (21) may be obtained from the generating function $\exp(-t^2)J_0(t\sqrt{1-4x^2})$. Equations (16) and (17) may also be found by using (29) and

$$\{x-P(x)\}^{2n+1} \doteq 0,$$

which is a special case of a known [5] result. Equations (20) and (21) may be obtained from (29) and the known [5] result

$$\{x-P(x)\}^{2n} \doteq \frac{(2n)!(x^2-1)^n}{2^{2n}(n!)^2}.$$

Equation (19) is a special case of (27), exhibited because the Laguerre polynomials are so much better known than the more general $g_n(x)$ of equation (27).

The relation (22) may be obtained by using the operator

$$B_3 = \sum_{k=0}^{\infty} \frac{H_k(y)D^k}{2^k k!}$$

on $H_n(x)$. It also follows from the generating function $\exp[2(x+y)t-2t^2]$, or from equations (7) and (29). From equations (8) and (22) we get (23) at once.

Equation (25) is most easily obtained from the generating function. The relation (25a) is a special case, $y=\sqrt{5}/2$, of (25), a particularly simple one with an interesting emphasis on the $\sqrt{5}$.

Equation (26) is an immediate consequence of (29). It may also be derived by applying

$$B_4 = \sum_{k=0}^{\infty} \frac{x^k(y-1)^k D^k}{k!}$$

to $H_n(x)$, or by using the generating function $\exp(2xyt-t^2)$.

We may use any of the three methods of this paper, together with formula (13) of [2], to obtain the relation (27) above. For relation (28) we employ the same methods with the added use of formula (31) of [2].

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3. Our (30) is used by F. H. Jackson, Proc. Lond. Math. Soc., (2), 3, 1905, 1–20. See his page 7.

4. H. Bateman, A generalization of the Legendre polynomial, Proc. Lond. Math. Soc., (2), 3, 1905, 111–123. See his formula (8).

5. In equation (8) of Notes on Legendre polynomials mentioned in reference [1] above, put $\alpha=\pi/2$ and $\cos\beta=x$.

DIFFERENTIAL GEOMETRY IN THE KASNER PLANE*

JOHN DE CICCO, Illinois Institute of Technology

1. Horn angles. A horn angle is defined as the configuration formed by two curves which pass through a point (the vertex) in a common direction. From ancient times, the qualitative aspect and quantitative nature of a horn angle had been a subject of serious consideration. Newton thought that the measure of a horn angle (of the first category) should depend on the curvatures of the two curves, calculated at the vertex.

In the nineteenth century, horn angles were revived as a concrete illustration of geometries that do not obey the Archimedian Axiom. It was noticed that different types of horn angle depending on order of contact had different qualities but it could not be decided what kind of suitable measure should be attached to each type.

* Presented to the American Mathematical Society, 1945.

In 1909, Kasner began to consider this question; the first paper appeared in the Proceedings of the Cambridge Congress of Mathematicians 1912. In his discussions from 1909–1912, Kasner showed that each category of horn angle (order of contact) has a suitable measure. This measure for each category is invariant under the total group of conformal transformations. It was proved that this invariant is essentially unique for each category.

Consider the simplest case, namely the horn angle of the first category (simple contact). This is formed by two distinct curves starting at a point in the same direction with different curvatures. The measure M_{12} of a horn angle of the first category is

$$(H) \qquad M_{12} = \frac{(\gamma_2 - \gamma_1)^2}{\left(\frac{d\gamma_2}{ds_2} - \frac{d\gamma_1}{ds_1}\right)},$$

where γ_1 and γ_2 are the curvatures of the two curves C_1 and C_2, and $d\gamma_1/ds_1$ and $d\gamma_2/ds_2$ are the rates of variation of the curvature with respect to the arc length, all of which are calculated at the vertex.

For a horn angle of category n, there is a measure M_{12} involving derivatives of order $(2n+1)$. This is invariant under the conformal group. It is essentially unique for each category. Any such invariant may serve to characterize the conformal group within the group of contact transformations of the plane.

Kasner has also studied the horn angle with respect to the equilong group which is roughly dual to the conformal group. The invariants discussed above are no longer useful but an analogous set of invariants have been found which are valid only for the equilong group.

2. The Kasner plane. In order to discuss the conformal geometry of horn angles of the first category, it is found convenient to introduce an associated plane which is called the *Kasner plane* or the *K_2-plane*. This is defined in the following manner. A simple horn-set consists of all the curves (third order differential elements) which possess a common point and a common direction. Let $x=\gamma$ denote the curvature and $y=d\gamma/ds$ the variation of curvature per unit arc length of any curve C of the simple horn-set, calculated at the common point. Any curve C (third order differential element) of the simple horn-set is given by an ordered pair of numbers (x, y). Thus a simple horn-set is a two-dimensional space which is termed the Kasner plane or the K_2-plane. Any point $P(x, y)$ of this plane, where x is the abscissa and y is the ordinate, corresponds to a curve C of the given simple horn-set where x is the curvature and y is the rate of variation of the curvature of C.

The group of conformal transformations operating on the curves of a simple horn-set induces the following three-parameter group G_3 of transformations

$$(1) \qquad X = mx + h, \qquad Y = m^2y + k,$$

where $m \neq 0$, h, k are constants, between the points (x, y) of the Kasner plane.

Thus this plane is a special type of affine plane. In preceding papers, the geometry of this *fundamental K-group* G_3 has been studied. In the present paper, we shall begin the study of the differential geometry of this group.

A *general line* consists of the ∞^1 points $P(x, y)$ which satisfy a linear equation of the form $y=px+r$, where $p\neq 0$ and r are constants. An *infinite line* (or *zero line*) is the totality of ∞^1 points $P(x, y)$ which satisfy the special linear equation $y=$const. (or $x=$const.). The infinite and zero lines are termed the *minimal lines* of the K_2-plane. Under the K-group G_3, any general line (or infinite line, or zero line) is converted into any other general line (or infinite line, or zero line).

Two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ of the K_2-plane possess the fundamental invariant

$$M_{12} = M(P_1, P_2) = \frac{(x_2 - x_1)^2}{(y_2 - y_1)}, \qquad M_{21} = -M_{12}. \tag{2}$$

This is called the *distance* M_{12} between the two points. Note that this corresponds to the measure M_{12} of the horn angle in the associated simple horn-set.

This distance is zero (or infinite) only for points along a zero line (or infinite line). It is indeterminate if and only if the two points coincide.

The distances between three points P_1, P_2, P_3, such that no two lie on the same minimal line, satisfy the triangular inequality

$$M_{12}M_{23}M_{31}(M_{12} + M_{23} + M_{31}) \leqq 0. \tag{3}$$

The equality is valid only in the case when the three points are collinear (thus obtaining a wide-open trihorn).

Two general lines $L_1: y=p_1x+r_1$ and $L_2: y=p_2x+r_2$, possess the invariant

$$\alpha_{12} = \alpha(L_1, L_2) = p_2/p_1, \qquad \alpha_{21} = 1/\alpha_{12}. \tag{4}$$

This is called the *dihorn angle* α_{12} between the two general lines. If the dihorn angle α_{12} is positive, then we define the *radian dihorn angle* θ_{12} by the formula

$$\theta_{12} = \theta(L_1, L_2) = \tfrac{1}{2}\log \alpha_{12} = \tfrac{1}{2}\log (p_2/p_1), \qquad \theta_{21} = -\theta_{12}. \tag{5}$$

3. The arc length of a curve in the Kasner plane. An arbitrary set of ∞^1 points in the K_2-plane is called a curve. This may be defined by the parametric equations $x=x(t)$, $y=y(t)$. From this it follows that a curve in the K_2-plane may be given either by the equation $y=y(x)$, or by the equation $x=x(y)$.

Of course, a line is a simple example of a curve in the K_2-plane. Another important example is a *parabolic-circle*. This may be defined as the locus of a point $P(x, y)$ which is at a fixed distance M from a fixed point $P_0(x_0, y_0)$. The point P_0 is called the center and M is termed the radius of the parabolic-circle. According to (2), the equation of any parabolic-circle is

$$(x - x_0)^2 = M(y - y_0). \tag{6}$$

By (2), it is found that the differential ds of arc length in the K_2-plane is given by

$$ds = \frac{dx^2}{dy}. \tag{7}$$

The K_2-plane is a special type of Finsler space. It is the best known example which is neither euclidean nor riemannian. The arc length is thus defined by

$$s = \int \frac{\left(\frac{dx}{dt}\right)^2}{\left(\frac{dy}{dt}\right)} dt = \int \frac{1}{\left(\frac{dy}{dx}\right)} dx = \int \left(\frac{dx}{dy}\right)^2 dy. \tag{8}$$

It follows from this that the extremals of the K_2-plane are the lines. Also transversality is obtained by taking half the slope. That is, the dihorn angle 1/2 is a right dihorn angle.

It is remarked that the arc s intercepted on a parabolic-circle of radius M by a radian-dihorn angle θ with vertex at the center, is $s = M\theta$.

4. The rac curvature R of a curve. In 1913, Kasner considered the *rac* curvature R of a curve in any space, which may be euclidean or riemannian or of the Finsler type. The *rac* curvature R at a given point P of a curve is defined to be the limit of the ratio of the arc PQ to the chord PQ as the point Q approaches P along the curve.

We shall consider the *rac* curvature R of a curve in the K_2-plane. Thus R is defined by the expression

$$R = \lim_{Q \to P} \frac{\text{arc } PQ}{\text{chord } PQ} = \lim_{(x_2, y_2) \to (x_1, y_1)} \frac{\int_{(x_1, y_1)}^{(x_2, y_2)} \frac{dx^2}{dy}}{\frac{(x_2 - x_1)^2}{(y_2 - y_1)}}. \tag{9}$$

A curve of the K_2-plane is termed an analytic element if the abscissa x and ordinate y of any point on the curve are defined by integral power series in a parameter t. Thus any analytic element is defined by

$$\begin{aligned} x &= x_0 + a_p t^p + a_{p+1} t^{p+1} + \cdots, \quad a_p \neq 0, \\ y &= y_0 + b_q t^q + b_{q+1} t^{q+1} + \cdots, \quad b_q \neq 0. \end{aligned} \tag{10}$$

Upon eliminating the parameter t, it is found that any analytic element is given by

$$y - y_0 = c_q (x - x_0)^{q/p} + c_{q+1} (x - x_0)^{(q+1)/p} + \cdots, \quad c_q \neq 0. \tag{11}$$

Under the group (1), the positive integers p and q are arithmetic invariants. Accordingly *the species* (p, q) is defined to consist of all analytic elements with fixed exponents (p, q) but with arbitrary coefficients $c_q, c_{q+1}, \cdots$. An analytic element is said to be regular if it belongs to one of the species $(1, q)$ or $(p, 1)$,

or if q/p (or p/q) is an integer and the development of y in terms of x (or x in terms of y) consists of only positive integral powers of x (or y). Otherwise it is said to be irregular.

THEOREM 1. *The rac curvature R of an analytic element of species (p, q) is given by the formula*

$$R = \frac{p^2}{q(2p - q)}. \tag{12}$$

This result may be contrasted with the corresponding one obtained by Kasner in the euclidean plane. If (x, y) are interpreted as minimal coordinates of a point so that the linear element is given by $ds^2 = dxdy$, then the *rac* curvature R of an analytic element of species (p, q) is given by the formula

$$R = \frac{2\sqrt{pq}}{p + q}. \tag{13}$$

The proof of our Theorem 1 is obtained by substituting the equations (10) defining an analytic element of species (p, q) into the formula (9) for the *rac* curvature.

From (12), we find that the ratio q/p in terms of the *rac* curvature R is given by

$$\frac{q}{p} = 1 \pm \sqrt{\frac{R - 1}{R}}. \tag{14}$$

Thus a finite number R is the *rac* curvature of an analytic element if and only if R is a non-zero rational number such that $(R^2 - R)$ is the square of a rational number. We make the following remarks.

(a) There are two distinct species of analytic elements with the given *rac* curvature $R > 1$.

(b) There is only one species, namely (p, p), of analytic elements with the given *rac* curvature $R = 1$. In this case the analytic element is tangent to neither of the minimal lines.

(c) There are no analytic elements with the given *rac* curvature R such that $0 \leqq R < 1$.

(d) There is only one species of analytic elements with the given *rac* curvature $R < 0$.

(e) For the species $(p, 2p)$, the *rac* curvature R is infinite.

5. The osculating parabolic-circles of a general curve of the K_2-plane. A curve such that its *rac* curvature R at any point is unity is termed a *general curve*. The tangent lines of a general curve are all general lines. In the neighborhood of any one of its points (x_0, y_0), a general curve may be given by $y = y(x)$ where $y(x)$ is a power series in integral powers of $(x - x_0)$ such that $y'(x_0) \neq 0$.

In the remainder of this paper, we shall study the differential geometry of general curves.

The parabolic-circle which has three consecutive points in common with the general curve $y=y(x)$ at a given point P is called the *osculating parabolic-circle* of the general curve at the point P. Its center is termed the *center of curvature* and its radius is called the *radius of curvature.*

The center of curvature (a, b) and the radius of curvature ρ of the osculating parabolic-circle at any point P of the general curve $y=y(x)$ are given by the equations

$$a = x - \frac{y'}{y''}, \qquad b = y - \frac{y'^2}{2y''}, \qquad \rho = \frac{2}{y''}. \tag{15}$$

THEOREM 2. *The necessary and sufficient condition that the* ∞^1 *parabolic-circles*

$$[x - a(t)]^2 = \rho(t)[y - b(t)], \tag{16}$$

be an osculating set of parabolic-circles is

$$a'^2 + b'\rho' = 0. \tag{17}$$

The curve to which these parabolic-circles (16) are the osculating parabolic-circles is given by the equations

$$x = a + \frac{\rho b'}{a'}, \qquad y = b + \rho\frac{b'^2}{a'^2}. \tag{18}$$

We prove our Theorem 2. The parabolic-circles (16) must be necessarily the osculating parabolic-circles of their envelope. Hence the parabolic-circles must have contact of the second order with their envelope. Therefore consecutive parabolic-circles of (16) must possess a common lineal-element. Imposing this last condition on the equations (16), we find

$$\begin{aligned} x &= a + \tfrac{1}{2}\rho p, & y &= b + \tfrac{1}{4}\rho p^2, \\ 0 &= a' + \tfrac{1}{2}\rho' p, & 0 &= b' + \tfrac{1}{4}\rho' p^2, \end{aligned} \tag{19}$$

where $p=dy/dx$ is the slope of the curve. Eliminating p from the last two equations, we obtain our condition (17). Solving the first of the last two equations for p and substituting this value of p into the first two equations, we obtain the equations (18). This completes our proof.

6. The curvature γ of a general curve. Let a general curve be given by the parametric equations $x=x(t)$, $y=y(t)$. Solving the formula (8) defining the arc length s, for the parameter t, it is seen that $t=t(s)$. Substituting this into the parametric equations of the curve, it is found that a general curve may be given by the parametric equations $x=x(s)$, $y=y(s)$, where the arc length s is the parameter. By (8), we obtain the following result.

THEOREM 3. *A general curve is given by the parametric equations* $x=x(s)$, $y=y(s)$, *where* s *is the arc length, if and only if*

$$\left(\frac{dx}{ds}\right)^2=\left(\frac{dy}{ds}\right). \tag{20}$$

The reciprocal of the radius of curvature ρ of a general curve is termed the *curvature* $\gamma=1/\rho$. From the definition of the radian dihorn angle θ, we obtain the following proposition.

THEOREM 4. *The curvature* γ *of the general curve* $y=y(x)$ *is equal to the rate of change of the radian dihorn angle that the general tangent line makes with a fixed general line, per unit arc length. The curvature* γ *is given by the equation*

$$\gamma=\frac{d\theta}{ds}=\frac{1}{2}\frac{d^2y}{dx^2}=\frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{2\left(\dfrac{dx}{dt}\right)^3}. \tag{21}$$

Now we shall prove the following fundamental theorem.

THEOREM 5. *Two general curves of the* K_2*-plane which have their curvatures* γ *the same functions of the arc length* s *are equivalent under the* K*-group* G_3. *Thus the intrinsic equation of a general curve is*

$$\gamma=\gamma(s). \tag{22}$$

Let us find all the general curves which satisfy the condition (22). By (20), (21), and (22), we find that all the curves which satisfy this condition are given by

$$\begin{aligned} x &= m\int_{s_0}^{s}\exp\left[2\int_{s_0}^{s}\gamma ds\right]ds+h \\ y &= m^2\int_{s_0}^{s}\exp\left[4\int_{s_0}^{s}\gamma ds\right]ds+k, \end{aligned} \tag{23}$$

where $m\neq 0$, h, k, are arbitrary constants. This means that as we change m, h, k, we obtain different curves which satisfy (22). But by (1), it is found that any two such curves are equivalent under the K-group G_3. This completes the proof of our Theorem 5.

7. The theory of evolutes and involutes. The locus of the centers of curvature of a given general curve is called the *evolute* of that curve. By a *normal* to a general curve at a point P, we shall mean the general line N which passes through the given point P and which is orthogonal to the general tangent line T of the curve. That is, the dihorn angle $\alpha(T, N)=1/2$.

THEOREM 6. *The envelope of the normals of a general curve is the evolute. Thus the normals of a given curve are tangent to its evolute.*

The normals of the general curve $y=y(x)$ are given by the equation

$$Y - y(x) = \tfrac{1}{2}y'(x)(X - x). \tag{24}$$

Upon finding the envelope of this family of lines, we obtain the first two of the equations (15). Thus our Theorem 6 is proved.

Let S denote the length of arc of the evolute of a given curve. Upon substituting this into (17), we see that the condition (17) may be written in the form

$$dS + d\rho = 0. \tag{25}$$

THEOREM 7. *The length of arc of the evolute is equal to the difference between the radii of curvature of the given curve which are tangent to this arc at its extremities. Also the radius of curvature of the given curve decreases as fast as the arc of the evolute increases.*

Let a curve E be given by the parametric equations $a=a(t)$, $b=b(t)$, where a and b are functions of the parameter t. Any curve I such that the given curve E is the evolute of I is called an *involute* of E. From (18) and (25), it is seen that all involutes I of the curve $E: a=a(t)$, $b=b(t)$, are given by the equations

$$\begin{aligned} x &= a - \frac{b'}{a'}\int_{t_0}^{t}\frac{a'^2}{b'}\,dt + C\frac{b'}{a'}, \\ y &= b - \frac{b'^2}{a'^2}\int_{t_0}^{t}\frac{a'^2}{b'}\,dt + C\frac{b'^2}{a'^2}, \end{aligned} \tag{26}$$

where C is an arbitrary constant. This shows that for a given curve E, there exist ∞^1 involutes.

THEOREM 8. *A given curve E of the K_2-plane possess ∞^1 involutes. These involutes are all parallel to each other.*

To finish the proof of Theorem 8, we shall consider a special type of lineal-element transformation in the K_2-plane. A *dilatation* D_k is the lineal element transformation which slides each element of the K_2-plane through a constant distance k in a direction orthogonal (that is, the dihorn angle 1/2) to itself. The equations of the dilatation D_k are

$$X = x + \tfrac{1}{2}kp, \qquad Y = y + \tfrac{1}{4}kp^2, \qquad P = p. \tag{27}$$

Every dilatation is a contact transformation. It carries every curve into a curve. We shall say that two curves of the K_2-plane are parallel if one curve can be carried into the other by a dilatation.

These considerations show that all involutes of a given curve are parallel. The proof of Theorem 8 is complete.

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THE $\alpha+\beta$ HYPOTHESIS AND RELATED PROBLEMS

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1. Introduction. Let $a_1<a_2<a_3\cdots$ be a set, A, of positive integers. Let $A(n)$ denote the number of integers of A which are not greater than n. The *asymptotic density* $\delta(A)$ of A is defined to be the lower limit of $A(n)/n$. This is a natural definition of density: the set of all (positive) integers has asymptotic density 1; the odd integers, $\frac{1}{2}$; the even integers, $\frac{1}{2}$; the integers which are multiples of 7, 1/7.

A less natural definition, but one which has attracted considerable attention, is the *greatest lower bound* of $A(n)/n$. This is the *Schnirelmann density* [14] of A, denoted here by $d(A)$. The odd integers have density $\frac{1}{2}$, clearly, but the even integers have density 0 under this definition. In fact, if a set A lacks the integer 1, then $d(A)=0$. Although this may seem artificial, Schnirelmann was led to the definition quite naturally by a study of certain problems in additive number theory, as we shall see later.

It is clear from the definitions that $0\leqq d(A)\leqq\delta(A)\leqq 1$.

Let B be the set of positive integers $b_1<b_2<b_3<\cdots$. The sum $A+B$ of the sets A and B is defined as the set of integers of the form a_i or b_j or a_i+b_j; that is, an integer x is in $A+B$ if x is in A, or if x is in B, or if x is the sum of an integer of A and an integer of B. For convenience we denote $d(A)$, $d(B)$, and $d(A+B)$ by α, β, and γ, respectively. The question of the relation of γ to α and β has been the subject of much investigation, culminating in Mann's proof [12] of the celebrated Khintchine conjecture [8], commonly known as the $\alpha+\beta$ hypothesis, which is that γ is not less than the minimum of 1 and $\alpha+\beta$. This paper outlines briefly, and not at all exhaustively, the history and present state of knowledge of this problem and some related ones. Some of the simplest proofs are given.

Landau [11] and Rohrbach [13] have summarized much of the principal work in this field up to 1937 and 1938 respectively.

2. Asymptotic density. First we mention some well known sequences of integers and their asymptotic densities. If A is the set of prime numbers, then $\delta(A)=0$ [9]. The set of square-free integers has density $6/\pi^2$ [7]. If A is either the set of squares or the integers which are expressible as sums of two squares, then $\delta(A)=0$. The set of integers which can be written as sums of three or fewer squares has density 5/6, as can be readily deduced from the fact that this set contains all integers except those of the form $4^a(7+8b)$.

Davenport [4] has shown that if A represents the set of integers which are sums of s or fewer cubes of positive integers, then $\delta(A)=1$ for $s=4$ but for no smaller value of s. Change cubes to fourth powers, change $s=4$ to $s=15$, and you get another result of Davenport [5]. In each of these cases the fact that the result is best possible can be proved by simple congruences. For example, no

integer of the form $15+16a$ can be a sum of 14 fourth powers, because any fourth power is congruent to 0 or 1 (mod 16).

Less is known about the higher powers. If $A_{k,s}$ represents the set of integers which are expressible as sums of s or fewer kth powers of positive integers, it is known that $\delta(A_{k,k-1})=0$; but whether $\delta(A_{k,k})$ is zero or positive is not known for $k\geqq 3$.

We mentioned in §1 the problem of the relationship between the Schnirelmann densities of A, B, and $A+B$. For asymptotic densities the $\alpha+\beta$ hypothesis is not true: consider the case, for example, where both A and B are composed of all positive even integers, so that $A+B$ is the same set, and we have $\delta(A)=\delta(B)=\delta(A+B)=\frac{1}{2}$.

However, Erdös [6] has proved the following result. Let α', β', and γ' stand for $\delta(A)$, $\delta(B)$, and $\delta(A+B)$. If $\gamma'\leqq 1$, $\alpha'\leqq\beta'$, and $a_1=1$ (recall that a_1 is the first integer of A), then $\gamma'\geqq\frac{1}{2}\alpha'+\beta'$. This result is best possible in the sense that sets A and B can be exhibited so that the equality sign holds in the conclusion. This can be done by taking A and B to be the same set, the set of integers which are congruent to 0 or 1 (mod 4), so that $A+B$ is composed of all integers congruent to 0, 1, or 2 (mod 4).

3. Two proofs. Henceforth we shall discuss only Schnirelmann density. Schnirelmann [14] and Landau [10] proved that $\gamma\geqq\alpha+\beta-\alpha\beta$. We now prove this result.

Proof. Consider any integer a in A such that $a+1$ is not in A. Let the next largest integer in A be $a+h+1$. Thus we have a gap of length h, that is, a gap of exactly h integers which are not in A. Consider now any integer x, not necessarily in A. Let there be m gaps of lengths $h_1, h_2, \cdots, h_m$ in A among the integers which are $\leqq x$. In case x is not in A, we shall take h_m to be the number of consecutive integers $x, x-1, x-2, \cdots$ which are not in A; thus h_m, unlike the other h's, may not represent the length of a *complete* gap in the integers of A.

Now since $A(x)$ denotes the number of integers in A which are $\leqq x$, we have

$$x - A(x) = h_1 + h_2 + \cdots + h_m. \tag{1}$$

Moreover, a and $a+h+1$ being in A (but no integer between these two), we can add the $B(h)$ integers of B which are $\leqq h$ to a to get $B(h)$ integers in $A+B$ between a and $a+h+1$. Doing this for all the gaps of lengths $h_1, h_2, \cdots, h_m$ we see that, denoting $A+B$ by C,

$$C(x) \geqq A(x) + B(h_1) + B(h_2) + \cdots + B(h_m). \tag{2}$$

But by definition $B(h)\geqq\beta h$, and hence (2) implies

$$C(x) \geqq A(x) + \beta(h_1 + h_2 + \cdots + h_m),$$

which combines with (1) to give

$$C(x) \geqq A(x) + \beta(x - A(x)) = (1 - \beta)A(x) + \beta x.$$

But $A(x) \geqq \alpha x$, so we have

$$C(x) \geqq (1-\beta)\alpha x + \beta x = x(\alpha+\beta-\alpha\beta),$$

which completes the proof.

As we have stated the $\alpha+\beta$ hypothesis it includes the result that if $\alpha+\beta \geqq 1$, then $\gamma=1$. This is very simple, and we prove it now.

Proof. We must prove that $A+B$ is the set of all integers. Suppose a certain integer n is not in $A+B$. Let $a_1<a_2<a_3<\cdots<a_k$ be the members of A which are $<n$. Then $A(n)=k$ and $k/n=A(n)/n \geqq \alpha$. Also it is clear that B lacks the $k+1$ integers $n-a_1, n-a_2, \cdots, n-a_k, n$. Hence β, the density of B, is at most $1-(k+1)/n$. Thus we have

$$\alpha \leqq \frac{k}{n}, \qquad \beta \leqq 1-\frac{k+1}{n}, \qquad \alpha+\beta \leqq 1-\frac{1}{n},$$

which contradicts the hypothesis.

4. Concerning Goldbach's hypothesis. Schnirelmann [14] proved that if A represents the set of all primes (we include 1 as a prime for convenience in this discussion), then, although $d(A)=0$, $d(A+A)$ is positive. By the Schnirelmann-Landau theorem of §3 (or a fortiori by the $\alpha+\beta$ hypothesis) it follows from $d(A+A)=d(2A)=\lambda>0$ that $d(4A) \geqq \lambda+\lambda-\lambda^2$ which exceeds $\frac{3}{2}\lambda$ if $\lambda<\frac{1}{2}$. By induction we know that there exists an n such that $d(nA)>\frac{1}{2}$ and the second result proved in §3 implies that $d(2nA)=1$. Thus Schnirelmann was able to show the existence of a constant $2n$ such that every integer is expressible as a sum of $2n$ primes. Later, Vinogradoff proved by different methods that every sufficiently large odd integer is a sum of three primes. Goldbach's hypothesis is that every even integer is a sum of two primes.

5. The $\alpha+\beta$ hypothesis. Khintchine [8] conjectured the $\alpha+\beta$ hypothesis (that $\gamma \geqq \alpha+\beta$ or $\gamma=1$, a best possible result), having proved it in the special cases $\alpha=\beta$ and $\alpha=1-2\beta$. There followed a series of papers proving partial or modified results.

The Schnirelmann-Landau result has already been mentioned in §3. Besicovitch [2] defined β^* as the greatest lower bound of $B(n)/(n+1)$, and proved that the density of integers of the form a_i or a_i+b_j is not less than $\alpha+\beta^*$, a result which is best possible.

Schur [15] proved that $\gamma \geqq \alpha/(1-\beta)$ or $\gamma=1$. Brauer [3] proved that $\gamma \geqq \frac{9}{10}(\alpha+\beta)$ or $\gamma=1$.

Landau [11, p. 7] posed the question of the as yet undecided problem and wrote: "Ich weiss es nicht: dies ungelöste Problem möchte ich dem Leser ans Herz legen."

Finally Mann [12] became interested in the problem while in attendance at a series of lectures on number theory by A. T. Brauer, and achieved the result that had eluded so many, the proof of the $\alpha+\beta$ hypothesis. Later Artin and Scherk [1] gave a simplification of Mann's proof.

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ELLIPTIC FERMAGORIC TRIANGLES

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1. Introduction. Let us consider the following numerical relations:

(1)
$$12 + 3 = 15. \qquad 12^2 + 5^2 = 13^2.$$

$$12^3 = (9 + \sqrt{5})^3 + (9 - \sqrt{5})^3.$$

$$12^3 + [9 - (432 + \sqrt{718065})^{1/3} - (432 - \sqrt{718065})^{1/3}]^3$$
$$= [9 + (432 + \sqrt{718065})^{1/3} + (432 - \sqrt{718065})^{1/3}]^3.$$

$$12^4 = [9 + (\sqrt{62856} - 243)^{1/2}]^4 + [9 - (\sqrt{62856} - 243)^{1/2}]^4.$$

$$12^4 + [9 - (144 + \sqrt{40419})^{1/3} - (144 - \sqrt{40419})^{1/3}]^4$$
$$= [9 + (144 + \sqrt{40419})^{1/3} + (144 - \sqrt{40419})^{1/3}]^4.$$

$$12^5 = [9 + (\sqrt{200340}/5 - 81)^{1/2}]^5 + [9 - (\sqrt{200340}/5 - 81)^{1/2}]^5.$$

These identities represent isoperimetric fermagoric triangles of constant perimeter 30, each of them sharing a common side 12. The first linear one may

be considered as a triangle of altitude zero and fermagoric order one.

If we take an ellipse of eccentricity 2/3, with major axis 18 and focal distance 12, and imagine the common leg 12 of each of these triangles placed on the segment between the foci, then their third vertex in every case will be located at a different point on the ellipse.

For this reason we have named such a family of isoperimetric fermagoric triangles, when their legs a, b, c satisfy the Fermat equation $a^n+b^n=c^n$, when they have one constant leg b or c, and when their perimeter $a+b+c$ is also constant, *elliptic fermagoric triangles.*

The generalized study of elliptic fermagoric triangles which we propose to initiate in this article, is possibly new and we shall see that it turns out to be interesting and instructive.

2. Elliptic triangles. In Fig. 1, let L be an ellipse of eccentricity $s/r<1$, where the major axis is $gh=2r^2$ and the focal distance is $ab=2rs$. Let the parameters r and s be two relatively prime integers of different parity, that is, not both odd. (It will follow later that our selection of the integer parameters r and s must be curiously restricted by certain necessary limitations.)

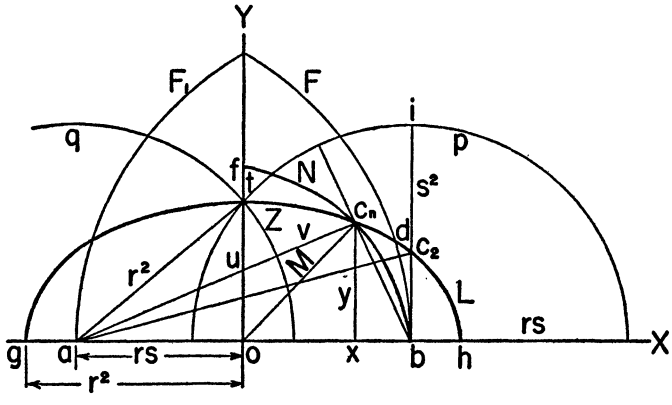


FIG. 1

In order to obtain families of isoperimetric triangles with constant perimeter $2rs+2r^2$ and each sharing a common leg $2rs$, it will suffice that their base be on the focal segment ab, and that their third vertex c_n be any point on the ellipse L.

If now we draw with radii r^2 and with centers at the foci a and b, the two focal circles p and q, these two circles intersect at the upper central point t of the ellipse. Then the two upper legs of our triangles are $r^2\pm v$, where v is the distance from c_n to each of the circles p and q, inasmuch as the ellipse L is the locus of all points equidistant from the two focal circles p and q.

So far we are dealing with an infinite family of elliptic triangles having one common leg $ab=2rs$ within the foci, and having the constant perimeter

$$2rs + (r^2 + v) + (r^2 - v) = 2rs + 2r^2 = 2r(r + s).$$

3. Elliptic fermagoric triangles. Let us now further require that the triangles be fermagoric of successive integral orders $n>0$.

For $n=1$, we shall have $ab+bh=ah$, or $2rs+(r^2-v)=(r^2+v)$, whence $v=rs$.

Now on the focus b draw the perpendicular to gh, (latus rectum), cutting the ellipse at point c_2 and circle p at point i. Then in the right triangle abc_2 we have $(2rs)^2+(r^2-v)^2=(r^2+v)^2$. Again solving for v we obtain $v=s^2$, hence $c_2i=s^2$, and in the triangle abc_2

$$(2rs)^2 + (r^2 - s^2)^2 = (r^2 + s^2)^2,$$

which is the classical Pythagorean equation, primitive if we make the parameters r and s relatively prime integers not both odd. We see now why for convenience we did so, and why we took the focal distance and the major axis of the ellipse L to be respectively $2rs$ and $2r^2$.

In order for us to continue finding fermagoric elliptic triangles of order n greater than 2, we will have to solve for v in terms of our parameters r and s, the respective Fermat equations

Type I. $(2rs)^n+(r^2-v)^n=(r^2+v)^n$,

Type II. $(2rs)^n=(r^2-v)^n+(r^2+v)^n$,

depending on whether (r^2+v) or $2rs$ is to be the generalized fermagoric hypotenuse of the triangle that we are seeking.

We have seen that whenever r and s are integers, the type I Fermat equation renders us integral values for $v_1=rs$, $v_2=s^2$.

It is obvious that if ever an elliptic fermagoric triangle of finite integral order n greater than 2 should exist having integral and primitive sides obtainable by means of proper selection of the parameters r and s, it would have to belong to one or the other of our two types, and that the value of v would have to be integral, for if c is even, a and b must be odd, and $v=(b-a)/2$ is an integer.

The numerical identities with which we started this article were obtained by the use of the above equations making $r=3$, $s=2$. Those corresponding to type II for $n=6$, 7, 8 and 9 can also be calculated with increasingly complicated finite numerical values for v under radicals.

For the time being we shall limit our generalized study of elliptic fermagoric triangles to the equations and triangles of type II and of odd fermagoric order $n=2m+1$, which of course include all prime exponents greater than 2. We will see later that every triangle of type I, if ever its three sides are possibly rational, may be transformed into one of type II of the same fermagoric order by a convenient change in the value of the parameters.

Notice now that all elliptic fermagoric triangles of type II must exist within the fermatic region bounded by the circular arcs F_1 and F.*

One shows easily that $(oY)^2=3r^2s^2$, $(ot)^2=r^4-r^2s^2$; hence for the existence of

* See the author's 'Fermagoric Triangles' (G. E. Stechert Co., N.Y.). This paper constitutes a kind of appendixed tenth chapter to that book, and must be read with a knowledge of the book.

fermagoric triangles of type II on the ellipse, it is necessary that $oY > ot$, which is equivalent to $r < 2s$, $s/r > 1/2$ or or $4 > (r/s)^2$.

We can now determine the least value of the exponent n for which such triangles exist, by computing the real fermagoric exponent for the triangle at point t where $v = 0$ and

$$(2rs)^n = (r^2)^n + (r^2)^n, \qquad (2s/r)^n = 2,$$

the result being

$$n = \log 2/(\log 2s - \log r).$$

For $n = 3$, the solution for v of the Fermat equation

$$(2rs)^3 = (r^2 + v)^3 + (r^2 - v)^3,$$

gives us, for example, the parametric identity

$$(2rs)^3 = [r^2 + \sqrt{r(4s^3 - r^3)/3}]^3 + [r^2 - \sqrt{r(4s^3 - r^3)/3}]^3,$$

where the value of v is an irrational function of the parameters.

We can generalize the fermatic equation of type II for all odd fermagoric exponents $n = 2m+1$ greater than 2, in the following convenient manner:

$$\begin{aligned} (2) \qquad 4^m r^n s^n - r^{2n} &= r^n(4^m s^n - r^n) \\ &= C_2^n (r^2)^{n-2}(v^2) \\ &\quad + C_4^n (r^2)^{n-4}(v^2)^2 \\ &\quad + C_6^n (r^2)^{n-6}(v^2)^3 \\ &\quad + \cdots + n r^2 (v^2)^m. \end{aligned}$$

Observe that for any odd fermagoric exponent $n = 2m+1$ greater than 2, the resulting fermatic equation (2) is of algebraic degree $m = (n-1)/2$ in v^2 as the unknown. It is solvable for real values of v in terms of radicals involving the parameters r and s, by elementary methods, when $n = 1$, 3, 5, 7 and 9. (Type II equations are also solvable for v in terms of the parameters in radicals for the even fermagoric exponents $n = 2$, 4, 6 and 8.)

For odd $n = 2m+1$ greater than 9 we shall be faced whith the solution of algebraic equations in v^2 as the unknown, of algebraic degrees $m = 5$ and higher. But remark that no matter how high be the algebraic degree m of the equation to be solved, its real root v^2 shall always be measured by the square on a vector $\mathbf{v}$ separating the upper vertex of the corresponding elliptic fermagoric triangle from the focal circles p and q.

Inasmuch as at the frontier F of the fermatic region the fermagoric exponent n is always infinite, the value of n increases on the ellipse from $n = \log 2/(\log 2s - \log r)$ at point t, to $n = \infty$ at point d, thus rendering interior triangles of type II. On the other hand the exterior triangles of type I start with exponent $n = 1$ at point h, increasing to $n = 2$ at point c_2 and finally again reaching ∞ at point d.

It is apparent that the fermagoric exponent n must be a hidden function of the length of the elliptic arc $tc_n = Z$. The writer invites collaboration in seek-

ing the secreted relation involved which so far has eluded his efforts to find, but the double periodicity of the fermagoric exponent n on the elliptic quadrant suggests a possible connection with elliptic functions.

The fact remains that there is a one to one correspondence between the points on the elliptic arc td within the fermatic region, and every real fermagoric exponent from the minimum $\log 2/(\log 2s - \log r)$ to infinity.

Let us note that in (2.) in order for v^2 to be positive we must have $4^m s^n - r^n$ positive. Whence a further necessary condition in the selection of our parameters r and s is that

$$4^m s^n > r^n. \text{ Hence } 4 > \left(\frac{r}{s}\right)^{n/m},$$

which is a more subtle requirement than the previously found inequality

$$4 > \left(\frac{r}{s}\right)^2, \text{ for } \frac{n}{m} = \frac{2n}{n-1} > 2.$$

This more precise inequality requirement also follows from the necessity that $ot < of$, where f is the Y intercept of the n-fermat N and $of = \sqrt{4^{(n-1)/n} - 1}\, rs$.

4. Geometry of elliptic triangles. Let x and y be the cartesian coordinates of the vertex c_n of any elliptic triangle, when the coordinate axes are the axes of the ellipse. Then

$$y^2 = (r^2 + v)^2 - (rs + x)^2$$
$$y^2 = (r^2 - v)^2 - (rs - x)^2.$$

By subtraction we get the following relation:

$$\frac{v}{x} = \frac{s}{r} = e. \tag{3}$$

By addition we obtain

$$r^2 s^2 + x^2 + y^2 = r^4 + v^2 = r^2 s^2 + M^2, \tag{4}$$

where M is the median oc_n.

By (3) we can now write the general Fermat equation of type II as follows:

$$(2rs)^n = (r^2 + ex)^n + (r^2 - ex)^n.$$

Therefore if ever this equation is possible in integers for $n > 2$ when $e = s/r$ is rational, the abscissa x of the c_n vertex of the corresponding elliptic triangle must be rational and such that $v = ex = sx/r$ is an integer.

Relation (4) proves the following theorem:

In the triangle aoc_n, whose area is half that of the elliptic triangle abc_n, the square on its base $ao = rs$, plus the square on its side $oc_n = M$, is equal to the sum of the squares on each of the two segments r^2 and v of its third side $ac_n = r^2 + v$. This relation becomes the Pythagorean theorem when $v = 0$, and the triangle is then the right triangle aot.

Many other interesting relations in elliptic triangles can be derived from the geometry of the ellipse. For instance, if ac_n is considered divided by the Y axis into two segments au and $uc_n=w$, then $w/v=(r^2+v)/(s^2+v)$ which is equivalent to $2w=(r^2+v)(1-(s^2-v)/(s^2+v))$, and to

$$(r^2 + 2v - w)^2 = r^4 + 2(2s^2 - r^2)w + w^2.$$

The ordinate y of the c_n vertex is

$$y = \sqrt{r^2 - s^2}\sqrt{r^2s^2 - v^2}/s = \sqrt{r^2 - s^2}\sqrt{r^2 - \left(\frac{v}{s}\right)^2},$$

which is rational when $v=s^2$ and $n=2$.

5. Reduction of case I to case II. Before we conclude, let us go back to the exterior elliptic fermagoric triangles of type I, when the vertex c_n is located on the elliptic arc dc_2 outside of the fermatic region, and (r^2+v) is the fermagoric hypotenuse.

In Fig. 2 let abc_n be such a triangle of order n greater than 2. Through point e where its longer side crosses the frontier arc F of the fermatic region, draw ef parallel to c_nb. Then triangle afe is similar to triangle abc_n and therefore it is also

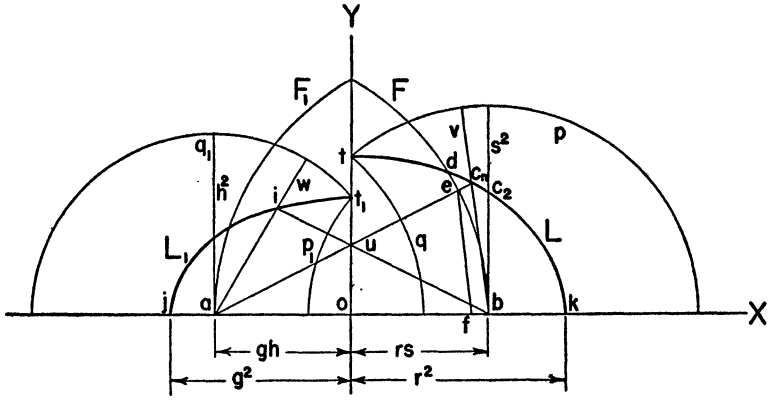


FIG. 2

fermagoric of nth order. Let u be the point where ac_n crosses the ordinate axis OY. Through b and u draw a line and on it make $bi=af$. Then triangle abi is equal to triangle aef, except that now the hypotenuse (longer side) lies on ab, and its upper vertex i rests within the fermatic region. In it

$$\frac{ai}{r^2 - v} = \frac{ab}{r^2 + v} = \frac{ib}{2rs},$$

where $ab=2rs$. Hence

$$ib = \frac{4r^2s^2}{r^2 + v}, \qquad ai = \frac{2rs(r^2 - v)}{r^2 + v}$$

and we have $(2rs)^n=[4r^2s^2/(r^2+v)]^n+[2rs(r^2-v)/(r^2+v)]^n$.

We may ask now: To what family of interior elliptic fermagoric triangles would the triangle *abi* belong?

To find out we must draw through point i and with focal distance ab, the second ellipse L_1 which will cut the Y axis at t_1 and the X axis at j. If then we make $oj=at_1=g^2$, and $ao=rs=gh$, the triangle *abi* will belong to the elliptic family $(2gh)^n=(g^2+w)^n+(g^2-w)^n$, which is of type II. We shall now show that if ever abc_n has rational sides, so will *abi* have. Therefore we can treat *abi* in the same way that we have treated those belonging to type II.

In the interior triangle $gh=rs$.

$$ib = g^2 + w = 4r^2s^2/(r^2 + v).$$
$$ia = g^2 - w = 2rs(r^2 - v)/(r^2 + v).$$
$$g^2 = rs[2rs + (r^2 - v)]/(r^2 + v).$$
$$w = rs[2rs - (r^2 - v)]/(r^2 + v).$$

Therefore if for some high value of n in the exterior triangle the root-vector $\mathbf{v}$ is ever rational, so will the values of gh, g^2, and w be in the similar interior triangle. The new parameters in the latter must be such that r, g, h and s diminish in value in that order. The eccentricity of the new ellipse L_1 is h/g, which must be greater than that of the first ellipse s/r.

The fact that gh, g^2 and w in $(2gh)^n=(g^2+w)^n+(g^2-w)^n$ would be rational, making this interior Fermat equation rational, does not imply that g is necessarily rational, nor that h and w are integers.

However, in our initial treatment of elliptic fermagoric triangles we chose to make the Fermat equation $(2rs)^n+(r^2-v)^n=(r^2+v)^n$ only for the convenience of getting $v=s^2$ in the case $n=2$, so that the resulting identity $(2rs)^2+(r^2-s^2)^2=(r^2+s^2)^2$ would result immediately in this classical Pythagorean solution. For either case we could just as well have started with $(2s)^n=(r+v_1)^n+(r-v_1)^n$ where $c=2s$, $(a+b)/2=r$, $(b-a)/2=v_1$, and $c^n=b^n+a^n$.

The eccentricity of the fermatic ellipse for case I would remain as before s/r, and the corresponding reduction to case II would be now $(2h)^n=(g+w_1)^n+(g-w_1)^n$ where h and g and w_1 would be rational if r, s and v_1 were integral. Now, if h and g were both rational, the eccentricity of the new ellipse L_1 would again be rational.

6. Conclusion. In this very preliminary study of the isoperimetric elliptic fermagoric triangles, we believe to have established that they are worthy of further study which we foresee may possibly be the source of new and interesting aspects of Number Theory as well as of Theory of Equations.

Their application to the study of Fermat's Problem depends on our finding more precise properties in those points on the ellipse corresponding to the apexes of fermagoric triangles of the higher orders, where the ellipse intersects the corresponding n-fermat ovals.

Perhaps we can develop relevant relations on these apex points and the Dedekind lattice of rational points on the plane, which may aid us in determining the possible rationality or irrationality of $r^2\pm v$ when $n>2$.

CONCERNING PEDAL CIRCLES AND SPHERES*

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1. Introduction. In the *Nouvelles Annales de Mathématiques* (1904, p. 400) M. T. Lemoyne announced the following theorem: *The radical axes of all pairs of the pedal circles, relative to a triangle ABC, of the points of a transversal Δ, are concurrent in a point ω.*

Analytical demonstrations† and geometrical solutions‡ of Lemoyne's theorem have occupied a number of geometers. We are unaware if the MONTHLY has published any of the demonstrations of this important proposition. The following, which uses only elementary geometrical properties, possesses the advantage of leading immediately to a theorem on pedal spheres, relative to a tetrahedron, of points on a line. A statement of this latter theorem was communicated to Cl. Servais (December, 1921) after having ourselves established a particular interesting case.§ The analytical and geometrical demonstrations of this space theorem as supplied by Cl. Servais are not simple.‖

2. Geometrical demonstration of the theorem of M. T. Lemoyne. Let $A_1B_1C_1$, $A_2B_2C_2$ be the pedal triangles, relative to a triangle ABC, of two points P_1, P_2 of any given line Δ, and let $A_1'B_1'C_1'$, $A_2'B_2'C_2'$ be their anticomplementary triangles.** Also, let D_1, D_2, D_3 be the points of intersection of the corresponding sides of triangles $A_1B_1C_1$ and $A_2B_2C_2$, and E_1, E_2, E_3 those of the corresponding sides of triangles $A_1'B_1'C_1'$ and $A_2'B_2'C_2'$. The lines D_1E_1, D_2E_2, D_3E_3 are concurrent in a point P, the *equicenter* of triangles $A_1B_1C_1$ and $A_2B_2C_2$.†† If P_1 remains fixed while P_2 describes the line Δ, the points A_2, B_2, C_2 describe similar ranges on BC, CA, AB, and the point P of intersection of the lines D_1E_1, D_2E_2, D_3E_3 remains fixed and has the same barycentric coordinates α, β, γ for the triangle $A_1B_1C_1$ and all the triangles $A_2B_2C_2$. Therefore, if through P one draws segments PA_3', PB_3', PC_3' equal and parallel to A_1A_2, B_1B_2, C_1C_2, the point P

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* Received in April, 1942. Translated from the French by Howard Eves.

† J. Neuberg, Bull. de l'Acad. royale de Belgique, 1910; Liénard, Mathesis, 1912, p. 238 and 1914, p. 33; R. Goormaghtigh, Mathesis, 1925, p. 196; R. Goormaghtigh, The Tohoku Mathematical Journal, 1926, p. 80.

‡ de Lépiney, Mathesis, 1914, p. 177; Gallatly, The Modern Geometry of the Triangle, 2nd ed., p. 51; R. Bouvaist, Nouvelles Annales, 1915, p. 555; V. Thébault, Nouvelles Annales, 1918, p. 293; V. Thébault, The Tohoku Mathematical Journal, vol. 19, p. 33.

§ V. Thébault, The Tohoku Mathematical Journal, 1919 and 1921.

‖ Cl. Servais, Bull. de l'Acad. royale de Belgique, 1922, p. 32; Cl. Servais, Mathesis, 1922, p. 87.

** Triangle $A'B'C'$ is *anticomplementary* to triangle ABC if A, B, C are the midpoints of $B'C'$, $C'A'$, $A'B'$ respectively. Tetrahedron $A'B'C'D'$ is *anticomplementary* to tetrahedron $ABCD$ if A, B, C, D are the centroids of the faces $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$ respectively. (H. Eves)

†† E. Duporcq, Intermédiaire des Mathématiciens, 1899, p. 98; Sollerstinsky, Mem. de la Soc. royale des Sciences de Liège, 3rd series, t. X, 1913.

It may be shown that the barycentric coordinates of P with respect to triangle $A_1B_1C_1$ are the same as those with respect to triangle $A_2B_2C_2$. Hence the name *equicenter*. (H. Eves)

is the barycenter of the points A_3', B_3', C_3' for the coefficients α, β, γ.* These coefficients are, then, proportional to the products $(B_1B_2)(C_1C_2)\sin A, \cdots$, that is, to $a\sec\theta_a$, $b\sec\theta_b$, $c\sec\theta_c$,† where a, b, c are the lengths of the sides BC, CA, AB, and θ_a, θ_b, θ_c the angles of like sense which these sides make with Δ.

Designating by (ω_1, ρ_1), (ω_2, ρ_2) the circles with centers ω_1, ω_2 and radii ρ_1, ρ_2, circumscribed about the triangles $A_1B_1C_1$, $A_2B_2C_2$, we have‡

$$\rho_1^2\sum\alpha = (\omega_1P)^2\sum\alpha + \sum(PA_1)^2\alpha,$$
$$\rho_2^2\sum\alpha = (\omega_2P)^2\sum\alpha + \sum(PA_2)^2\alpha.$$

Also, if M_a is the midpoint of A_1A_2, P_a the projection of P on A_1A_2, we find by simple geometry that

$$\sum(PA_1^2 - PA_2^2)\alpha = \sum(A_1A_2)(M_aP_a)\alpha.$$

On the other hand let M be the midpoint of P_1P_2, which projects into M_a on A_1A_2 and into M_a' on PA_3'. We have $M_aP_a = M_a'P$ and, therefore,

$$\sum(A_1A_2)(M_aP_a)\alpha = \sum(PA_3')(M_a'P)\alpha.$$

Now, being given any n points whatever, $A_1, A_2, \cdots, A_n$, and the barycenter, G, of these points for coefficients $\alpha_1, \alpha_2, \cdots, \alpha_n$, and if $l_1, l_2, \cdots, l_n$ are the projections of any vector whatever on $GA_1, GA_2, \cdots, GA_n$, then we have, in both magnitude and sign,§

$$\sum\alpha_1(GA_1)l_1 = 0.$$

We then have

$$\sum(PA_3')(M_a'P)\alpha = 0,$$

and therefore

$$(\omega_1P)^2 - \rho_1^2 = (\omega_2P)^2 - \rho_2^2,$$

which proves that P is the center of a circle (P, σ) orthogonal to circle $A_1B_1C_1$ and all the circles $A_2B_2C_2$. This proves Lemoyne's theorem.

For the square of the radius of the circle (P, σ) we have

$$\sigma^2 = (a_1^2\beta\gamma + b_1^2\gamma\alpha + c_1^2\alpha\beta)/(\alpha+\beta+\gamma)^2,$$

a_1, b_1, c_1 being the lengths B_1C_1, C_1A_1, A_1B_1. One can verify that the power of P with respect to the circles $A_1B_1C_1$, $A_2B_2C_2$ is equal to $-2d\delta$, d and δ being the distances of the circumcenter of ABC and of P from Δ, so that P coincides with the orthopole of Δ with respect to triangle ABC.

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* See, *e.g.*, exercise 8 on page 94 of M'Clelland's Geometry of the Circle. (H. Eves)

† $[(B_1B_2)(C_1C_2)\sin A, \cdots] = [(B_1B_2)(C_1C_2)a, \cdots] = [P_1P_2\cos\theta_bP_1P_2\cos\theta_ca, \cdots]$
$= [a\sec\theta_a, \cdots]$. (H. Eves)

‡ See, *e.g.*, art. 55, p. 99, of M'Clelland's Geometry of the Circle. In this article replace P and O by ω_1 and P respectively. (H. Eves)

§ R. Blanchard, Journal de Vuibert, t. 50, p. 12.

3. Geometrical demonstration of the theorem of Servais and Thébault. *The pedal spheres, relative to a tetrahedron* $T \equiv ABCD$, *of the points of a given line* Δ, *have a common orthogonal sphere.*

Let $T_1 \equiv A_1B_1C_1D_1$, $T_2 \equiv A_2B_2C_2D_2$ be the pedal tetrahedra, relative to tetrahedron T, of the points P_1, P_2 on an arbitrary line Δ. If we designate by a, b, c, d the lines of intersection of the corresponding faces of the two tetrahedra T_1 and T_2, and by a', b', c', d' the lines of intersection of the corresponding faces of the tetrahedra anticomplementary to T_1 and T_2, then the planes (a, a'), (b, b'), (c, c'), (d, d') are copunctual at a point P which has the same barycentric coordinates for the tetrahedra T_1 and T_2, and the point P remains fixed when, with P_1 fixed, P_2 describes the line Δ. This point P is the *equicenter* of the tetrahedra T_1 and T_2.* If through P one draws the segments PA_3', PB_3', PC_3', PD_3' equal and parallel respectively to A_1A_2, B_1B_2, C_1C_2, D_1D_2, P is the barycenter of the points A_3', B_3', C_3', D_3' for the coefficients α, β, γ, δ, the barycentric coordinates of P for the tetrahedra T_1 and T_2. Designating the spheres circumscribing T_1 and T_2 by (ω_1, ρ_1), (ω_2, ρ_2), we then have

$$\rho_1^2 \sum \alpha = (\omega_1 P)^2 \sum \alpha + \sum (PA_1)^2 \alpha,$$
$$\rho_2^2 \sum \alpha = (\omega_2 P)^2 \sum \alpha + \sum (PA_2)^2 \alpha,$$

whence, as in the plane case, M being the midpoint of P_1P_2, M_a' its orthogonal projection on PA_3',

$$\sum (PA_1^2 - PA_2^2)\alpha = \sum (A_1A_2)(M_aP_a)\alpha = \sum (PA_3')(M_a'P)\alpha = 0,$$

and finally

$$(\omega_1 P)^2 - \rho_1^2 = (\omega_2 P)^2 - \rho_2^2,$$

which proves that P is the center of a fixed sphere orthogonal to the pedal sphere of P_1 and to those of all the points P_2 of Δ.

If we designate by Ω_A, Ω_B, Ω_C, Ω_D the trihedral angles $(P-B_3'C_3'D_3')$, $(P-C_3'D_3'A_3')$, $(P-D_3'A_3'B_3')$, $(P-A_3'B_3'C_3')$, whose edges are parallel to (B_1B_2, C_1C_2, D_1D_2), $\cdots$, and by θ_a, θ_b, θ_c, θ_d the angles of like sense which Δ makes with the faces BCD, CDA, DAB, ABC of T, the barycentric coordinates α, β, γ, δ of the center P of the sphere (P, σ) orthogonal to the spheres (ω_1, ρ_1), (ω_2, ρ_2) are equal to $(B_1B_2)(C_1C_2)(D_1D_2) \sec \theta_a \sin \Omega_A$, $\cdots$. Finally, for the square of the radius of the sphere (P, σ), we have

$$\sigma^2 = \sum \alpha\beta(A_1B_1)^2/(\sum \alpha)^2,$$

a formula which does not seem susceptible to transformation into a simple relation among the elements of the fundamental tetrahedron T.

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* J. Neuberg, Mem. de la Soc. royale des Sciences de Liège, 1913, *loc. cit.*

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

Beginning with the August-September number of the MONTHLY the editor of this department will be Professor E. F. Beckenbach, University of California at Los Angeles, Los Angeles 24, California. Manuscripts should henceforth be sent to him.

ON THE EQUATION $\phi(x)=2m$

V. L. KLEE, JR., University of Virginia

Carmichael proved (*Bull. Amer. Math. Soc.*, 13 (1907), 241–243) that all solutions of the equation

$$\phi(x) = 2m \tag{1}$$

(where m is odd and >1 and ϕ is the Eulerian function) are of the form p^a, $2p^a$, where p is a prime of the form $4s-1$. Using this result we have at once the following

THEOREM. *The number of distinct solutions of* (1) *is exactly twice the number of ways in which m can be expressed in the form* (F) $n(2n+1)^k$, *where* $2n+1$ *is prime and k is an integer* $\geqq 0$.

For let $x=p^a=(4s-1)^a$ be a solution of (1). Then $\phi(p^a)=(4s-2)(4s-1)^{a-1}=2m$, whence, letting $n=2s-1$, we have $m=n(2n+1)^{a-1}$. Conversely, if $n(2n+1)^k=m$ is in the form (F), let $x_1=(2n+1)^{k+1}$ and $x_2=2x_1$. Then $\phi(x_1)=\phi(x_2)=2m$.

As corollaries we have:

I. *For* (1) *to have no solutions it is sufficient that m have no divisor* $d>1$ *for which* $2d+1$ *is prime.*

(Hence $\phi(x)$ is excluded from the value $2m$ for $m=7, 13, 17, 19, 31, \cdots$.)

II. *If* $m=\Pi p_i^{a_i}$, *where each prime* p_i *is of the form* $3k_i+1$, (1) *has no solutions.*

Since any product of numbers of the form $3k+1$ is itself of that form, any divisor of m must be of the form $3k+1$. Since $2(3k+1)+1$ is divisible by 3, the only expression of m in the form (F) is $1\cdot 3^0$, whence $m=1$, which contradicts the original assumption that $m>1$. (Hence $\phi(x)$ is excluded from the value $2m$ if m is any product formed from the factors 7, 13, 19, 31, $\cdots$.)

III. *If* $m=p^2$, *where p is a prime* >3, (1) *has no solution.*

We have $2p^2+1=2(p-1)(p+1)+3$, which is divisible by 3. (Since p must be of one of the forms $3k+1$ and $3k-1$.) Then if $2p+1$ is not prime, III follows from I. But if $2p+1$ is prime it is impossible to have $p^2=p(2p+1)^k$ unless $p=1$ and $k=0$. (Hence $\phi(x)$ is excluded from the value $2m$ for $m=25, 121, 289, \cdots$.)

IV. *The number of solutions of* (1) *is either* 0, 2, *or* 4.

Using the theorem, we need only show that m can be expressed in the form (F) in not more than two ways. Let $m=n_1(2n_1+1)^{k_1}=n_2(2n_2+1)^{k_2}$ be distinct expressions of m in the form (F). Clearly $(2n_1+1)^{k_1}$ divides either n_2 or $(2n_2+1)^{k_2}$. If it divides n_2 we have $n_2=q(2n_1+1)^{k_1}$, so $n_2>n_1$. But we must also have, $n_1=q(2n_2+1)^{k_2}$, whence $n_1>n_2$, which is a contradiction. Hence $(2n_1+1)^{k_1}$ divides $(2n_2+1)^{k_2}$. But then $n_1=n_2$. Hence for $k>0$, m has at most one expression in the form (F), and for $k=0$ the only possible expression is $m(2m+1)^0$.

THE SINE AND COSINE AS PROJECTION FACTORS

A. R. JERBERT, University of Washington

The following discussion of the sine and cosine as projections of a unit vector on the axes leads to a simple proof of the addition theorems for the sine and cosine. In figure 1 the terminal sides of *any* angle θ, and an equally general

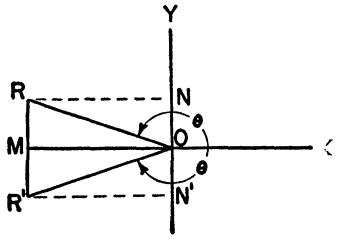


FIG. 1

angle $-\theta$, are evidently mirror images of each other in the x-axis. Therefore, OM is their common projection, so that, with $OR=OR'=1$,

$$\cos\theta = OM = \cos(-\theta). \tag{1}$$

Since MR and MR' extend in opposite directions from the "mirror," we have the equally evident, and equally general,

$$\sin(-\theta) = -\sin\theta. \tag{2}$$

The generality of the equations,

$$\begin{aligned} \sin(\theta+180) &= -\sin\theta \\ \cos(\theta+180) &= -\cos\theta, \end{aligned} \tag{3}$$

is also intuitive, since adding 180° to an angle reverses the terminal side and its projections on the two axes.

Cos θ is the *fundamental* projection ratio in the sense that it projects the terminal side of an angle θ, upon its initial side, which is the $+x$-axis if the angle is in "standard" position. Sin θ, on the other hand, projects the same unit vector on a line which is 90° *in advance*, namely the $+y$-axis. In figure 1, *e.g.*, $\sin\theta=ON$ is the projection of OR on OY. Evidently, the same projection results from replacing $\theta=\angle XOR$, by $\theta-90=\angle YOR$ and employing the cosine in place of the sine. Thus,

$$\sin \theta = \cos (\theta - 90). \tag{4}$$

Since $(\theta-90)$ is *any* angle, we may denote it by α, so that equation (4) becomes

$$\sin (\alpha + 90) = \cos \alpha. \tag{5}$$

Letting $\alpha=\beta+90$, we have,

$$\sin (\beta + 180) = \cos (\beta + 90),$$

so that by reference to equation (3)

$$-\sin \beta = \cos (\beta + 90). \tag{6}$$

Cosine and sine projections which are directed oppositely to OX and OY are, nevertheless, identified with these directions, respectively, since the associated minus sign supplies the "correction" as in equations (3). With this understanding it can be stated, without exception, that $\cos \theta$ and $\sin \theta$ project the terminal side of any angle θ, upon its initial side, and a line 90° in advance of the initial side, respectively.

In figure 2 the angles α, β, and $(\alpha+\beta)$ are identified by drawing a curved arrow for β, and simply labeling the terminal sides of the other two, since they

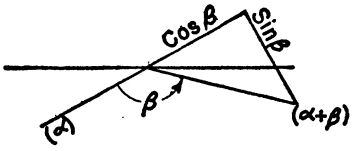


FIG. 2

are in "standard position." The functions of θ and $\theta \pm k360$ are identical so that there is no loss of *generality* in considering α and β as positive and less than 360°. The sine of $(\alpha+\beta)$ is the y-projection of its terminal side, 1. We may therefore compute this function by projecting the component vectors, $\cos \beta$ and $\sin \beta$, on the y-axis. Since the terminal side of α is initial for β, these vectors, in view of the remark following equation (6), are in the α and $\alpha+90$ directions; so that multiplication by $\sin \alpha$ and $\sin (\alpha+90) = \cos \alpha$, respectively, accomplishes the projection. Hence,

$$\sin (\alpha + \beta) = y\text{-proj. } (\cos \beta, \sin \beta) = \cos \beta \sin \alpha + \sin \beta \cos \alpha.$$

The remaining addition theorems may be obtained, in the usual manner, by taking advantage of equations (1) to (6).

Note by the Editor. Professor R. A. Johnson has called my attention to the fact that the proof of Feuerbach's theorem by H. F. Sandham, this MONTHLY, vol. 52, 1945, p. 571, is given in his *Modern Geometry*, p. 200, §321. The proof, he believes, is an old one, but he is unable to give a reference. R. Goormaghtigh also notes that this proof is given in practically the same form by W. J. Hodgetts in the *Mathematical Gazette*, 1944, p. 198. M. J. W.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

An Historical and Analytical Bibliography of the Literature of Cryptology. By J. S. Galland. Evanston, Northwestern University, 1945. 8+209 pages. $5.00.

The scope of this book is described very well in the following quotation from the preface:

"This volume is a register of the most important works that have been written, not only on the subject of cryptography, but also on its manifestations in related fields. Hence, the items presented will be seen to vary considerably in nature and content: some are scientific and technical treatises dealing with the application of cryptography to military, diplomatic, and commercial uses; others dwell on the more popular and more limited manifestations of this science, and appear in a variety of disparate forms,—in that of cuneiform, runic, and other kinds of hieroglyphic writings, in literary anagrams and acrostics, in symbolism of colors, gems, emblems, and insignia, in the jargon of children as well as that of thieves, in various types of universal languages and stenographic systems, in tramps' signs and semaphoric signals, and also in the crude devices and ingenious stratagems which were employed during primitive ages as a means of concealing and conveying thought and information."

The volume being reviewed consists of between 1600 and 1700 references to items on the above subjects. Some references are to short notes, while others are to works of several hundred pages. The references are listed alphabetically,—mostly by author's name, but occasionally by subject, or by the name of the company sponsoring the publication. Many of the listings are followed by comments concerning the referenced work. From these comments, which account for the "Analytical" in the title of the book, it is clear that the author devoted a great amount of work to the task of making the volume as useful as possible.

Since this review is written primarily for teachers of mathematics, it should be pointed out that, although the book contains a few references to mathematical articles having cryptology or cryptography in the title, those interested in the application of mathematics to cryptology may find that the volume is not as useful as might be desired from this point of view.

It seems to the reviewer that this book will be particularly useful to those interested in cryptology from the historical point of view, or as a recreation. The author's comments concerning the referenced works should be, in many cases, very helpful in estimating the probable usefulness of the item in question.

(Opinions and assertations expressed in this review are the private ones of the reviewer and are not to be construed as official or reflecting the views of the United States Navy Department or of the United States Naval Service at large.)

A. W. LANDERS, Lt. Comdr., USNR

Algebra. A Second Course. By R. O. Cornett. New York and London, McGraw-Hill Book Co., Inc., 1945. 13+313 pages. $2.00.

This text, which is planned for students who have had one year of high school algebra, could be used satisfactorily with students of this group who wish to study elementary mathematics in college for engineering or for later specialization in science or mathematics. It might serve as a text for those who are to study mathematics only as a part of their general education if the department prefers the more traditional course for this purpose.

Aside from the first two chapters there is no extreme departure from the traditional procedure or content. The intent of these two chapters to provide "an understanding of purpose and significance in advance of details" is certainly commendable. Only experience will determine if this can be done in so short a space. Chapter 1, Mental Gymnastics, contains numerous problems intended to assist the student in the development of what the author calls basic mental skills. Problems are planned to demonstrate visualization, manipulation of mental images, memorizing, systematic thinking, and concentration and endurance. In Chapter 2, the author compares a "systematic trial" method of solving problems involving two unknowns with the algebraic method. Uses of formulas in science and engineering are also brought out in this chapter.

Although the contents of chapters 3–10 are quite elementary, the book does contain sufficient material for a semester's work in algebra. There are 21 chapters in all. An indication of the extent of the material is given by chapter headings such as simple systems involving quadratics, systems leading to an equation of fourth degree, imaginary numbers, trigonometry, extraneous roots and irrational equations, progressions and the binomial theorem. The book contains the tables necessary for the topics covered. Answers to odd-numbered exercises are given. No use of determinants is made in the text.

There are numerous innovations which are worthy of mention. For example, students are led to solve stated problems involving two unknowns first by use of two variables, later being shown the possibility of using only one variable as a short cut. The space, through three chapters, given to quadratic equations in two variables in which pairs of equations are solved graphically and algebraically should give assistance in later work in analytic geometry and the calculus. In the chapter on the algebraic solution, frequent reference is made to the graphs of the preceding chapter.

Exercises on the use of graphs in the solution of stated problems are found in the chapter on Graphical Methods. Listed as optional topics in this chapter are the determination of the maximum or minimum value of a function from the graph and functional notation.

The organization of topics related to roots and radicals is different in that there is a chapter on square roots and radicals preceding quadratic equations and a later chapter on exponents and roots preceding logarithms. More complete than that usually found in freshman texts is the explanation of interpolation errors. In the chapter on logarithms, the exponential form is used even for rather difficult computations before the logarithmic notation is introduced.

Special techniques employed which are helpful include use of double asterisks (**) at the end of sentences to direct the student to stop reading and think carefully through what has just been discussed; classification of stated problems; use of the symbol $\doteq$ for approximate relationships; and review questions at the ends of chapters. The setting off of rules and of special formulas in boxes is desirable but is perhaps overdone in the more elementary parts of the text. The use of boys' names in the discussions on pages 84 and 254 seems a little juvenile for a college text.

The reviewer is most critical of what seems to him to be a carelessness in certain statements or expressions which might lead to misunderstanding on the part of the student even though later discussions of the same topics are faultless. This criticism is especially applicable to the discussion of roots of quadratic equations at the top of page 90 and to some of the later references to imaginary numbers as on page 154 and 187. Other examples are found in the paragraph on equations involving fractions on page 125 and the reference to the recording of an angle measurement on page 242.

The cover and whole make-up of the book are attractive. The pages are not crowded and are very readable. There are a number of misprints in the text, but none of a serious nature were noted. All things considered this book is a worthwhile addition to the present supply of algebra texts. Its strength lies in its fulfillment of a new need in the college field, through its elementary approach and its special emphasis on motivation. The practical selection of topics and carefully chosen exercises and applications are factors enhancing its value.

J. R. MAYOR

Elements of Calculus. By W. A. Granville, P. F. Smith, and W. R. Longley. Boston, Ginn and Co., 1946. 11+549 pages. $3.75.

The book bearing the above title is a revision of *Elements of the Differential and Integral Calculus* by the same authors. The only essential way in which it differs from the latter book is in the order of arrangement of the contents.

In the revised edition chapters dealing with indefinite and definite integrals, together with integration as a summation process and the Fundamental Theorem of the Integral Calculus, precede the differentiation of transcendental functions. This arrangement makes the revised edition particularly adaptable to courses in which the early introduction of both differentiation and integration is a desideratum.

H. P. EVANS

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Department of Mathematics, Oregon State College, Corvallis, Oregon.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 726. *Proposed by W. Nicholson, Chicago, Illinois*

If p and N are positive integers, $p>1$, show that N^p is the sum of N consecutive odd integers.

E 727. *Proposed by Frederick Mosteller, Princeton, N. J.*

What is the probability that a King and a Jack will appear side by side in a shuffled pack?

E 728. *Proposed by Victor Thébault, Tennie, Sarthe, France*

With the vertices A, B, C of an equilateral triangle as centers draw the circles (A), (B), (C) which are concurrent at the center O of the triangle, and then draw an arbitrary circle (D) passing through O. Show that the length of one of the common tangents to the circles (A) and (D), (B) and (D), (C) and (D) is equal to the sum of the lengths of the other two.

E 729. *Proposed by F. J. Duarte, Caracas, Venezuela*

Let x, y, z be three real positive numbers such that $x^3+y^3=z^3$, and set

$$\lambda = \sqrt[3]{3(x+y)}/(\sqrt[3]{z-x}+\sqrt[3]{z-y}).$$

Show that

$$(\sqrt[3]{2}+\sqrt[3]{4})/2 < \lambda < \sqrt[3]{3}.$$

E 730. *Proposed by J. H. Butchart, Arizona State College*

The interior angle bisectors of a triangle meet the noncorresponding sides of the medial triangle in six points which lie in pairs on the lines joining the points of tangency of the inscribed circle.

SOLUTIONS

A Ball in a Vase

E 687 [1945, 578]. *Corrected. Proposed by Victor Thébault, Tennie, Sarthe, France*

A heavy ball is gently dropped into a vase full of water, in the shape of a segment of a paraboloid of revolution. The size of the vase is given; that of the ball is such as to cause the maximum displacement. Find the radius of the ball.

Solution by E. P. Starke, Rutgers University. Let an axial cross section of the parabolic vase have the equation $x^2=2py$, and let its depth be a. Since the curvature of the parabola decreases monotonically as the distance from the vertex increases, the larger the ball the higher it will come to rest in the vase. Further, if a ball rests on the rim of the vase, not tangent to the parabola, a slightly smaller ball will sink farther in and displace more water. Hence, to determine a maximum, we need consider only balls which are tangent to the parabola and tangent to or intersecting the water surface.

If the ball is tangent to the parabola at the point (x_1, y_1), $y_1 \leqq a$, its center is at $(0, y_1+p)$ because of the subnormal property of the parabola. Suppose r is the radius of the ball and h is the depth of the submerged part. We then have

$$a = y_1 + p - r + h, \qquad r^2 = x_1^2 + p^2 = 2py_1 + p^2. \tag{1}$$

Now the volume of a spherical segment of radius r and height h is

$$V = \pi h^2(3r - h)/3. \tag{2}$$

Noting from (1) that

$$dy_1/dr = r/p, \qquad dh/dr = 1 - r/p,$$

we obtain from (2), by differentiation and simplification,

$$dV/dr = \pi hr(h + 2p - 2r)/p.$$

The usual tests for a maximum are satisfied when $2r=2p+h$, whence *the desired ball extends to a height* $2p$ *(the latus rectum of the generating parabola) above the surface of the water.* The value of r in terms of the given dimensions of the vase is found by eliminating h and y_1; it is

$$r = (2ap + 4p^2)^{1/2} - p.$$

Also solved by W. B. Campbell, M. P. de Regt, Roy Dubisch, R. E. Gaines, S. T. Parker, J. V. Pennington, Nathan Schwid, Elijah Swift, P. D. Thomas, and W. B. Zacharias.

Heronian Triangles with Sides in Arithmetic Progression

E 695 [1945, 516]. *Proposed by H. L. Lee, University of Tennessee*

Find triangles whose sides are integers in arithmetic progression, and whose areas are integers.

both odd; $k=1/3$ when n is a multiple of 3; $k=1/6$ when both conditions hold; $k=1$ in all other cases.

An interesting special case arises if the sides are to be consecutive integers. Then $d=1$ and (1) becomes

$$a^2 - 3x^2 = 4.$$

The left member cannot be divisible by 4 except when a, x are both even: say $a=2A$, $x=2X$. If A_n, X_n is a solution of $A^2-3X^2=1$, so also is A_{n+1}, X_{n+1}, where

$$A_{n+1} = 2A_n + 3X_n, \qquad X_{n+1} = A_n + 2X_n. \tag{2}$$

From $A_0=1$, $X_0=0$ all solutions can be found by successive applications of (2). The first few solutions yield $a=4$, 14, 52, 194.

Also solved by F. A. Alfieri, Murray Barbour, P. T. Bateman, J. H. Cross, Monte Dernham, Daniel Finkel, B. A. Hausmann, W. L. Johnson, J. B. Kelly, M. Kirk, W. G. McGavock, Eric Michalup, C. D. Olds, S. T. Parker, P. A. Piza, L. S. Shively, R. H. Urbano, and the proposer. A number of these solutions gave only partial results and some of the solutions arrived too late to be considered for publication.

C. D. Olds referred to vol. II of Dickson's *History of the Theory of Numbers* for further problems of this type. Eric Michalup recorded 513 primitive solutions. L. S. Shively directed attention to the special solutions (939, 1876, 2813) and (193, 194, 195). The first is interesting because the triangle is so slender, the largest angle being approximately 175° 25′. The second is interesting because the triangle is so nearly equilateral. Lest someone may think that there is not a more nearly equilateral triangle, Shively also mentioned the triangle (805307497, 805308626, 805309755) with area 280818256141860864, inradius 232472576, and angles (to the nearest tenth of a second) 59° 59′ 59.5″, 60° 00′ 00.0″, 60° 00′ 00.5″.

Approximate Construction of Regular Pentagon

E 697 [1945, 578]. *Proposed by C. A. Murray, West Texas State College*

A certain geometry text raises the question whether the following procedure will inscribe a regular n-gon in a circle: AB being a diameter of the circle, construct an equilateral triangle ABC. Divide AB into n parts and let D be the second point of division from A. Draw CD, producing it to cut the circle at E. Is AE the side of a regular n-gon inscribed in the circle? For n equal to 3 or 4 the answer is readily affirmative. Does the procedure yield a regular pentagon for n equal to 5? If not, give a measure of the error.

I. *Solution by J. H. Cross, Lubbock, Texas.* The procedure does not give a regular pentagon; the arc so obtained is short by 2′ 48″. To see this let O be the center of the circles and r its radius. Draw EO. Evidently $AC=2r$, $AD=4r/5$, $DO=r/5$, $EO=r$, and $\sphericalangle DAC=60°$. From the law of tangents $\sphericalangle ADC=96°$ 35′ 13″ $=\sphericalangle EDO$. From the law of sines $\sphericalangle DEO=11°$ 27′ 35″. Hence $\sphericalangle AOE=71°$ 57′ 12″ instead of the required 72°.

II. *Solution by E. D. Schell, Arlington, Va.* Let the circle be of unit radius with its center at the origin. For $n=5$, the coordinates of C and D are $(0, \sqrt{3})$ $(-1/5, 0)$. The line CD has the equation $5\sqrt{3}x-y+\sqrt{3}=0$ and intersects the circle $x^2+y^2=1$ in the point E with coordinates

$$\{(-15-\sqrt{73})/76,\ \sqrt{3}(1-5\sqrt{73})/76\}.$$

The point A has the coordinates $(-1, 0)$, and the distance AE is found to be $\sqrt{(61-\sqrt{73})/38}=1.17491$. The side of a pentagon in a circle of unit radius, however, is $\sqrt{(5-\sqrt{5})/2}=1.17558$. The estimate is close enough to be useful since the error is only 7 parts in 10,000.

III. *Solution by Elmer Latshaw, Philadelphia, Pa.* Denote by θ the central angle subtended by side AE of the inscribed n-gon. By algebra and trigonometry we find

$$\cos\theta = (n-4)[3n+(n^2+16n-32)^{1/2}]/4(n^2-2n+4).$$

Solving this equation for various values of n gives:

n	θ by formula	$360°/n$	error per side
3	120.000°	120.000°	0.000°
4	90.000°	90.000°	0.000°
5	71.953°	72.000°	−0.047°
6	60.000°	60.000°	0.000°
7	51.528°	51.429°	0.099°
8	45.186°	45.000°	0.186°
9	40.278°	40.000°	0.278°
10	36.356°	36.000°	0.356°
20	18.633°	18.000°	0.633°

The procedure gives an exact construction for $n=3$, 4, and 6. For $n=5$ the error is small and $-0.047°$ per side; the error is positive and increases with n when $n>6$.

Also solved by G. W. Grotts, S. T. Parker, P. A. Piza, W. A. Rees, and R. H. Urbano.

A Construction in the Complex Plane

E 698 [1945, 578]. *Proposed by J. M. Feld, Brooklyn College*

Let the complex numbers a and b represent two points on the Gauss plane. If $\rho=re^{i\alpha}$, (r and α real), show how one can construct the point corresponding to $z=(a+\rho b)/(1+\rho)$.

Solution by E. A. Hedberg, M. I. T. Radar School, Boston. Solving the given relation for $-\rho$ we have

$$(z-a)/(z-b) = -\rho = re^{i(\alpha+\pi)},$$

whence

$$|z-a|/|z-b| = r, \tag{1}$$

$$\arg(z-a) - \arg(z-b) = \alpha + \pi + 2n\pi, \tag{2}$$

where n is an integer.

The locus represented by equation (1) is a circle with its center on the line ab and dividing the segment from a to b internally and externally in the ratio r. The locus of equation (2) is an arc of a circle constructed upon segment ab as chord and tangent to the half line aR, where the positive angle baR is equal to α. Equation (2) is satisfied for any point on the arc lying on the same side of ab as aR. Therefore the intersection of this arc with circle (1) is the required point z.

If ρ is real, so that $\alpha = n\pi$, then z divides the segment ab internally or externally in the ratio r.

Also solved by G. W. Grotts, H. A. Luther (three ways), S. T. Parker and the proposer.

A Property of Some Cevians

E 699 [1945, 578]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3 be the feet of the altitudes, the symmedians, and the cevians through the circumcenter, on the sides BC, CA, AB of a triangle ABC. (a) The lines B_1C_1, B_2C_2, B_3C_3 are concurrent in a point M_1; C_1A_1, C_2A_2, C_3A_3 in a point M_2; A_1B_1, A_2B_2, A_3B_3 in a point M_3. (b) Triangle $M_1M_2M_3$ is homological to, and circumscribes, triangle ABC.

Solution by H. E. Fettis, Dayton, Ohio. Let H, K, O be respectively the orthocenter, symmedian point, and circumcenter of triangle ABC. Since the isogonal conjugates of these points are on a line, the points themselves are on a conic through the vertices of the triangle. Therefore $B(HKOA) = C(HKOA)$, and since the ranges $(B_1B_2B_3A)$ and $(C_1C_2C_3A)$ are sections of these pencils, $(B_1B_2B_3A) = (C_1C_2C_3A)$, whence B_1C_1, B_2C_2, B_3C_3 are concurrent at a point M_1. Similarly, C_1A_1, C_2A_2, C_3A_3 are concurrent at M_2; A_1B_1, A_2B_2, A_3B_3 are concurrent at M_3.

Now, let BM_1 intersect AC in N_2, and let BM_3 intersect AC in N_2'. We have

$$(N_2'B_1B_2B_3) = (BA_1A_2A_3) = A(BHKO) = C(BHKO),$$

and

$$(N_2B_1B_2B_3) = (BC_1C_2C_3) = C(BHKO),$$

so that $(N_2'B_1B_2B_3) = (N_2B_1B_2B_3)$, and N_2' coincides with N_2. Therefore B is on M_3M_1. Similarly, A is on M_2M_3, and C is on M_1M_2.

Further, let AM_1, BM_2 intersect in a point P, and let PM_3 cut BA, BC, AC in Q_3, R_3, R_3' respectively. Then

$$B_1(A_1C_1BA) = A(M_3M_1BN_2) = (M_3PQ_3R_3),$$

and

$$A_1(B_1C_1AB) = B(M_3M_2AN_1) = (M_3PQ_3R_3').$$

But $B_1(A_1C_1BA)=A_1(B_1C_1AB)$, since these are two harmonic pencils. Therefore R_3 and R_3' coincide at C and triangles ABC and $M_1M_2M_3$ are in perspectivity at P. The axis of perspectivity is $N_1N_2N_3$, the trilinear polar of P for triangle ABC.

Also solved by R. Bouvaist, J. H. Butchart, and L. M. Kelly.

Editorial Note. R. Bouvaist, employing barycentric coordinates, established the more general theorem: *Given a triangle ABC and three points D, E, F in its plane, whose cevians meet BC, CA, AB in D_a, D_b, D_c; E_a, E_b, E_c; F_a, F_b, F_c respectively. If the trilinear polars of D, E, F are concurrent, then*

$$D_bD_c,\ E_bE_c,\ F_bF_c \text{ are concurrent in } R,$$

$$D_cD_a,\ E_cE_a,\ F_cF_a \text{ are concurrent in } S,$$

$$D_aD_b,\ E_aE_b,\ F_aF_b \text{ are concurrent in } T.$$

Further, if two of the points D, E, F are isogonal conjugates for triangle ABC, then RST is homological to, and circumscribes, ABC.

By a slight improvement of Bouvaist's proof we may show that the restriction that two of the points D, E, F be isogonal conjugates is unnecessary, and that *RST is homological to, and circumscribes, ABC for any three points, D, E, F whose trilinear polars are concurrent.*

Following is an indication of the proof of the modified version of Bouvaist's generalization. If (x_D, y_D, z_D), (x_E, y_E, z_E), (x_F, y_F, z_F) are projective coordinates of D, E, F, the condition of the concurrency of their trilinear polars is

$$\begin{vmatrix} 1/x_D & 1/y_D & 1/z_D \\ 1/x_E & 1/y_E & 1/z_E \\ 1/x_F & 1/y_F & 1/z_F \end{vmatrix} = 0,$$

which may be written as

$$\begin{vmatrix} -1/x_D & 1/y_D & 1/z_D \\ -1/x_E & 1/y_E & 1/z_E \\ -1/x_F & 1/y_F & 1/z_F \end{vmatrix} = 0,$$

from which it follows that D_bD_c, E_bE_c, F_bF_c are concurrent in a point R. Similarly, D_cD_a, E_cE_a, F_cF_a are concurrent in a point S and D_aD_b, E_aE_b, F_aF_b are concurrent in a point T.

We readily obtain the following coordinates for R and S:

$$R:\ (1/y_Dz_E - 1/y_Ez_D,\ 1/x_Dz_E - 1/x_Ez_D,\ 1/y_Dx_E - 1/y_Ex_D),$$

$$S:\ (1/y_Dz_E - 1/y_Ez_D,\ 1/x_Dz_E - 1/x_Ez_D,\ 1/x_Dy_E - 1/x_Ey_D),$$

whence we see that R, S, C are collinear. Similarly, S, T, A and T, R, B are collinear, and RST circumscribes ABC.

Finally, RA and SB intersect in the point

$$P: \quad (1/y_D z_E - 1/y_E z_D,\ 1/z_D x_E - 1/z_E x_D,\ 1/x_D y_E - 1/x_E y_D).$$

Since the coordinates of P are symmetrical in those of D and E it follows that RST and ABC have P as center of perspectivity.

Needless to say, Fettis's proof also is capable of some generalization.

Kelly used trilinear coordinates. In terms of the notation of this note he first showed that *if, D, E, F are any three points in the plane of triangle ABC such that D_bD_c, E_bE_c, F_bF_c are concurrent in a point R, then D_cD_a, E_cE_a, F_cF_a are concurrent in a point S and D_aD_b, E_aE_b, F_aF_b are concurrent in a point T, and triangle RST is homological to, and circumscribes, triangle ABC.* He then completed the solution of the proposed problem by showing that the orthocenter, symmedian point, and circumcenter of ABC are three points like D, E, F.

Butchart established the first part of the theorem by cross-ratio methods. For the second part he referred to example 12, page 390 of Smith's *Conic Sections* (new edition, 1927).

Multinomial Coefficients

E 700 [1945, 579]. *Proposed by Arnold Dresden, Swarthmore College*

If h and i are respectively positive and non-negative integers, then

(a) $$\sum \frac{1}{p_0!p_1!\cdots p_i!} = \frac{(h+i-1)!}{i!(h-1)!h!},$$

where the sum on the left is extended over all sets of non-negative integers $p_0, p_1, \cdots, p_i$ which satisfy the conditions $p_0+p_1+\cdots+p_i=h$, $p_1+2p_2+\cdots+ip_i=i$;

(b) under the further restriction $i \geqq h-1$,

$$\sum \frac{1}{p_1!\cdots p_{i+1}!} = \frac{i!}{(i+1-h)!(h-1)!h!},$$

where the sum on the left is extended over all sets of non-negative integers $p_1, \cdots, p_{i+1}$ for which $p_1+\cdots+p_{i+1}=h$, $p_1+2p_2+\cdots+(i+1)p_{i+1}=i+1$.

Solution by D. W. Alling, Rochester, N. Y. In the identity

$$(1+x+\cdots+x^i)^h = (1-x^{i+1})^h(1-x)^{-h}$$
$$= (1-x^{i+1})^h \sum_{k=0}^{\infty} \binom{h+k-1}{k} x^k,$$

the coefficient of x^i on the right side is $\binom{h+i-1}{i}$, while by the multinomial theorem, the coefficient of x^i on the left side is $\sum h!/(p_0!p_1!\cdots p_i!)$, where the summation and the p's have the same meaning as defined in part (a) of the problem. Hence

$$\sum \frac{1}{p_0!p_1!\cdots p_i!} = \frac{(h+i-1)!}{i!(h-1)!h!}.$$

Similarly, comparing the coefficients of x^{i+1} in the identity

$$(x + x^2 + \cdots + x^{i+1})^h = x^h(1 - x^{i+1})^h(1 - x)^{-h}$$
$$= x^h(1 - x^{i+1})^h \sum_{k=0}^{\infty} \binom{h + k - 1}{k} x^k,$$

we have

$$\sum \frac{h!}{p_1!p_2! \cdots p_{i+1}!} = \binom{h + (i + 1 - h) - 1}{i + 1 - h} = \binom{i}{i + 1 - h},$$

which has meaning only if $h \leqq i+1$, and where the summation and the p's have the same meaning as defined in part (b) of the problem.

Also solved in the same way by H. S. Grant, M. S. Knebelman, E. D. Schell, F. Underwood, and the proposer. Schell pointed out that the necessary expansion theorems are given in Chrystal's *Algebra*, Vol. II, pp. 15 and 193.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4206. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider spheres with centers at the vertices of a tetrahedron $ABCD$ and radii equal respectively to k times the sum of the squares of the three opposite edges. Show that the sum of the squares of the distances from the four vertices to the center of the sphere orthogonal to the four spheres is equal to

$$[2(4k + 1)]^2R^2 - 2k(k + 1)\Sigma,$$

where R is the radius of the circumsphere and Σ means the sum of the squares of the six edges. Consider particular cases.

4207. *Proposed by K. W. Miller, Chicago, Ill.*

Consider the determinant $|a_{ij}| = \Delta$ of order n where

$$a_{ij} \neq 0, \quad \text{when} \quad (i - j)^2 \leqq k^2$$
$$a_{ij} = 0, \quad \text{when} \quad (i - j)^2 > k^2$$
$$1 \leqq i, j \leqq n$$
$$0 \leqq k \leqq (n - 1), \quad k \text{ arbitrary but fixed,}$$

that is, an nth order determinant Δ in which $(2k+1)$ consecutive diagonals symmetrical about and including the principal diagonal have *no* zero elements, while all other elements are zero in the two equal triangular corners of the determinant at each side of the central diagonal band. How many non-zero terms are there in the completely expanded determinant?

4208. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given an orthocentric tetrahedron, if two isogonal conjugate points are also conjugate with respect to the circumsphere, their pedal sphere is orthogonal to the sphere whose center is the complement of the orthocenter for the linear net determined by the spheres circumscribed and conjugate.

SOLUTIONS

De L'Hospital's Rule

4151 [1945, 163]. *Proposed by B. M. Stewart, Michigan State College*

Let O be a point at which a given curve has a second derivative; let the tangent and normal at O serve respectively as x, y axes, the equation of the curve becoming $y=f(x)$; and let P; (x, y) be a point on the curve, say with positive x. Denote the arc length OP by s and locate the point S: $(s, 0)$. The line SP intersects the y-axis in the point B:$(0, b)$. If R indicates $[1+(y')^2]^{3/2}/y''$, show that the limiting position of the point B as P approaches O is such that $\lim b = 3 \lim R$.

This is a generalization of a problem in the calculus of Granville, Smith, and Longley, 1934, p. 177, ex. 20.

Solution by L. A. Santaló, Rosario, Argentina. Let $x=x(s)$, $y=y(s)$ be the parametric equations of the curve $y=f(x)$ with the arc length s as parameter. If we denote by accents derivatives with respect to s, it is well known that $(x')^2+(y')^2=1$ and $x'x''+y'y''=0$. From this and from $y'((0)=0$ and $R=1/y''$, we deduce $x''(0)=0$. Hence

$$(1) \qquad x(0) = y(0) = 0; \quad x'(0) = 1, \; y'(0) = 0; \quad x''(0) = 0, \; y''(0) = 1/R_0;$$

where

$$R_0 = \lim_{s\to 0} R.$$

From the equation of the straight line SP we deduce $b=OB=sy/(s-x)$. Applying l'Hospital's rule twice, we have

$$\lim_{s\to 0} b = \lim_{s\to 0} \frac{sy}{s-x} = \lim_{s\to 0} \frac{y+sy'}{1-x'} = \lim_{s\to 0} \frac{2y'+sy''}{-x''}.$$

From the fact that $x'x''=-y'y''$, and from (1), we deduce that

$$\lim_{s\to 0} b = \lim_{s\to 0} \frac{2y'x'+sy''x'}{y'y''} = 2R_0 + \lim_{s\to 0} \frac{sx'}{y'},$$

and, if we apply this rule again, we find that

$$\lim_{s\to 0} b = 3R_0.$$

This problem may be generalized in the following way. Let us consider two plane curves C, C_1 tangent at O, and let R and R_1 be their radii of curvature at O. Let P be a point on C and P_1 a point on C_1 with the condition that arc length $OP=s$ equals arc length OP_1. The straight line of PP_1 meets the normal at O in point B; then we have

$$\lim_{s\to 0} OB = \frac{3}{\frac{1}{R}+\frac{1}{R_1}}. \tag{2}$$

If $R_1=\infty$ we obtain the proposed problem. If $R=R_1$, the formula (2) is not applicable. In this case, if $(dR/ds)_0 \neq (dR_i/ds)_0$, we find that

$$\lim_{s\to 0} OB = \tfrac{4}{3}R. \tag{3}$$

From (2) and (3) may be deduced a geometrical example referring to a function of two variables for which the order of limits cannot be interchanged. Let us consider the curves C and C_1 tangent at O with $R\neq R_1$, and suppose that we wish the limiting position of the point B as $R_1\to R$ and $s\to 0$. If we take first $s\to 0$ and then $R_1\to R$, by (2) it follows that lim $OB=3R/2$. But if we take first $R_1\to R$ and then $s\to 0$, we have by (3) lim $OB=4R/3$.

All these results and the proposed problem are treated in our paper *Algunas propriedades infinitesimales de las curvas planas*, Mathematicae Notae, Año I, pp. 128–144, 1941.

Solved also by Mrs. R. C. Buck, Howard Eves, J. F. Heyda, J. B. Kelly, A. Sisk, C. E. Springer, J. T. Webster, and the proposer.

Editorial Note. The solutions by Eves and Springer used formulas for the coordinates x, y of a point on a curve in terms of the arc length parameter s, the first referring to Graustein's *Differential Geometry*, p. 39, equations (46), and the second to Eisenhart's *An Introduction to Differential Geometry*, p. 26. The remaining solvers use three applications of de l'Hospital's theorem in a manner somewhat similar to the above solution.

We may also use vector methods as follows. Consider the curve as concave upward at O passing through P in its neighborhood, say to the right; and let $\mathbf{t}$ and $\mathbf{n}$ be the unit vector tangent and normal at O forming a right hand system. Let $\mathbf{r}$ be the vector of P, the origin of vectors being arbitrarily chosen, where $\mathbf{r}$ is a function of s, the arc length to P with the positive sense $\widehat{OP}$. Then at O we find the values

$$\mathbf{t} = d\mathbf{r}/ds = \mathbf{r}';\ \mathbf{t}' = \mathbf{r}'' = \kappa\mathbf{n},\ \kappa = 1/R;\ \mathbf{n}' = -\kappa\mathbf{t};\ \mathbf{t}'' = \kappa'\mathbf{n} - \kappa^2\mathbf{t};$$
$$\mathbf{t}''' = (\kappa'' - \kappa^3)\mathbf{n} - 3\kappa\kappa'\mathbf{t};\ \textit{etc.} \tag{1}$$

Using these results we have

$$\overrightarrow{OP} = d\mathbf{r} = \mathbf{t}ds + \frac{\kappa\mathbf{n}}{2!}(ds)^2 + \frac{(\kappa'\mathbf{n} - \kappa^2\mathbf{t})}{3!}(ds)^3 + \cdots \tag{2}$$

Since $\overrightarrow{OS} = \mathbf{t}ds$, the sum of the remaining terms of (2) is $\overrightarrow{SP}$; and then $b\mathbf{n} = \mathbf{t}ds + \overrightarrow{ASP}$. This last equation gives

$$\begin{aligned} 1 &= \frac{A(ds)^2}{3!}\left[\kappa^2 + \tfrac{3}{4}\kappa\kappa' ds + \cdots\right]; \\ b &= \frac{A(ds)^2}{2}\left[\kappa + \frac{\kappa' ds}{3} + \cdots\right]; \\ b &= 3R\,\frac{\left[1 + \dfrac{R\kappa'}{3}ds + \cdots\right]}{\left[1 + \dfrac{3R\kappa'}{4}ds + \cdots\right]}. \end{aligned} \tag{3}$$

The limit value of b is now easily obtained.

Trigonometric Expansion of Binomial Coefficients

4152 [1945, 163]. *Proposed by William J. Taylor, Washington, D. C.*

Prove the following trigonometric expansion for the binomial coefficient

$$\frac{N!}{\left(\frac{N+x}{2}\right)!\left(\frac{N-x}{2}\right)!} = \frac{2^N}{N}\sum_{m=1}^{N}\left(\cos\frac{m\pi}{N}\right)^N \cos\frac{m\pi x}{N}, \qquad -N < x < N.$$

I. *Solution by J. B. Kelly, Hampton, Va.* Set $w = e^{\pi i/N}$, then the right member in the problem becomes

$$\begin{aligned} &\frac{1}{2N}\sum_{n=1}^{N}(w + w^{-n})^N(w^{nx} + w^{-nx}), \\ &\quad = \frac{1}{2N}\sum_{n=1}^{N}\sum_{k=0}^{N}\binom{N}{k} w^{n(N-2k)}(w^{nx} + w^{-nx}), \\ &\quad = \frac{1}{2N}\sum_{k=0}^{N}\binom{N}{k}\sum_{n=1}^{N}\left[w^{n(N-2k+x)} + w^{n(N-2k-x)}\right]. \end{aligned} \tag{1}$$

Since $-N < x < N$, we have

$$\sum_{n=1}^{N} w^{n(N-2k+x)} = 0, \tag{2}$$

unless $N - 2k + x = 0$, in which case the sum in (2) is N. Similarly,

$$(3) \qquad \sum_{n=1}^{N} w^{n(N-2k-x)} = 0, \qquad -N < x < N,$$

unless $k=(N-x)/2$, and the sum in (3) is again N. Hence (1) becomes

$$\frac{1}{2N}\left[N\binom{N}{\frac{N+x}{2}} + N\binom{N}{\frac{N-x}{2}}\right] = \frac{N!}{\left(\frac{N+x}{2}\right)!\left(\frac{N-x}{2}\right)!},$$

and the proof is complete.

II. *Solution by C. D. Olds, San Jose State College, Calif.* Let $w=e^{2\pi i/N}$, and use the binomial theorem to obtain in turn the expansions

$$(1+w)^N = \binom{N}{0} + \binom{N}{1}w + \binom{N}{2}w^2 + \cdots + \binom{N}{N}w^N,$$

$$(1+w^2)^N = \binom{N}{0} + \binom{N}{1}w^2 + \binom{N}{2}w^4 + \cdots + \binom{N}{N}w^{2N},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(1+w^N)^N = \binom{N}{0} + \binom{N}{1}w^N + \binom{N}{2}w^{2N} + \cdots + \binom{N}{N}w^{N^2}.$$

Multiply the rows through respectively by $w^{-r}, w^{-2r}, \cdots, w^{-Nr}$, where r is an integer, $0<r<N$, and then add. We obtain at once

$$\sum_{m=1}^{N}(1+w^m)^N w^{-rm} = N\left[\binom{N}{r} + \binom{N}{N+r} + \cdots\right] = N\binom{N}{r},$$

since

$$\binom{N}{kN+r} = 0.$$

On the other hand

$$\sum_{m=1}^{N}(1+w^m)^N w^{-rm} = \sum_{m=0}^{N}(w^{m/2}+w^{-m/2})^N \omega^{m(N-2r)/2},$$

$$= \sum_{m=0}^{N}\left(2\cos\frac{m\pi}{N}\right)^N\left[\operatorname{cor}\frac{m(N-2r)\pi}{N} + i\sin\frac{m(N-2r)\pi}{N}\right]$$

$$= N\binom{N}{p}.$$

Equating real and imaginary parts we get

$$\binom{N}{r} = \frac{2^N}{N}\sum_{m=1}^{N}\left(\cos\frac{m\pi}{N}\right)^N \cos\frac{m(N-2r)\pi}{N}.$$

Letting $x=N-2r$ we see that this is the expression we set out to verify.

Remarks: The same method can be used to show that if $q<N$ and $0\leqq r\leqq q-1$, then

$$\binom{N}{r}+\binom{N}{q+r}+\binom{N}{2q+r}+\cdots=\frac{1}{q}\sum_{m=0}^{q-1}\left(2\cos\frac{m\pi}{q}\right)^N\cos\frac{m(N-2r)\pi}{q}.$$

This leads to the interesting special cases:

$$\binom{N}{0}+\binom{N}{3}+\binom{N}{6}+\cdots=\frac{1}{3}\left(2^N+2\cos\frac{N\pi}{3}\right),$$

$$\binom{N}{1}+\binom{N}{4}+\binom{N}{7}+\cdots=\frac{1}{3}\left(2^N+2\cos\frac{(N-2)\pi}{3}\right),$$

and so on. This expression was first obtained, I believe, by C. Ramus, *Journ. f. Math.* vol. 11 (1834), p. 353.

Solved also by the professor.

Editorial Note. The result in Old's remark may be obtained by Kelly's method. The integers r, q, N are such that $1\leqq q<N$, $0\leqq r\leqq q-1$. Set $w=e^{\pi i/q}$, then

$$\begin{aligned}\frac{1}{q}\sum_{m=0}^{q-1}&(2\cos m\pi/q)^N\cos m(N-2r)\pi/q\\ &=\frac{1}{2q}\sum_{m=0}^{q-1}(w^{-m}+w^m)^N\left[w^{m(N-2r)}+w^{-m(N-2r)}\right]\\ &=\frac{1}{2q}\sum_{m=0}^{q-1}\sum_{k=0}^{N}{}_NC_k\left[w^{2(N-r-k)m}+w^{2(r-k)m}\right]\\ &=\frac{1}{2q}\sum_{k=0}^{N}{}_NC_k\sum_{m=0}^{q-1}\left[w^{2(N-r-k)m}+w^{2(r-k)m}\right].\end{aligned}\tag{1}$$

The second sum consists of two geometric series of q terms each, and in first, if $N-r-k$ for the fixed k is neither zero nor a multiple of q, the sum of the series is zero. If on the contrary it is zero or a multiple q, the sum is q, and in this case k has the values $N-r$, $N-r-q$, $N-r-2q$, $\cdots$, $N-r-[(N-r)/q]q$. In a similar manner the second series is zero, if $r-k$ is neither zero nor a multiple of q; if it is, then the sum is again q, and $k=r$, $r+q$, $r+2q$, $\cdots$, $r+[(N-r)/q]q$. Hence

$$\sum_{t=0}^{[(N-r)/q]}{}_NC_{r+tq}=\frac{2^N}{q}\sum_{m=0}^{q-1}\left(\cos\frac{m\pi}{q}\right)^N\cos\left[\frac{m(N-2r)\pi}{q}\right].\tag{2}$$

Tetrahedron and Rolling Sphere

4153 [1945, 163]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A sphere (S), radius r, rolls on the plane of the face BCD of the tetrahedron

$ABCD$ so that its center S lies on a fixed sphere concentric with (O) the circumsphere of $ABCD$. Show that: (1) The tetrahedron $BCDS$ is inscribed in a fixed sphere (S'). (2) The sphere tangent interiorly to (S) and passing through the points B, C, D envelopes a fixed sphere concentric with (O). If OS' is prolonged by $S'A'=r/2$, the sphere with center A' and passing through B, C, D is orthogonal to (S).

Solution by Howard Eves, College of Puget Sound.

(1) As (S) rolls in the prescribed manner, the center S traces a circle σ on the sphere (O). This circle σ and the circle α, passing through B, C, D, have a common axis. Therefore σ and α are cospherical on a sphere (S').

(2) It is clear that all spheres passing through B, C, D and interiorly tangent to the various positions of (S) coincide, and the theorem of part (2) of the problem becomes trivial.

(3) Let OS' cut the plane BCD in point K. Denote $S'K$ by k, and let s be the radius of (S'), and t the radius of the sphere (A') having center A' and passing through B, C, D. Let L be the foot of the perpendicular from S on OS'. Then

$$\begin{aligned} t^2 &= A'K^2 + KD^2 = A'K^2 + (S'D^2 - S'K^2) \\ &= (s^2 - k^2) + (k + r/2)^2 = s^2 + r^2/4 + kr. \end{aligned}$$

Also

$$\begin{aligned} A'S^2 &= A'L^2 + SL^2 = A'L^2 + (S'S^2 - S'L^2) \\ &= (3r/2 + k)^2 + [s^2 - (r + k)^2] \\ &= s^2 + 5r/4 + kr. \end{aligned}$$

It follows that

$$A'S^2 = t^2 + r^2,$$

whence (A') and (S) are orthogonal.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

A conference on Algebra will be held at the University of Chicago during the week of July 15–19, 1946. A program is being planned consisting of approximately twelve invited addresses. Several of the sessions will emphasize recent contacts of algebra with other fields such as topology, geometry and function theory. There will also be a session devoted to pure algebra.

The Society for the Promotion of Engineering Education has instituted a George Westinghouse Award in Engineering Education to be given to the college or university teacher adjudged to have contributed most to the successful teaching of engineering students. The amount of the award is $1,000, to be conferred annually. For details inquire of Dr. Harry S. Rogers, Polytechnic Institute of Brooklyn.

Dr. H. W. Alexander has been appointed to a professorship at Adrian College, Adrian, Michigan.

Professor L. D. Ames of the University of Southern California has retired.

Professor H. F. Bohenblust of Indiana University has been appointed to a professorship at California Institute of Technology.

Professors E. T. Browne and M. A. Hill and Associate Professor E. A. Cameron have returned to the University of North Carolina.

Assistant Professor J. H. Curtiss of Cornell University has been appointed assistant to the director of the National Bureau of Standards in Washington, D. C.

Assistant Professor W. J. Dixon of the University of Oklahoma has been promoted to an associate professorship.

Dr. D. C. Duncan of Los Angeles City College has been appointed chairman of the mathematics department at East Los Angeles Junior College.

Associate Professor Samuel Eilenberg of the University of Michigan has been appointed to a professorship at Indiana University.

Captain H. T. Engstrom, U.S.N.R. has been awarded the rank of Officer in the Military Division of the Order of the British Empire.

Dr. Benjamin Epstein has been appointed mathematician in the *Coal* Research Laboratory of Carnegie Institute of Technology—not in the *Cost* Research Laboratory as was previously announced.

Dr. R. W. Erickson has been appointed to an associate professorship at Carleton College, Northfield, Minnesota.

Professor V. G. Grove and Assistant Professors A. W. Jones, C. P. Wells and J. W. Zimmer have returned to Michigan State College from leaves of absence.

Assistant Professor S. G. Hacker has been promoted to an associate professorship at Washington State College.

J. F. Hubbard, formerly of the W. B. Rogers School, Hyde Park, Boston, is now teaching in the School for Veterans in the High School of Commerce Building, Boston.

Professor H. K. Justice of the University of Cincinnati has been appointed assistant dean and director of admissions in the College of Engineering.

Assistant Professor Mark Kac of Cornell University and Associate Professor J. W. T. Youngs of Indiana University have been awarded Guggenheim Fellowships.

Dr. E. S. Kennedy of the University of Alabama has been appointed to an adjunct professorship at the American University of Beirut, Beirut, Lebanon.

Dr. P. A. Lagerstrom of Bell Aircraft Corporation has been appointed research aerodynamicist with the Douglas Aircraft Company of Santa Monica, California.

Dr. Walter Leighton of the Rice Institute has been appointed to a professorship at Washington University.

Professor W. T. Martin of Syracuse University has been appointed to a professorship at Massachusetts Institute of Technology.

Associate Professor G. M. Merriman of the University of Cincinnati has been promoted to a professorship.

Assistant Professors Ivan Niven and A. H. Smith of Purdue University have been promoted to associate professorships.

P. B. Norman has been appointed to an assistant professorship at the School of Engineering of Pratt Institute.

Dr. M. J. Norris has been appointed to an assistant professorship at the College of St. Thomas, St. Paul, Minnesota.

Associate Professor V. C. Poor of the University of Michigan has retired with the title of professor emeritus.

Drs. Maxwell Reade and G. S. Young of Purdue University have been promoted to assistant professorships.

Assistant Professor Henry Scheffé of Syracuse University has been appointed to an associate professorship in engineering at the University of California at Los Angeles.

Dr. R. W. Shephard of Bell Aircraft Corporation has been appointed to an assistant professorship at Purdue University.

W. F. Smith has been appointed to an assistant professorship at the University of Detroit.

Professor J. L. Synge of Ohio State University has been appointed head of the applied mathematics division of Carnegie Institute of Technology. Professor Tibor Radó has been appointed chairman of the department of mathematics at Ohio State University.

Assistant Professor F. J. Taylor of the College of St. Thomas has been promoted to an associate professorship.

Associate Professor R. M. Trimble of the University of North Carolina has been promoted to a professorship.

Drs. R. W. Wagner and E. P. Vance of Oberlin College have been promoted to assistant professorships.

Associate Professor D. L. Webb of Texas Technological College has been appointed to an assistant professorship at the University of Arizona.

Lieutenant Commander R. H. Wilson, Jr. of the United States Naval Academy has been appointed to an assistant professorship at Temple University.

Dr. P. M. Young has returned to Miami University as an assistant professor.

The following appointments to instructorships are announced:
Agricultural and Mechanical College of Texas: Dr. R. E. Basye
College of St. Thomas: Eric Clamons, Donald Lewis, Dr. Takashi Terami
Lehigh University: S. D. Bernardi, Dr. H. K. Brown
Missouri School of Mines: Neil Lockwood
North Carolina State College: S. R. Baker
Oregon State College: Dr. J. C. R. Li
Pratt Institute (School of Engineering): G. C. Helme
Purdue University: Dr. Bernard Dimsdale, H. L. Harter, J. P. Wayne, Dr. John Dyer-Bennet, G. L. Kinnett, Dr. Sam Perlis, G. C. Sipple
Rutgers University: Dr. Lee Byrne
Syracuse University (Endicott Branch): Frances M. Wright
Tulane University: E. P. Miles, Jr.
The University of Buffalo: H. W. Baeumler, Ruth A. Brendel, Irene L. Fisher, Mrs. Magdalene M. Hettler, C. E. Maley, Audrey L. Strabel, Mrs. Rosemary R. Trautman
University of Michigan: Dr. Samuel Kaplan
University of Missouri: L. M. Kelly
University of Pennsylvania: Dr. Y. C. Wong
The University of Rochester: Norman Gunderson
University of Southern California: Dr. P. J. Kelly

Dr. Fletcher Durell died March 25, 1946. He was a charter member of the Association.

Professor Laurence Hadley of Purdue University died March 21, 1946. He was a charter member of the Association.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE GRADUATE RECORD EXAMINATION

The Graduate Record Examination is a project begun in 1937 under the sponsorship of the Graduate Schools of Harvard, Yale, Princeton, and Columbia Universities in collaboration with the Carnegie Foundation for the Advancement of Teaching. Since that time the growth of the project has been so rapid that the Graduate Record Examination has become an important, and a controversial, feature of American academic life. The following paragraphs have been selected from reports published by the Graduate Record Office and by the Carnegie Foundation for the Advancement of Teaching.

"During a period of confusion in the calculation of time credits and course credentials on which progress in American colleges and universities has traditionally rested, this Examination is believed to offer an element of stability theoretically sounder than the customary measures when these were used alone and certainly deserving of thoroughgoing trial in the present emergency. Broadly covering the student's total current academic resources, the results provide a series of comparable measures that make his available equipment from whatever source immediately intelligible at any institution.

"Early in 1944, foreseeing the problem of the returning veteran provided with partial or irregular credentials of the old type, a group of graduate schools familiar with the tests (Harvard, Yale, Michigan, Wisconsin) urged that they be made generally available. The proposal finally issued by them to their colleagues in other schools was to the effect that in return for the widespread requirement or recommendation of the Examination to applicants, the Carnegie Foundation would undertake to make it readily accessible throughout the country at a small fee, reimbursing a part of the cost.

"The suggestion that the Examination could be conveniently taken and the results submitted with the student's application before admission appealed strongly to the graduate schools at large, especially to those receiving students from a distance or from little-known institutions. By June, 1944, the cooperating graduate schools numbered 57, all of which either required or invited the applicant for admission to submit results of the Examination as auxiliary credentials. During the year now closing (June, 1945) this number has increased to 98, of which 38 make the Examination a requirement either of all or of certain classes of its applicants.

"In addition to the 98 sponsoring institutions there are (June 30, 1945) 110 other colleges or universities at which official examiners under contract with

the Foundation are available to give the tests. Of these 208 institutional centers, 11 are in Canada.

"Although the Examination originated from the need of graduate schools of liberal arts and sciences for a trustworthy picture of the student's underlying education, it is obvious to those familiar with it that its usefulness is much wider. Any professional institution of higher than freshman level—medicine, law, business, engineering—that is concerned about the intellectual attainment and emphases of its prospective students can get accurate and comprehensive information from these tests. Likewise in most cases norms will be found indicating the performance of large numbers of students having in view the particular professional goal desired.

"All told, during the past year, 1944–45, 6,535 students took the Graduate Record Examination. Of these, 1,213 were individual applicants tested at over 200 different centers—many of them at the central office in New York—while 5,322 were in 101 institutional groups tested for the routine purposes of the institution concerned. These bring to a total of more than 45,000 the number of complete scorings of the G.R.E. that have been made since its inception.

"The Graduate Record Examination is administered in two parts. The general or elementary part covers seven subject matter fields and a general vocabulary test. This part of the Examination is taken by all students without regard to their major subject. Separate scores are reported for each subject as follows: Mathematics, Physics, Chemistry, Biology, Social Studies, Literature, Fine Arts, and Verbal Factor. These scores which constitute a record of general education may be graphically shown in the Report Chart, which is made available to the student.

"In addition to the general tests, the record will include scores on a 90-minute specialized examination suitable for testing students in their subjects of major study. At the time of registering for the Examination, the student indicates his choice of Advanced Test. Such tests are offered in History of Fine Arts, Biological Science, Chemistry, German, Economics, French, Geology, History, Government, Psychology, Literature, Mathematics, Physics, Engineering, Sociology, and Philosophy. Each student elects one advanced subject test.

"The questions employed in the tests which comprise the Graduate Record Examination require answers that are unequivocal and admit of precise tabulation. Most of the questions are written in the familiar multiple-choice form; some require the student to complete a statement.

"The Examination is administered as a unit in two half-day sessions. The sessions are scheduled for separate days to minimize fatigue. The actual working time is about three hours per session; the formal schedule requires a morning or afternoon of four hours to allow for a mid-session intermission and additional time for students who may require more than normal time. The separate tests are given within specified time limits. These time limits are generous, and most students will have finished as much of the test as their knowledge permits before the time limit is reached.

"Primary emphasis in the General Mathematics Test is placed on skill in quantitative thinking. The problems presented in the test vary widely in their derivation but the emphasis is generally of a practical nature. The test includes problems dealing with the interpretation of data in graphical and tabular form, with inconsistencies in numerical data, with the transformation of verbal descriptions into mathematical descriptions and the reverse transformations, with evaluation of the soundness of conclusions drawn from numerical data, with the recognition of assumptions that must be made before valid conclusions can be drawn, and with similar accepted outcomes for the general development of college students. The items in this test were selected from a variety of situations which the advanced college student is likely to meet during his college study. Quantitative problems from various fields of study are included."

Inquiries regarding the Examination may be addressed to the Secretary of the Graduate Record Examination, 437 West 59th Street, New York 19, N. Y.

THE VETERAN EDUCATION PROBLEM

The number of veterans desiring to take advantage of the educational provisions of Public Laws 346 and 16 has far exceeded all of the early estimates. By February 28, a total of almost 1,500,000 applicants had been received by the Veterans Administration; the trend is indicated by the fact that 340,000 of these applications were received in January and 456,000 in February. Not all of these "cases in file" are now in education and training; some have completed or discontinued their training, and many have applied to receive their certificate of eligibility and entitlement but have not yet enrolled in an educational institution. Dr. Francis J. Brown, consultant for the American Council on Education, recently stated that "four million veterans will take advantage of educational benefits in five years—if the benefits are available."

On the basis of the number of applicants, a previous estimate of 750,000 veterans enrolled in our colleges and universities by September, 1946, appears to be conservative. So, too, does the estimate that the total enrollment in higher education by next fall will be at least 25% above the previous peak enrollment of 1,480,000 in 1939–40. The total number will also be increased by at least 100,000 high school graduates if Selective Service is modified on May 15 to include only men over 20 years of age. Never have the colleges faced so serious a responsibility nor so great an opportunity! U. S. Solicitor General J. H. McGrath analyzed the situation now facing colleges and universities in the following words: "The great bottleneck now is housing, and it undoubtedly will continue to vex us; the next problem will be where to find instructors even if we do find more housing."

The editor of this section recently made a survey of a large number of mathematics departments to ascertain plans and policies for taking care of the greatly increased enrollment in mathematics courses. It appears up to the present time that most departments have succeeded in obtaining sufficient staff members to handle the large teaching load; frequently, however, it has been necessary to

increase slightly the credit-hour load of each instructor and the class size. In general, the maximum class size seems to be 30 or 35. Regular staffs are being augmented in many cases by the return to the class room of married women who formerly taught mathematics. Some of the better prepared high school teachers are being drawn into the college field, and more graduate students are being given teaching assignments. Since a large part of the heavy load at the present time is to be found upon the elementary level, such solutions of the teaching problem seem to be working out very well, although, as one person points out, "the general lowering of the academic level of teachers does not do justice to the minority of superior students who would profit by contact with teachers of higher academic qualifications." Several institutions are developing plans for the closer supervision of inexperienced instructors.

Some institutions are making extensive and reasonably effective use of guidance and placement examinations in the registration of students. In spite of all precautions, however, many institutions find during the early weeks of a semester that a large number of students are improperly registered, so there is a considerable amount of re-registration after a term starts. To accommodate those veterans who are unprepared to handle college work, a large number of review courses are being conducted. These courses may take up high school algebra, plane and solid geometry, and even arithmetic. At least one institution reports the operation of special tutoring sessions at which veterans may obtain help.

To take care of the shortage in class rooms, several institutions have extended the length of the teaching day. Late afternoon and night classes seem to be common. In some institutions the noon hour is being used.

Advanced undergraduate students are generally being used to advantage. For example, many senior students are being assigned as assistants to instructors having large classes. One correspondent reported that he knew of some mathematics majors at the senior level who taught classes in freshman mathematics at the conclusion of the first World War. He indicated his belief that a good student at the senior level would derive much benefit from teaching one freshman course, and "he would probably do a good job of it."

One large midwestern institution is operating seven large classes in mathematics during the present term in addition to many sections of the usual size. Three of the large classes are five-hour courses in college algebra and trigonometry, and four of them are five-hour courses in analytic geometry; each large class contains 100 to 125 students. On four days of the week, the instructor of a large class meets the group as a unit in a lecture period; on the fifth day, each class is divided into four sections, all under the supervision of the instructor who has given the lectures. The problem of scheduling the four sections on the fifth day was easily solved by analyzing the vacant periods of the class members on that day. An instructor handling one of the large classes has his teaching load figured at 12 hours for one group. Also, a student assistant is assigned to each large class.

Virtually all institutions are seeking additional staff members for the fall term, since most colleges and universities are assuming that there will be at least a 25% increase in enrollment.

THE LOUISIANA-MISSISSIPPI EDUCATIONAL COMMITTEE

W. V. PARKER AND H. T. KARNES, Louisiana State University

In 1941, the Mathematics Section of the Louisiana College Conference, the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics, and the Louisiana-Mississippi Section of the Mathematical Association of America created a standing Joint Committee, composed of two members from each of the organizations. This Committee was charged with the study of the curricula of high schools and colleges, and was asked to make recommendations to the proper authorities, from time to time, concerning mathematical requirements for graduation and the training of teachers of secondary mathematics.

The Joint Committee was very active through the year 1942. During the war years little work was done. However, the Committee was reactivated at the recent Ruston meeting (March 22–23, 1946). At that time, representatives from the Mathematics Section of the Louisiana Teachers Association and the Louisiana Parent-Teachers Association were added to the Committee. Now there are three members upon the Committee from each of the five groups.

The most outstanding work accomplished by the Committee in 1941–42 was concerned with the secondary field and teacher training. The Committee had many sessions with the Superintendent of Education of Louisiana, his department heads, and several of his committees. These deliberative sessions were scheduled after the Joint Committee had made an extensive study of the problems at hand. The bibliography of the Committee consists of approximately twenty studies devoted to problems concerning secondary mathematics.

Before the Joint Committee was instituted, the following conditions existed in Louisiana:

1. Students could graduate from high school with only one unit of mathematics; this unit could be in algebra, general mathematics, or arithmetic.

2. Teachers needed only six semester hours of college mathematics in order to teach high school mathematics.

In 1942, the following changes were made by the State Department of Education:

1. The number of units in mathematics required for graduation from high school was increased to three. (The Committee had asked for two.) War conditions had an influence here. It is understood that this requirement will be reduced to two units.

2. The number of units of college mathematics required for certification to teach mathematics in high school was increased to eighteen. The Committee had asked for 24 hours, including 9 hours in college algebra, trigonometry, and analytic geometry (or general mathematics), 6 hours in differential and integral

calculus, 3 hours in the history of mathematics, 3 hours in college geometry, and 3 hours in the theory of equations or differential equations.

3. All teachers, irrespective of the level at which they wish to teach, must have six semester hours of college mathematics.

Because of this third requirement, the colleges in Louisiana are trying to develop a course primarily for those students who are preparing to teach in the elementary grades. Such a course would include some history of mathematics, some drill in the fundamental operations, and a survey of elementary mathematics, possibly including some calculus.

It is undoubtedly true that some of the changes made in 1942 were a result of war-time conditions and the realization of the inadequacy of mathematical training in the high schools, but it is believed that they were partially a result of the efforts of this Committee. The Committee is now trying to forestall the apparent tendency to return to the conditions that prevailed in the pre-war years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

The following eighty persons have been elected to membership on applications duly certified:

M. L. ANTHONY. U. S. Navy. 9923 Rutland St., Detroit, Mich.

J. G. BALLINGER, B.S.(Notre Dame; Illinois) Teacher, High School, Chicago, Ill.

A. F. BARTHOLOMAY, A.M.(Syracuse) Asst. Prof., Keuka Coll., Keuka Park, N. Y.

P. T. BATEMAN, A.M.(Pennsylvania) Lecturer, Statistics, Bryn Mawr Coll.; Grad. Student, Univ. of Pennsylvania, Philadelphia, Pa.

R. H. BING, Ph.D.(Texas) Asst. Prof., Univ. of Texas, Austin, Tex.

G. F. BRADFIELD, M.S.(Northwestern Univ.) Instr., Illinois Inst. of Tech.; Northwestern Univ.

L. P. BURTON, A.M.(Alabama) Instr., Univ. of Alabama, University, Ala.

M. I. CHERNOFSKY, M.B.A.(New York Univ.) 1419 Carroll St., Brooklyn 13, N. Y.

HELEN E. CLARKSON, A.M.(Duke) Adjunct Prof., Univ. of South Carolina, Columbia, S. C.

H. J. COHEN, A.B.(C.C.N.Y.) Grad. Asst., Univ. of Wisconsin, Madison 6, Wis.

B. H. COLVIN, Ph.D.(Wisconsin) Instr., Univ. of Wisconsin, Madison 6, Wis.

I. D. COOK, B.S.(Univ. of Maryland) Salesman, Automotive Parts, System Brakes, Baltimore, Md.

BYRON COSBY, JR., Ph.D.(Chicago) Instr., U. S. Naval Acad., Annapolis, Md.

SISTER AGNES THERESE DIMOND, A.M.(Columbia) Asso. Prof., Coll. of Mount St. Joseph, Mount St. Joseph, Ohio

NELLE C. DOUGLAS, M.Ed.(Duke) Instr., Univ. of South Carolina, Columbia, S. C.

H. P. EDMUNDSON. 1st Lt., Sig. Co. 539 N. Kilkea Drive, Los Angeles 36, Calif.

M. D. EULENBERG, A.M.(Loyola Univ.) Instr., Evening School, Illinois Inst. of Tech., Chicago, Ill.

H. E. FETTIS, A.B.(Wittenberg) Aero. Engr., Air Tech. Service Command. 931 Five Oaks Ave., Dayton 6, Ohio

REV. F. J. FISCHER, M.S.(DePaul) Instr., DePaul Univ., Chicago, Ill.

HARLEY FLANDERS. Student, Univ. of Chicago, Chicago, Ill.

ATHANASIUS FLORIS, Grad. Engr.(Tech. Hochschule, Munich) Asst. to Structural Engr., Pacific Elec. Ry. Co. 748 S. Kingsley Drive, Los Angeles 5, Calif.

M. K. FORT, JR., A.M.(Virginia) Instr., Univ. of Virginia, Charlottesville, Va.

MARY L. FOSTER, M.S.(Louisiana State) Adjunct Prof., Univ. of South Carolina, Columbia, S. C.

A. M. FREEMAN. Dir., Math. Lab., Boston Fiduciary and Research Associates, Boston, Mass.

SADIE C. GASAWAY, M.S.(Illinois) Instr., Tennessee A. and I. State Coll., Nashville, Tenn.

M. J. GOTTLIEB, Ph.D.(Washington Univ.) Asst. Prof., Washington Univ., St. Louis, Mo.

G. W. GROTTS, M.S.(Illinois) Instr., Univ. of Tennessee, Knoxville, Tenn.

W. S. GUSTIN, A.B.(U.C.L.A.) Asst., Univ. of California at Los Angeles, Los Angeles 24, Calif.

R. V. GUTHRIE, JR., M.S.(Iowa) Prof., Southwestern Louisiana Inst., Lafayette, La.

ISRAEL HALPERIN, Ph.D.(Princeton) Asso. Prof., Queen's Univ., Kingston, Ont., Canada

KATHARINE E. HAZARD, Ph.D.(Chicago) Instr., New Jersey Coll. for Women, Rutgers Univ., New Brunswick, N. J.

E. W. HAZLETON, A.M.(Chicago) Teacher, Calumet High School, Chicago, Ill.

REV. J. J. HENNESSEY, A.M.(Woodstock Coll.) Prof., Woodstock Coll., Woodstock, Md.

R. T. HOOD. Student, Oberlin Coll., Oberlin, Ohio

N. C. HUNSAKER, A.M.(California) Asst. Prof., Utah State Agric. Coll., Logan, Utah

H. D. HUSKEY, Ph.D.(Ohio State) Instr., Univ. of Pennsylvania, Philadelphia, Pa.

L. A. JEHN, B.M.E.(Dayton) Instr., Univ. of Dayton, Dayton, Ohio

C. A. JOHNSON, A.M.(Northwestern Univ.) Instr., Univ. of Missouri, School of Mines and Met., Rolla, Mo.

W. B. KEHL, A.M.(Harvard) Instr., Georgia School of Tech., Atlanta, Ga.

P. J. KELLY, Ph.D.(Wisconsin) Univ. of Southern California, Los Angeles, Calif.

E. R. KIELY, A.M.(Fordham) Prof., Iona Coll., New Rochelle, N. Y.

V. L. KLEE, JR., A.B.(Pomona) Grad. Student, Univ. of Virginia, Charlottesville, Va.

SISTER M. PACHOMIA LACKAY, A.M.(Missouri) Coll. of St. Teresa, Kansas City, Mo.

E. PAULINE LACY, Ed.M.(Harvard) Prof., William Woods Coll., Fulton, Mo.

L. C. LEITHOLD, JR., A.B.(California) Teaching Asst., Univ. of California, Berkeley 4, Calif.

C. C. LIN, Ph.D.(Calif. Inst. of Tech.) Asst. Prof., Brown Univ., Providence, R.I.

D. E. MARRS, A.M.(California) Teacher, Senior High School, Pittsburg, Calif.

SISTER MARY AMATOR, A.M.(Pennsylvania) Asst. Prof., Immaculata Coll., Immaculata, Pa.

E. L. MCDOWELL. Student, Illinois Inst. of Tech., Chicago, Ill.

ELSA H. MUUSS, M.S. in Educ.(Northwestern), A.M.(Loyola Univ.) Teacher, Chicago City Coll., Wright Branch, Chicago, Ill.

S. B. MYERS, Ph.D.(Harvard) Asso. Prof., Univ. of Michigan, Ann Arbor, Mich.

H. M. NAHIKIAN, Ph.D.(North Carolina) Lt. Comdr., U.S.N.R. Instr., U. S. Naval Acad., Annapolis, Md.

A. B. NEALE, B.S. in M.E.(Detroit Inst. of Tech.) Research and Develop. Engr., Watson Labs. P.O. Box 889, Red Bank, N. J.

W. J. NEMEREVER, A.M.(Iowa State) Aerodynamicist, Curtiss-Wright Corp., Airplane Div., Columbus, Ohio

J. J. NEWMAN. Student, Harvard Univ., Cambridge, Mass.

BART PARK, M.S.(Mich. Coll. of Mining and Tech.) Asso. Prof., Michigan Coll. of Mining and Tech., Houghton, Mich.

R. S. PATE, Ph.D.(Illinois) Asso. Prof., Univ. of South Carolina, Columbia, S. C.

R. P. PETERSON, JR. Student, Univ. of California at Los Angeles, Los Angeles, Calif.

PASQUALE PORCELLI. Student, Illinois Inst. of Tech., Chicago, Ill.

V. J. RATHBUN, B.S.(Univ. of Dayton) Instr., Colegio San Jose, Rio Piedras, Puerto Rico

O. W. RECHARD, A.B.(Wyoming) Research Asst., Univ. of Wisconsin, Madison 6, Wis.

DOROTHY L. REES, A.M.(Texas) Instr., Trinity Univ., San Antonio 1, Tex.

SISTER MARY EDMUND RICE, Ed.M.(Cincinnati) Business Mgr., Instr., Our Lady of Cincinnati Coll., Edgecliff, Walnut Hills, Cincinnati 6, Ohio

R. A. ROSENBAUM, A.B.(Yale) Asst. Prof., Reed Coll., Portland 2, Ore.

MARIAN C. ROSENBECK, M.S.(DePaul) Instr., Chicago City Coll., Wilson Branch, Chicago, Ill.

LEILA R. RUBASHKIN. Student, Hunter Coll., New York, N. Y.

R. G. SCHAWALDER, B.S.(Univ. of Dayton) Head of Dept., Mount St. John Prep. School, Dayton, Ohio

O. F. G. SCHILLING, Ph.D.(Marburg, Germany) Asso. Prof., Univ. of Chicago, Chicago 37, Ill.

ABRAHAM SEIDENBERG, Ph.D.(Johns Hopkins) Instr., Univ. of California, Berkeley 4, Calif.

AUDREY L. SELITZKY. Student, Univ. of Chicago, Chicago, Ill.

H. G. S. SHARMA. Prof., Lingaraj Coll., Belgaum, India

ERNST SNAPPER, Ph.D.(Princeton), Asst. Prof., Univ. of Southern California, Los Angeles 7, Calif.

W. H. SPRAGENS, JR., M.S.(Kentucky) Asso. Prof., Georgia State Woman's Coll., Valdosta, Ga.

O. M. THOMAS, A.M.(Colo. St. Coll. of Educ.) Instr., U. S. Naval Acad., Annapolis, Md.

F. B. THOMPSON. Student, Univ. of California at Los Angeles, Los Angeles, Calif.

H. J. VANDORT, A.B.(West. Mich. Coll. of Educ.) Principal, High School, LeRoy, Mich.

C. J. WEBER. Trust Officer, Chase National Bank, New York, N. Y.

C. P. WELLS, Ph.D.(Iowa State) Asst. Prof., Michigan State Coll., East Lansing, Mich.

G. S. WHITEHEAD, A.M.(Georgia; Oxford Univ.) Bay Pines, Fla.

H. J. ZIMMERBERG, Ph.D.(Chicago) Instr., North Carolina State Coll., Raleigh, N. C.

W. B. CARVER, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Twenty-eighth Summer Meeting, Ithaca, New York, August 19–20, 1946.
Thirtieth Annual Meeting, Swarthmore, Pa., December 26–27, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., October, 1946
ILLINOIS, Peoria, May 9–10, 1947
INDIANA, Terre Haute, October 18, 1946
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Claremont, March 8, 1947
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

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by John A. Northcott

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A SIMPLIFIED APPROACH TO CAUCHY'S INTEGRAL THEOREM*

D. V. WIDDER, Harvard University

1. Introduction. Cauchy's integral theorem is the basic theorem of the complex integral calculus. It is unfortunate that the usual proofs are so involved with topological questions arising from the nature of the path of integration that the essential simplicity of the underlying ideas becomes obscured. One way of avoiding the difficulties is to restrict the nature of the curves used. But this procedure seems undesirable beyond a certain point since one is thereby hampered in the applications. Surely one wants the curve to be at least as general as the usual "regular" curve, which has a continuously turning tangent except at isolated points, where it may have "corners." Also it is desirable to permit the curve to cut itself, so that there will be no difficulty later in showing that certain integrals are independent of the path.

The present note avoids most of the topological problems by first showing the existence of the indefinite integral and then by using it to compute the integral of Cauchy's theorem. It thus becomes unnecessary to approximate the regular curve by a polygonal line. Other parts of the paper are standard; they are added here for the reader's convenience.

Two possible approaches are contained herein, corresponding to the two usual assumptions about the integrand $f(z)$. If the hypothesis is that $f(z) \in C^1$, the proof is based on Green's theorem (for rectangles only); if it is assumed only that $f(z)$ has a derivative, the proof of Theorem 1 must be altered, as in §7, for example. The rest of the discussion remains the same, so that we also arrive at a simplified demonstration of the Cauchy-Goursat theorem.

2. Definitions. We shall use the following notations:

R: A rectangle with sides parallel to the axes
D: A bounded simply-connected domain
L: A regular arc
Γ: A regular closed curve
P: A polygon with sides parallel to the axes
A: The class of analytic functions (possessing a derivative in a domain)
C^1: The class of functions posesssing a continuous derivative.

We make a few clarifying remarks about the topological objects. Every point of D is the center of some circle lying in D. Every pair of points in D can be joined by a broken line (with a finite number of segments) in D. In fact the segments may be taken parallel to the axes. A regular arc L may be given parametrically,

$$x = g(t), \qquad y = h(t) \qquad \alpha \leqq t \leqq \beta, \tag{1}$$

where $g(t)$ and $h(t) \in C^1$ in $\alpha \leqq t \leqq \beta$. The curve Γ is made up of a finite number of regular arcs. Note that it may cut itself or indeed have several of its regular arcs coinciding in part or entirely. If each of the regular arcs of Γ is a straight-

* Presented at the Summer Meeting of the Association, Ithaca, N. Y. August 20, 1946.

line segment parallel to an axis, then $\Gamma = P$. We are assuming Jordan's theorem for a simple polygon P.

3. Existence of the definite integral. It is important for our method to have clearly in mind the reduction of a complex integral to real integrals. Let $f(z)$ be continuous on L. Adopting the usual definition of the integral of $f(z)$ over L from the point $a = g(\alpha) + ih(\alpha)$ to the point $b = g(\beta) + ih(\beta)$ we reduce it at once to line integrals

$$(2) \qquad \int_L f(z)dz = \int_L u(x, y)dx - v(x, y)dy + i\int_L v(x, y)dx + u(x, y)dy.$$

The latter reduce, by use of equations (1), to Riemann integrals as follows,

$$(3) \qquad \int_L f(z)dz = \int_\alpha^\beta [u(g, h)g' - v(g, h)h']dt + i\int_\alpha^\beta [v(g, h)g' + u(g, h)h']dt.$$

Since L is regular the integrands are continuous, so that the integral of $f(z)$ over L exists.

If $f(z)$ is a derivative, $f(z) = F'(z)$, then the integral (2) can be integrated, whatever L may be. For, set $F(z) = U(x, y) + iV(x, y)$. Then

$$F'(z) = u + iv = U_x + iV_x = V_y - iU_y.$$

Hence equation (3) becomes

$$\int_L f(z)dz = \int_\alpha^\beta [U_x(g, h)g' + U_y(g, h)h']dt + i\int_\alpha^\beta [V_x(g, h)g' + V_y(g, h)h']dt.$$

Each integrand on the right is a derivative, so that

$$\begin{aligned}\int_L f(z)dz &= U(g(\beta), h(\beta)) - U(g(\alpha), h(\alpha)) + i[V(g(\beta), h(\beta)) - V(g(\alpha), h(\alpha))] \\ &= F(b) - F(a).\end{aligned}$$

The integral (2) is thus seen to be independent of the path L if the integrand is a derivative.

4. Cauchy's theorem for a rectangle.

THEOREM 1. *If $f(z) \in C^1$ in D and if R lies in D, then*

$$\int_R f(z)dz = 0.$$

Since $f(z) \in C^1$ the Cauchy-Riemann equations,

$$u_x = v_y, \qquad u_y = -v_x,$$

hold throughout D. Applying Green's formula for a rectangle to the line integrals (2), we have

$$\int_R f(z)dz = \iint (-v_x - u_y)dS + i\iint (u_x - v_y)dS = 0,$$

where the double integrals are extended over the interior of R. The proof is complete.

5. Cauchy's theorem for a polygon P. We shall now alter Theorem 1 by substituting P for R as the path of integration. We consider two cases.

Case I. P is simple. By Jordan's theorem P has an interior inside D. Extend each side as far as possible into the interior of P. We thereby divide P into a number of rectangles R *inside* D. The proof is completed by use of Theorem 1.

Case II. P is not simple. Since P is closed it may be continuously and completely traced, with a definite sense of description, from a given initial point back to that point. Certain points will be met more than once. If as the tracing proceeds the curve backtracks, retracing a line segment, that segment may be removed. Or if the curve comes back to a point already traced, thus completing a simple closed polygon, that polygon may be removed. In either case the integral extended over the portion removed is zero, and the remaining portion, if any, is closed. Since P has a finite number of sides, all of the curve will eventually be removed by the two processes described. This completes the proof.

6. The existence of an indefinite integral.

THEOREM 2. *If* $f(z) \in C^1$ *in* D, *then* $F(z)$ *exists such that* $F'(z) = f(z)$ *in* D.

We exhibit $F(z)$ explicitly. Choose a point a of D and set

$$F(z) = \int_a^z f(t)dt,$$

where the path of integration is a polygonal line with segments parallel to the axes. The integral is independent of the path by virtue of §4, and $F(z)$ is single-valued. To compute the derivative $F'(z)$ at an arbitrary point z_0 of D we have

$$\frac{\Delta F}{\Delta z} - f(z_0) = \frac{F(z_0 + \Delta z) - F(z_0)}{\Delta z} - f(z_0) = \frac{1}{\Delta z}\int_{z_0}^{z_0+\Delta z} [f(t) - f(z_0)]dt,$$

where the path of integration l may be taken as two legs of a right triangle when $|\Delta z|$ is sufficiently small. Hence

$$\left|\frac{\Delta F}{\Delta z} - f(z_0)\right| \leqq \max_{t \in l} |f(t) - f(z_0)| \,(|\Delta x| + |\Delta y|)/|\Delta z|.$$

By the continuity of $f(z)$ at z_0 it is clear that $F'(z_0) = f(z_0)$, and the proof is complete.

7. Cauchy's theorem for a regular curve.

THEOREM 3. *If $f(z)\in C^1$ in D and if Γ lies in D then*

$$\int_\Gamma f(z)dz = 0.$$

For, let L be one of the regular arcs of Γ. Then by §2 and Theorem 2

$$\int_L f(z)dz = F(b) - F(a). \tag{4}$$

If $L=\Gamma$, then $b=a$, and we have finished. If not, write equation (4) for each of the regular arcs of which Γ is composed, and add. The sum must be zero since Γ is closed.

8. The Cauchy-Goursat theorem.

THEOREM 4. *If $f(z)\in A$ in D and if Γ lies in D, then*

$$\int_\Gamma f(z)dz = 0.$$

We begin by proving two elementary preliminary results.

LEMMA 1. $\int_R dz=0$.

LEMMA 2. $\int_R z dz=0$.

If we have already proved Theorem 1 these results are obvious since the integrands surely belong to C^1. It is useful, however, to have independent proofs so that Theorem 4 can be proved without the use of Green's theorem. Let $z_0, z_1, \cdots, z_n=z_0$ be distinct points of R with

$$|z_j - z_{j-1}| \leqq \delta \qquad j = 1, 2, \cdots, n.$$

Then by definition of an integral

$$\int_R dz = \lim_{\delta\to 0} \sum_{j=1}^{n} (z_j - z_{j-1}) = z_n - z_0 = 0$$

$$\int_R z dz = \lim_{\delta\to 0} \sum_{j=1}^{n} z_j(z_j - z_{j-1}) = \lim_{\delta\to 0} \sum_{j=1}^{n} z_{j-1}(z_j - z_{j-1}).$$

Since each summation on the right approaches a common value, half their sum must approach the same value:

$$\int_R z dz = \lim_{\delta\to 0} \frac{1}{2} \sum_{j=1}^{n} (z_j^2 - z_{j-1}^2) = \frac{1}{2}(z_n^2 - z_0^2) = 0.$$

We turn now to the proof of Theorem 4. Clearly we need only prove the result for a rectangle R. Of course we cannot now use Green's theorem since we do not know that the partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous. Let us begin by joining the mid-points of the sides of R by straight lines, thus dividing R into four rectangles of half the size. If

$$\left| \int_R f(z)dz \right| = G,$$

then for at least one of the small rectangles, say R_1, we must have

$$\left| \int_{R_1} f(z)dz \right| \geqq \frac{G}{4}.$$

Now divide R_1 into four rectangles of half the size. For one of these, say R_2,

$$\left| \int_{R_2} f(z)dz \right| \geqq \frac{G}{4^2}.$$

Continue the process. We obtain a sequence of rectangles each inside and half the size of its predecessor. There must exist a unique point z_0 inside all of these. If k is the perimeter of R, then $k/2^n$ is the perimeter of R_n. Moreover,

$$\left| \int_{R_n} f(z)dz \right| \geqq \frac{G}{4^n}.$$

Now since $f'(z_0)$ exists the function

$$\phi(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$$

tends to zero as z approaches z_0. By Lemmas 1 and 2

$$\int_{R_n} f(z)dz = \int_{R_n} \phi(z)(z - z_0)dz.$$

If ϵ is an arbitrary positive number, we can determine an integer m so large that $|\phi(z)| < \epsilon$ for z on R_m. When z is on R_m it is evident that $|z - z_0|$ is less than the perimeter of R_m, so that

$$\frac{G}{4^m} \leqq \left| \int_{R_m} f(z)dz \right| \leqq \epsilon \frac{k}{2^m} \frac{k}{2^m}.$$

Since ϵ is arbitrary it follows that $G = 0$, and the result is proved.

THE ACCURACY OF LINEAR INTERPOLATION

P. M. HUMMEL, University of Alabama

1. Introduction. The student of mathematics is introduced to linear interpolation very early in his studies and yet almost nowhere is anything said concerning the accuracy of the values thus obtained. The purpose of this article is to give simple, yet fairly sharp, upper and lower limits for the error in the interpolated value. By the error we mean the difference between the value obtained by interpolation and the true value, that is,

$$\text{Error} = \text{interpolated value} - \text{true value}.$$

The results that follow require that the function and its first three derivatives be continuous and that the second and third derivatives be of constant sign throughout the interval of interpolation. However, these conditions usually hold since the interval of interpolation is generally small.

2. Bounds on the error. We shall establish the following theorem.

THEOREM. *Let $f(x)$ be a function which together with its first three derivatives is continuous throughout the interval $a \leqq x \leqq b$. Moreover let $f''(x)$ and $f'''(x)$ be of constant sign throughout the interval. If $f(a)$, $f(b)$, and x are given and $f(x)$ is determined by linear interpolation, the error satisfies the inequality*

$$\frac{(b-x)(x-a)}{(b-a)^2}\left[f(b)-f(a)-(b-a)f'(a)\right] \leqq \text{Error} \leqq \frac{(b-x)(x-a)}{(b-a)^2}\left[(b-a)f'(b)-f(b)+f(a)\right],$$

or this inequality reversed according as $f'''(x)$ is positive or negative.

Proof. The proof is given first for the case where $f''(x)$ and $f'''(x)$ are both $\geqq 0$.

Using Taylor's finite expansion for $f(a)$ about the point x it is easily verified that

$$(1)\quad f(x)-f(a)-(x-a)f'(x) = -\tfrac{1}{2}(x-a)^2 f''(\theta_1), \qquad a < \theta_1 < x.$$

$$(2)\quad f(x)-f(a)-(x-a)f'(x)+\tfrac{1}{2}(x-a)^2 f''(x) = \tfrac{1}{6}(x-a)^3 f'''(\theta_2), \quad a < \theta_2 < x.$$

Now consider the auxiliary function

$$(3)\qquad F(t) = \frac{f'(t)}{t-a} - \frac{f(t)-f(a)}{(t-a)^2}.$$

In view of (1), $F(t) = \frac{1}{2}f''(\theta_1)$, $a < \theta_1 < t$, so that $F(t) \geqq 0$ for $a < t \leqq b$.

If we differentiate $F(t)$ we obtain

$$F'(t) = \frac{2}{(t-a)^3}\left[f(t)-f(a)-(t-a)f'(t)+\tfrac{1}{2}(t-a)^2 f''(t)\right].$$

In view of (2), $F'(t)=\frac{1}{3}f'''(\theta_2)$, $a<\theta_2<t$. Consequently $F'(t)\geqq 0$ and $F(t)$ is non-decreasing for increasing t.

We next verify that

$$\int F(t)dt = \frac{f(t)-f(a)}{t-a} + C,$$

and consequently

$$\int_x^b F(t)dt = \frac{f(b)-f(a)}{b-a} - \frac{f(x)-f(a)}{x-a}, \quad \text{and} \quad \int_a^x F(t)dt = \frac{f(x)-f(a)}{x-a} - f'(a).$$

Now the integral $\int_x^b F(t)dt$ cannot exceed $(b-x)\cdot\max F(t)$ and since $F(t)$ is non-decreasing the maximum value of $F(t)$ is $F(b)$. Thus we have

$$\frac{f(b)-f(a)}{b-a} - \frac{f(x)-f(a)}{x-a} \leqq (b-x)\left[\frac{f'(b)}{b-a} - \frac{f(b)-f(a)}{(b-a)^2}\right], \tag{4}$$

or after rearranging and multiplying by $(x-a)$,

$$\frac{f(b)-f(a)}{b-a}(x-a) + f(a) - f(x) \leqq \frac{(b-x)(x-a)}{(b-a)^2}[(b-a)f'(b) - f(b) + f(a)].$$

The left member of this last inequality is the error in the interpolated value and hence the upper limit stated in the theorem is established.

To prove the lower limit we start with the inequality

$$\frac{\int_a^x F(t)dt}{x-a} \leqq \frac{\int_x^b F(t)dt}{b-x},$$

which is obviously true since each member represents the average value of $F(t)$ between the limits of integration and $F(t)$ is non-decreasing. Evaluating these integrals we get

$$(b-x)\left[\frac{f(x)-f(a)}{x-a} - f'(a)\right] \leqq (x-a)\left[\frac{f(b)-f(a)}{b-a} - \frac{f(x)-f(a)}{x-a}\right].$$

We next add

$$(b-x)\left[\frac{f(b)-f(a)}{b-a} - \frac{f(x)-f(a)}{x-a}\right]$$

to both members of the last inequality and get

$$(b-x)\left[\frac{f(b)-f(a)}{b-a} - f'(a)\right] \leqq (b-a)\left[\frac{f(b)-f(a)}{b-a} - \frac{f(x)-f(a)}{x-a}\right].$$

From this inequality we easily obtain

$$\frac{(b-x)(x-a)}{(b-a)^2}[f(b)-f(a)-(b-a)f'(a)] \leqq \frac{f(b)-f(a)}{b-a}(x-a)+f(a)-f(x).$$

This last inequality establishes the lower limit stated in the theorem thus completing the proof for the case where $f''(x)$ and $f'''(x)$ are both $\geqq 0$.

If $f''(x) \geqq 0$ and $f'''(x) \leqq 0$, then $F(t)$ is a non-increasing function and all inequalities starting with (4) are reversed. If $f''(x) \leqq 0$, we take $F(t)$ as the negative of the right member of (3) and proceed as before.

3. Concluding remarks. Clearly the lower limit may be used as a correction factor to obtain an improved value and a corresponding improved upper limit.

When an upper limit for the absolute value of the error is sufficient, only one member of the inequality need be evaluated since it is well known that the value obtained by interpolation is too large (small) when the second derivative is positive (negative) and the proper member of the inequality is easily determined. Moreover, it is easily shown that the quadratic factor $(b-x)(x-a)/(b-a)^2$ has a maximum value $=\frac{1}{4}$ and hence the upper limit for the absolute value of the error can be simplified, though slightly weakened, by replacing this quadratic factor by $\frac{1}{4}$.

THE NODES OF SOME CUBIC PENCILS

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1. Introduction. A well known theorem states that all cubics through eight given points have a ninth point in common, thus forming a pencil of cubics. This paper deals with three such pencils, in each of which the cubics have simple contact at three given points and pass through three other given related points. In each case a number of the cubics are nodal (in addition to the degenerate cubics, excluded from this discussion); this number is determined, together with the position of the nodes. In the third case the nodes lead to still further pencils and they are also linked to the reciprocal of the second case considered.

2. The pencil containing four nodal cubics. A given cubic curve (non-singular or nodal) is cut by any straight line in three points, D, E, F, and if L, M, N are points of contact of tangents drawn to the cubic from D, E, F respectively, then either they are the points of contact of a conic with the cubic or they are themselves collinear. The points L, M, N counted twice and the points D, E, F form a base for a pencil of cubics. Let the triangle ABC formed by the three tangents be taken as triangle of reference, and consider first the case in which a conic may be drawn to touch the cubic at the points L, M, N. The conic and the line DEF will have equations of the form

$$C \equiv a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0,$$

and

$$lx + my + nz = 0.$$

The cubic pencil is then

$$C(lx + my + nz) + \lambda xyz = 0,$$

and it is possible to choose the co-ordinate scales so that for some value of the ratios $l:m:n$ to be determined the unit point will be a node of a cubic of the pencil. The conditions for a node at (1, 1, 1) are

$$-2a(-a+b+c)(l+m+n) + lC_1 + \lambda = 0,$$

and two similar expressions, where $C_1 \equiv a^2+b^2+c^2-2bc-2ca-2ab$. Solving these for the ratios $l:m:n$ we have

$$l:m:n=(a-b+c)(a+b-c):(-a+b+c)(a+b-c):(-a+b+c)(a-b+c).$$

THEOREM. *The double lines of the involution pencil at a node O determined by the quadrilateral formed by the three given cubic tangents and DEF are the tangents to the conic LMN from the point O.*

The equation of OD is

$$-2ax + (a+b-c)y + (a-b+c)z = 0,$$

and the double lines of the involution cut AB in the pair of points given by

$$ax^2 - (a+b-c)xy + by^2 = 0, \qquad z = 0.$$

But the pair of tangents from O to the conic are given by

$$C \cdot C_1 = [\textstyle\sum ax(-a+b+c)]^2,$$

which cut AB in the same two points.

If the two tangents from a point O to a given conic are harmonically separated by the joins of O to two fixed points A, D, the locus of O is a second conic (the director circle when A, D are taken as circular points). The possible nodal positions are therefore the four (common) points of intersection of the three conics so derived from A, D; B, E and C, F.

3. The pencil containing three nodal cubics. In the case in which L, M, N are collinear, let the equation of the line LMN, referred to the same triangle as above, be

$$P \equiv ax + by + cz = 0,$$

and choose the scale of co-ordinates so that the equation of DEF is

$$Q \equiv x + y + z = 0.$$

The family of cubics is then

$$P^2 \cdot Q + \lambda xyz = 0,$$

and the nodes of the family are given by

$$P^2 + 2aP \cdot Q + \lambda yz = 0,$$

and two similar expressions. Eliminating λ gives three equations of the form

$$(x - y)\cdot P + 2(ax - by)\cdot Q = 0, \tag{1}$$

defining a pencil of conics. The nodes are therefore given by the ratios

$$ax{:}by{:}cz = y + z{:}z + x{:}x + y,$$

and are the common points of the pencil (1) other than the intersection of P and Q. They lie also on the conic

$$ax(y - z) + by(z - x) + cz(x - y) = 0,$$

circumscribing the triangle of reference, and whose tangent at A (say) is the harmonic conjugate with respect to AB, AC of the join of A to the point of intersection of P and Q.

4. The reciprocal theorems. It follows from Plücker's numbers, or by direct analysis as in Section 6, that the reciprocal of a nodal cubic is a tricuspidal quartic. Hence by reciprocating Section 2 it is possible to draw four tricuspidal quartics to touch three lines d, e, f concurrent in a point Q, and also to touch three other lines l, m, n at their intersections R, S, T with d, e, f respectively, so long as these lines are tangents to a conic at R, S, T. The intersections of the conic with the bitangents (one to each quartic) are the double points of the involution range determined on the bitangent by the quadrangle $QRST$.

Section 3 shows that three tricuspidal quartics can be drawn to touch three lines d, e, f, concurrent in Q, and the three lines l, m, n at R, S, T defined as above, if these lines are themselves concurrent in a point P. The three bitangents are tangents to the conic inscribed in the triangle RST, and whose point of contact with RS (say) is the harmonic conjugate with respect to R and S of the intersection of RS with PQ.

5. The pencil containing six nodal cubics. A closely allied pencil of cubics is given by cubics which touch three concurrent lines PR, PS, PT at R, S and T and which also pass through three points A, B and C, one on each side of the triangle RST. A necessary condition for the existence of the pencil is that AR, BS and CT must be concurrent in Q (say), and conversely. For, taking RST as triangle of reference, any cubic through these points has an equation of the form

$$x^2(ay + bz) + y^2(cz + dx) + z^2(ex + fy) + \lambda xyz = 0,$$

and the tangents at R, S, T are concurrent if $ace+bdf=0$. Again the join of R to A, the third intersection of ST with the cubic, and the two similar lines, have equations

$$cy + fz = 0, \qquad bx + ez = 0, \qquad ax + dy = 0,$$

and these are also concurrent if $ace+bdf=0$.

It is useful to consider two systems of co-ordinates, taking either ABC or RST as the triangle of reference and in each case choosing Q as the unit point.

This is equivalent to $\rho x' \equiv -x+y+z$, $\rho y' \equiv x-y+z$, $\rho z' \equiv x+y-z$, a transformation from the first to the second.

(a) *ABC as triangle of reference.*

With this system of co-ordinates the cubic has the equation

$$a(z+x)(x+y)(y-z)+b(x+y)(y+z)(z-x) + c(y+z)(z+x)(x-y)+\lambda(y+z)(z+x)(x+y)=0.$$

The nodes of the family are given by three equations of the type

$$a(y-z)(2x+y+z)-b(y+z)(2x+y-z) + c(y+z)(2x-y+z)+\lambda(y+z)(2x+y+z)=0,$$

which, by addition and subtraction, reduce to three of the type

$$a(z+x)(x+2y-z)-b(y+z)(2x+y-z)+\lambda(y+z)(z+x)=0.$$

These give equations independent of λ of the type

$$a(x+y)(z+x)(x+2y-z)-b(y+z)(x+y)(x+2y+z) + c(z+x)(y+z)(-x+2y+z)=0,$$

whence, if (f, g, h) is any possible position of a node,

$$a:b:c = f(g+h)^2 : g(h+f)^2 : h(f+g)^2.$$

Thus the equation of the cubic with a node at (f, g, h) is

$$\sum f(g+h)^2(z+x)(x+y)(y-z)+(g-h)(h-f)(f-g)(y+z)(z+x)(x+y)=0,$$

and the co-ordinates of P are $(-a+b+c, a-b+c, a+b-c)$.

It is evident from the ratios $a:b:c$ that a second cubic may be drawn with a node at $(1/f, 1/g, 1/h)$; and the points (f, g, h) and $(1/f, 1/g, 1/h)$ are the double points of the involution range determined on their join by the quadrangle $QRST$. With this position of P the ratio of the x and y co-ordinates of the other nodes are given by

$$x^2+y^2-2pxy=0,$$

where p is a root of the equation

$$[(h+f)(h+g)p+(h-f)(h-g)]^2-8fgh^2(p+1)=0.$$

The six nodal positions thus determined, together with the points A, B, C, form a base for a pencil of cubics, of which

$$\sum (b-c)x(y+z)^2=0,$$

is one having a node at Q.

(b) *RST as triangle of reference.*

From the first part of this Section the equation of the cubic may be written

$$x^2(cy - bz) + y^2(az - cx) + z^2(bx - ay) + \lambda xyz = 0,$$

and P is the point (a, b, c).

The nodes are given by equations of the type

$$2x(cy - bz) - cy^2 + bz^2 + \lambda zy = 0;$$

that is, they lie on cubics of type

$$- ayz(2y - z) + bzx(z - 2x) + cxy(x + y) = 0.$$

Hence there is a node at (f, g, h) if

$$a:b:c = f^2(-f + g + h):g^2(f - g + h):h^2(f + g - h),$$

and the corresponding cubic has equation

$$\sum f^2(-f + g + h)yz(y - z) - 2(g - h)(h - f)(f - g)xyz = 0.$$

Again, the six nodal positions and R, S, T form a base for a pencil of cubics. A particular one of this pencil is

$$\sum ayz[y(a - b) + z(c - a)] = 0,$$

which passes through the nodes and through P, Q, R, S, T.

Other cubic base sets of nine points are formed by A, B, C, Q, R, S, T and any pair of nodes chosen as above, since any cubic through the first seven points is self-conjugate under a transformation I_2 with these seven points as invariant and F-points.*

6. The relation between the nodes of Section 5 and the figure of Section 4. Reciprocating the results of Section 3 with respect to the conic $x^2+y^2+z^2=0$, the class cubic $(al+bm+cn)^2(l+m+n)+\lambda lmn=0$, is a tricuspidal quartic with bitangent (l, m, n) if

$$al:bm:cn = m + n:n + l:l + m.$$

This quartic touches the cubic of Section 5 at R, S and T, and also touches the lines QR, QS, QT.

The join of the corresponding nodal points (f, g, h) and $\{f(-f+g+h), g(f-g+h), h(f+g-h)\}$ of the cubic pencil of Section 5(b) has tangential co-ordinates $\{gh(g-h), hf(h-f), fg(f-g)\}$, and these are the co-ordinates of a bitangent of the quartic if

$$a:b:c = f^2(-f + g + h):g^2(f - g + h):h^2(f + g - h),$$

which are also the conditions for the cubic to have a node at (f, g, h).

THEOREM. *The six nodal positions of Section* 5 *may be paired in such a way that their joins are the three bitangents of the tricuspidal quartics given in the second part of Section* 4, *and the nodes themselves are the double points of the involution defined on the bitangent by the quadrangle QRST.*

* Hudson, Cremona Transformations. pp. 50–52.

A MANUAL FOR YOUNG TEACHERS OF MATHEMATICS

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1. Introduction. The novice who attempts to do his first job without the advice and counsel of older, experienced men, is not acting wisely. The young teacher is no exception. Today in America there are hundreds of young men and young women who are facing their first college classes in mathematics. Many of them do not have as much training as they should have and many have been in our armed forces or other war work so long that their mathematics is rusty from lack of use. This monograph is an effort to help them. We attempt to set forth in a somewhat orderly fashion the experience of nearly a half century of teaching, working and playing with college boys. They are the best type in America and, perhaps, anywhere else. A small percentage are brilliant, many more are bright enough to do good work and, of course, a few are poorly equipped mentally. The lazy we have always with us!

These American students take discipline well and punishment in a manly way. They are inherently honest. The opportunity to work with them is a privilege which is not to be taken lightly. The rewards to the teacher are many. They keep coming over the years and often from surprising quarters. A word of appreciation from an old student is a great part of the satisfaction one gets from living and working with youth.

Teaching is to a large extent a matter of the personality of the teacher. No fixed set of rules can be given to ensure success. Every young teacher should have a definite idea of how he is going to proceed. A good way to begin is to follow the methods of some good teacher he has known, or to read some good text on teaching and follow its suggestions. All such methods, however, are merely for a beginning. He must study his own personality and equipment as objectively as possible and gradually make his initial attempts conform to a conscious method best suited to him and probably to him alone.

2. Class discipline. The instant an instructor walks into a class room the students begin to size him up. On the part of the student this is for the most part unconscious. As a class they will soon find out all his weak points. If his weak point is discipline the class will soon become a rabble. A young teacher must be very sure that there is good behavior in his class from the first day to the last day of the term. To obtain it he should discipline himself. Let him speak distinctly and loud enough to be heard in the rear of the room. It is usually best to stand while speaking. Require attention from all the students while you are explaining even if you are talking to one student. His difficulty is likely to be the same as that of many others. No whispering should be tolerated while the instructor is talking and no newspaper reading in class.

All the students should work all the time of the class hour. If only a part of them are at the board give those seated a list of problems to work or an article to study. It may be possible to allow a little whispering by those seated while a part are at the board but the instant the instructor begins to explain a problem

all should be quiet and listen to him. There is always a chance that they may learn something.

It is a good plan to have students write their problems or questions on the board before the class begins. The students should be told to have them on the board ready for the instructor when he comes in. This saves time. The first ten or fifteen minutes may well be spent answering these questions.

It is much better to be too strict at first and perhaps ease up a bit later. A class once out of hand is very hard to control or re-discipline. Finally, make very few rules of behavior. Never make a statement as to discipline or class behavior unless you are sure you can and will enforce it. This last is a good rule to follow all your life whenever you have to organize and control a group of human beings.

3. Board work and homework. Work to be done at home and to be handed in should be assigned as often as the teacher can read it. It is not wise to assign work to be handed in unless you read it. All such work should be corrected and returned. In homework each problem should be stated at the beginning exactly as it is in the text. This is important for several reasons. One is that the student is more likely to understand what his problem or theorem means if he writes it out in full. Another reason is that the person who reads the work later will not have to guess which problem the student is trying to solve from the too often queer hieroglyphics that are on the page. If another reason is needed it is that it is good English to state first what is to be done and then proceed in an orderly fashion to carry through the work. A part of every problem is to find and apply a check to ensure accuracy.

Every piece of homework and every exercise done at the board should follow this procedure: write out what is to be done, draw and explain a figure, write out the solution, find and apply a check. Some teachers object that it takes too long to do problems this way. The answer is that it is better for a student to do one exercise carefully than to do five problems sloppily and write them up carelessly.

The symbols used in mathematics are nothing but abbreviations for words, phrases or sentences. They should be punctuated as such. For this reason it is not desirable to begin a sentence with a mathematical symbol for one cannot begin such a sentence with a capital letter without changing the notation. Mathematics is thought of by most freshmen as a mixture of x's, square roots, exponents, plenty of $+$ and $-$ signs, and with a few equal signs thrown in for variety or good measure. To them, and unfortunately to some of their high school teachers, there is nothing resembling logic in it. It is difficult to change their point of view. A program of neat, careful work such as is outlined above will at first discourage them but applied gently and persistently will teach them that mathematics is a process of thinking and the symbols are only for the purpose of helping with this thinking. Properly trained students will eventually take pride in the fact that they look for the answer in the back of the book only to see if the author got it right.

4. Quizzes and examinations. The quizzes and examinations are a necessary evil. Their chief value lies in the fact that they are the best way known to make the student take a review and get a comprehensive view of a chapter or, in the case of final examinations, of a book. They emphatically should not be used as the only means of determining a student's grade. The term grade should be made up, for the most part, from the instructor's day by day contact with the student, from oral quizzes, from blackboard work, and even from the type of questions that the student asks. Quizzes and examinations are merely a check on these. Students should know that their grades are so made up with the exception that it is probably not wise to tell them that the type of questions they ask helps the instructor to make up the grade. To tell them that would limit the number of questions they would ask and partly stifle their self expression.

A quiz should be given at a natural division of the subject matter. Usually this is at the end of a chapter. In most good texts a summary and review is given at the end of each chapter. This summary may be used for a day of review before a quiz. Sometimes the instructor may write on the board a list of ten or twelve questions and spend an hour helping the students answer them. Then a quiz may be made up from these. It is best to change the numbers in any numerical problems in this list of questions for an obvious reason.

The making up of a good quiz is an art. It is more important than most young instructors think it is. It should be constructed so that the best students may finish it in thirty or forty minutes and have time to go over it looking for faulty expression or numerical mistakes. The average student may finish most of it and even the poor student who has worked hard may finish two-thirds of it. In every quiz there should be a question or part of a question requiring some independent thinking. This serves to help pick out the brilliant or good students.

5. The instructor's preparation for class. We could begin and end this paragraph with the dogmatic statement that no instructor, old or young, should go before a class without preparation. But since the human race, from Adam down to the present, does not like to accept a blunt statement which sounds like a command it is perhaps better to amplify and clarify a bit.

One of the most embarrassing things that can happen to a young instructor is to get "stuck" on a problem or to find that he does not fully understand the text. He is making bad matters worse if he tries to pass his difficulty off by saying that the problem is not a good one or that he doesn't agree with the text. The students very quickly sense his difficulty and lose their respect for him. He is bluffing and they know it. It is better to admit lack of preparation or better still to study the text carefully. Read it as a whole to get the author's point of view, then read each assignment just before going to class. This isn't enough. A large percentage of the problems and exercises should be solved. This will save time in class and will prevent a situation like the one described above.

It is a good habit to quiz the class orally for a few minutes at the beginning of each class hour. Such a quiz can be done successfully only if the instructor knows what articles are in the day's assignment.

After a recitation which consists of an oral quiz and board work the instructor should record his opinion of those students whose answers he remembers. Some of this may be done in class if it does not interfere with class instruction. You will find that quiz grades and examination grades are usually a confirmation of your composite opinion.

6. The honor system. In a large number of our colleges and universities, particularly in the South, an honor system is in operation. This system is usually conducted by the students themselves under faculty supervision. The maintenance of an honor system is excellent training for students since it inculcates ideas of truth and honor and gives the whole student body practice in good citizenship. The existence of an honor system does not mean that the instructor writes the questions on the board and then leaves the room. He must do more than that. There are three things he should do:

1. He should explain it to the students, which implies that he reads any pamphlets that are available.

2. He should remove temptation as far as that is possible by asking them to put away all books, papers and notes and to sit on alternate seats. If the class is large they can move the chairs about so that it is not easy to see what others are doing.

3. He should explain that quiz grades and examination grades are not important enough to make it pay to cheat. Quiz grades are merely a check on the grades obtained day by day from board work and oral quizzes. He should convince them that he has complete confidence in them, that the arrangements are made not because he suspects them but to protect them against what may be unjust accusations.

Cheating is largely the fault of too large classes or of a wrong attitude on the part of the instructor. The vast majority of students are honest. A great many are young, immature and weak. They should be protected as far as possible but punished resolutely and irrevocably when found guilty.

Finally no honor system ever continues to work by itself. There must be a continual infusion of new strength into it year after year. This must be done either by a faculty member who believes in it, or from a group of students. This group of students often gets its ideals and enthusiasm from a faculty member in whom they have confidence and with whom they may associate and talk freely.

7. Conducting a class. There is no set rule for conducting a class. Every teacher has to devise his own technique. This paragraph is intended to give the young teacher something to go by till he can get on his feet and begin to develop his own methods. There are a few ideas, however, that are always good and should be followed even after he has had experience.

After the first day has passed, the text books having been announced, and an assignment made, let him begin by asking for questions on the lesson. This can be followed by an oral quiz. Perhaps only a few questions are necessary at first. They should be directed to finding out who have read the book and what

was learned from the reading. A constant pressure should be kept up to make them read the text. It may well be a gentle pressure for a week or so but increase it until you can get them to reproduce some paragraphs in their own words orally or in board work. Students will get very little value from a course in Freshmen mathematics unless they learn to read a scientific book. It is quite possible that the greatest educational value they will get is the ability to read a closely written scientific text.

When the questions have been answered and the oral quiz ended send as many as possible to the board. While they are getting up and moving to their places give those who are seated a list of exercises to work. Insist that they work. It is often a good plan to assign numbers to those who are at the board, and give a different problem to the even and odd numbers. This, in part, prevents blind copying by adjoining students. Have them write the problem exactly as in the text and require neat work.

A good instructor does very little talking himself. He will explain just enough to keep his students from getting too discouraged. Students gain strength, power and skill in mathematics by doing it themselves. They can no more learn mathematics by listening to someone explain it than they can learn to play baseball by sitting in the bleachers. It is for this reason that the lecture method is a poor way to teach mathematics.

8. The content of an algebra course. Any course in college algebra must have a review of factoring and fractions. This is particularly true in the Southern states where the high school preparation is not quite as good as it is in the North and West. Exponents should be studied from a more mature viewpoint than is usually possible in high school. Particular attention must be given to fractional and negative exponents. We rarely use radical signs in more advanced courses for the fractional exponents are better and simpler. In fact the long lists of exercises involving square roots, cube roots, *etc.* are rapidly disappearing from school and college texts, as they should, for they are confusing and useless.

The idea of a function and its graph should be the central part of most of the work. This includes the linear function, its graph and its slope; the quadratic function, its graph, its zeros, and the slope of the tangent line. The same procedure continues with the cubic functions. Only enough of the theorems on the polynomials need to be given to enable the student to understand Newton's method for solving an equation of any degree. Particular stress should be placed on the factor theorem and the remainder theorem.

Logarithms should be studied as a continuation of exponents and enough attention be given to the binomial theorem to enable students to expand any binomial. If there is time the progressions may be studied with a few of their applications to business problems. The chapter on variation will help students in many of their science courses.

9. Trigonometry. Trigonometry consists of two distinct parts. One is purely arithmetical, the other analytical. The arithmetical part consists of the solution

of triangles, both right and scalene and of the applications to various types of mensuration. The analytical part is concerned with the generalized definitions of the trigonometric functions and the many formulas and other identities which follow from them. The two parts could be taught separately but it is not the custom to do so. The analytical part of trigonometry is actually a branch of analytic geometry being derived from the rectangular coordinates of a point.

For a short course, such as we usually have, it saves time to give the generalized definitions at the beginning. The terms "opposite side over the hypotenuse," and "adjacent side over the hypotenuse" are not good terms to use since they cease to have a meaning for an angle larger than 90°. The definitions that should be given are in terms of x, y, r and θ, where x and y are the rectangular coordinates of a point. It is important to note that x and y carry their own signs. From these definitions there follow the three reciprocal relations; and using the theorem of Pythagoras we get three more formulas involving the squares of the functions and finally the formula $\tan\theta=\sin\theta/\cos\theta$. This makes a total of seven elementary formulas. They are sufficient to solve the right triangle. Two weeks should be enough to master this.

In the proof of the addition formulas it is desirable, perhaps even necessary, to consider directed lines and projections. These addition formulas lead us to the formulas for solving any triangle whatever, and the number of identities involving two or more angles is endless. It is necessary to prove a large number of these identities in order to handle them when they occur again in analytic geometry and calculus.

No course is complete without a study of the graphs and the periodicity. Enough trigonometric equations should be given to familiarize the student with a new type of equation and, of course, the inverse functions which occur in later mathematics must be studied.

Trigonometry, in so far as we are concerned with its numerical side, is ideal for the application of a check. A good check formula is one which has not been used in the computations of the unknown sides or angles and it should contain as many as possible of the quantities computed. For these requirements Mollweide's identity is perfect,

$$\frac{a-b}{c}=\frac{\sin\dfrac{A-B}{2}}{\cos\dfrac{C}{2}}$$

for it is adapted to logarithmic computations and it contains all the sides and all the angles. We suggest that you have it proved and use it as a check on the solution of the triangle in every possible case.

10. Analytic geometry. Analytic geometry is the capstone of elementary mathematics. Here for the first time the student has an opportunity to put into use all the skills which he has been learning all through high school and through the first two courses in college. If he can use elementary algebra rapidly and

effectively and is equally sure of his trigonometry, then analytic geometry should be a genuine pleasure to him. Unfortunately a very large percentage of the students have to struggle so hard with the elementary algebra and trigonometry that they have no strength left to understand or enjoy analytics. Perhaps the most they get is a good review.

It is in analytics that the student should be required to read the book and report to the instructor. He has already acquired the vocabulary and techniques. The pressure to make him read with understanding and to do some calculations between the lines ought to be increased. If he doesn't learn to read a scientific book in this course the chances are that he will never be able to do so.

VOLUME COORDINATES

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1. Introduction. In this paper, some results concerning the geometry of a tetrahedron are set forth. Because of the employment of volume coordinates (an extension of areal coordinates to space) it is deemed appropriate to include in the first part of the paper some discusson of the type of coordinates used. The linear transformation connecting volume coordinates and rectangular cartesian coordinates in ordinary euclidean space is exhibited in Section 2. Some analytical results which are of use later are given in Sections 3 and 4. Although volume coordinates have been known for a long time, they probably have not appeared before in the more tractable form of the tensor notation which is employed here.

Volume coordinates are particularly useful in the consideration of ratios of volumes of tetrahedra. Examples of this are shown in Sections 5 and 6. A pencil of quadric surfaces relative to any tetrahedron is discussed in Section 7.

2. The transformation relating volume coordinates and cartesian coordinates. In order to gain symmetry in notation in ordinary solid analytical geometry, it is sometimes convenient to use four coordinates in the analysis, where it is understood that the fourth coordinate can always be replaced by unity. This device will be used in what follows.

Let the rectangular cartesian coordinates of the four vertices $P_\sigma(\sigma=1, 2, 3, 4)$ of a tetrahedron T be denoted by $a_\sigma^\beta(\beta=1, 2, 3, 4)$, and let the cartesian coordinates of any point P in the space be denoted by ξ^β. The volume of the tetrahedron T is given by $\Delta/6$, where Δ denotes the value of the determinant $|a_\sigma^\beta|$, the vertices of T being ordered so that Δ is positive.

The first volume coordinate x^1 of the point P is defined by

$$\Delta x^1 = \begin{vmatrix} \xi^1 & \cdots & \xi^4 \\ a_2^1 & \cdots & a_2^4 \\ \cdots & \cdots & \cdots \\ a_4^1 & \cdots & a_4^4 \end{vmatrix}. \tag{1}$$

If A_α^β is used to represent the cofactor of a_β^α in Δ divided by the determinant Δ, then equation (1) may be written in the form

$$x^1 = \xi^1 A_1^1 + \xi^2 A_2^1 + \xi^3 A_3^1 + \xi^4 A_4^1,$$

or, more conveniently, as

$$x^1 = A_\alpha^1 \xi^\alpha.$$

Thus, the volume coordinates x^β of the point P are related to the rectangular cartesian coordinates ξ^α of P by the equations

$$x^\beta = A_\alpha^\beta \xi^\alpha. \tag{2}$$

A repeated index, as α in equations (2), always indicates summation over the range $1, \cdots, 4$ unless the contrary is stated.

If the point P is inside the tetrahedron T, and the tetrahedra with vertices P, $P_\sigma(\sigma = 1, 2, 3, 4)$ are oriented so that the coordinates x^β given by (2) are all positive, then it follows that

$$x^1 + x^2 + x^3 + x^4 = 1 \tag{3}$$

for every point P inside T. The relation (3) holds for every point in the finite space if the proper sign is attached to the volume of a tetrahedron. Thus, for example, if P and the vertex P_1 of T are on the same side of the plane determined by the vertices P_2, P_3, P_4, then x^1 is positive, but if P and P_1 are on opposite sides of this plane, x^1 is negative.

Attention is called to the fact that if one multiplies equations (2) by a_β^γ and sums on β, he obtains

$$a_\beta^\gamma x^\beta = a_\beta^\gamma A_\alpha^\beta \xi^\alpha,$$

in which the sum $a_\beta^\gamma A_\alpha^\beta$ is 1 or 0, according as $\gamma = \alpha$ or $\gamma \neq \alpha$. Hence, the rectangular cartesian coordinates of the point P are given in terms of the volume coordinates x^β of P by the equations

$$\xi^\gamma = a_\beta^\gamma x^\beta. \tag{4}$$

3. The volume of a tetrahedron. If the rectangular cartesian coordinates of the vertices Q_σ of any tetrahedron S be denoted by ξ_σ^α, then, by use of equations (4), the volume of S is given by

$$\frac{1}{6}\left|\xi_\sigma^\alpha\right| = \frac{1}{6}\left|a_\beta^\alpha x_\sigma^\beta\right| = \frac{1}{6}\left|a_\beta^\alpha\right| \cdot \left|x_\sigma^\beta\right| = \frac{\Delta}{6}\left|x_\sigma^\beta\right|, \tag{5}$$

where $\Delta/6$ is the volume of the tetrahedron T with vertices P_σ.

The volume coordinates of the vertices P_σ of the tetrahedron T, which is taken as the tetrahedron of reference, are given by

$$P_1(1, 0, 0, 0), \quad P_2(0, 1, 0, 0), \quad P_3(0, 0, 1, 0), \quad P_4(0, 0, 0, 1).$$

4. Formulas in volume coordinates. The various formulas of ordinary solid analytical geometry can be written in terms of volume coordinates by use of the transformation given by equations (2). For example, the formula for the square of the distance d between two points with volume coordinates x_1^α and x_2^α is given by

$$d^2 = -\tfrac{1}{2}(p_{\alpha\beta})^2(x_1^\alpha - x_2^\alpha)(x_1^\beta - x_2^\beta),$$

where $p_{\alpha\beta}$ is the length of the edge $P_\alpha P_\beta$ of the tetrahedron T. Of course, $p_{\alpha\beta}$ is zero when $\alpha=\beta$. It should be observed that any equation in volume coordinates can be rendered homogeneous by use of the relation (3), which holds for all finite points. The coordinates of points at infinity satisfy the relation $x^1+x^2+x^3+x^4=0$.

Because some of the formulas in volume coordinates are known,* space will be given here to the development of only two formulas, which are believed to be new. These formulas for the angle between two planes and the distance from a point to a plane are used to obtain an invariant relation relative to a tetrahedron which may be of some interest.

The planes given by $x^\sigma=0$, $x^\tau=0$ in volume coordinates are represented by $A_\alpha^\sigma\xi^\alpha=0$, $A_\alpha^\tau\xi^\alpha=0$ in cartesian coordinates, as is seen from equations (2). The angle $\Phi_{\sigma\tau}$ between the planes is given by

$$\cos\Phi_{\sigma\tau} = \frac{\sum_{i=1}^{3} A_i^\sigma A_i^\tau}{\sqrt{\sum_{i=1}^{3} A_i^\sigma A_i^\sigma}\sqrt{\sum_{i=1}^{3} A_i^\tau A_i^\tau}}. \tag{6}$$

In order to evaluate $\sum_{i=1}^{3} A_i^\sigma A_i^\sigma$ in terms of geometric magnitudes in the tetrahedron T, we note that the distance d_σ from the vertex P_σ, with cartesian coordinates $(a_\sigma^1, a_\sigma^2, a_\sigma^3, a_\sigma^4)$, to the opposite face of T, with equation $A_\alpha^\sigma\xi^\alpha=0$, is given by

$$d_\sigma = \frac{A_\alpha^\sigma a_\sigma^\alpha}{\sqrt{\sum_{i=1}^{3} A_i^\sigma A_i^\sigma}} = \frac{1}{\sqrt{\sum_{i=1}^{3} A_i^\sigma A_i^\sigma}} \qquad (\sigma \text{ not summed}).$$

Hence,

$$\sum_{i=1}^{3} A_i^\sigma A_i^\sigma = \frac{1}{(d_\sigma)^2} \qquad (\sigma = 1, 2, 3, 4). \tag{7}$$

If the area of the face of T opposite P_σ is denoted by K_σ, then the volume of T is $d_\sigma K_\sigma/3$. Since this $=\Delta/6$, we have $\Delta=2d_\sigma K_\sigma$ for any σ. Therefore, we may write

* For instance, "Solid Geometry," by Frost and Wolstenholme, Macmillan, 1863. This reference was furnished by the referee of this paper, but the author did not have access to it.

$$\sum_{i=1}^{3} A_i^{\sigma} A_i^{\sigma} = 4\left(\frac{K_\sigma}{\Delta}\right)^2. \tag{8}$$

It follows from (6), (7), (8) that

$$\sum_{i=1}^{3} A_i^{\sigma} A_i^{\tau} = \frac{4}{\Delta^2} K_\sigma K_\tau \cos \Phi_{\sigma\tau} = \frac{\cos \Phi_{\sigma\tau}}{d_\sigma d_\tau} \quad (\sigma, \tau \text{ not summed}). \tag{9}$$

The angle ϕ between any two planes with equations $M_\alpha x^\alpha = 0$, $N_\alpha x^\alpha = 0$ is found from their cartesian equations $M_\alpha A_\sigma^\alpha \xi^\sigma = 0$, $N_\alpha A_\sigma^\alpha \xi^\sigma = 0$ to be given by

$$\cos \phi = \frac{\sum_{i=1}^{3} M_\alpha N_\beta A_i^\alpha A_i^\beta}{\sqrt{\sum_{i=1}^{3} M_\alpha M_\beta A_i^\alpha A_i^\beta}\ \sqrt{\sum_{i=1}^{3} N_\alpha N_\beta A_i^\alpha A_i^\beta}} = \frac{M_\alpha N_\beta (K_\alpha * K_\beta)}{\sqrt{M_\alpha M_\beta (K_\alpha * K_\beta)}\sqrt{N_\alpha N_\beta (K_\alpha * K_\beta)}}, \tag{10}$$

in which use is made of (9) with $K_\sigma K_\tau \cos \Phi_{\sigma\tau}$ written as $K_\sigma * K_\tau$. The planes are perpendicular if, and only if, the double sum $M_\alpha N_\beta (K_\alpha * K_\beta)$ vanishes.

The distance D from any point with volume coordinates y^α to the plane with equation $M_\alpha x^\alpha = 0$ is given by

$$D = \frac{M_\alpha y^\alpha}{\sqrt{\sum_{i=1}^{3} M_\alpha M_\beta A_i^\alpha A_i^\beta}} = \frac{\Delta}{2} \frac{M_\alpha}{\sqrt{M_\alpha M_\beta (K_\alpha * K_\beta)}}. \tag{11}$$

The distances D_σ of the vertices P_σ of the tetrahedron T from the plane $M_\alpha x^\alpha = 0$ are

$$D_\sigma = \frac{\Delta}{2} \frac{M_\sigma}{\sqrt{M_\alpha M_\beta (K_\alpha * K_\beta)}} \qquad (\sigma = 1, 2, 3, 4),$$

from which it is seen that

$$D_\sigma D_\tau (K_\sigma * K_\tau) = \left(\frac{\Delta}{2}\right)^2. \tag{12}$$

By the fact that $\Delta = 2 d_\sigma K_\sigma$ for any σ, the identity (12) may be written in the form

$$\frac{D_\sigma D_\tau}{d_\sigma d_\tau} \cos \Phi_{\sigma\tau} = 1. \tag{13}$$

The result (13) may be stated in the following form. *If the distances from the vertices of any tetrahedron to any plane are divided by the altitudes of the tetrahedron*

from the respective vertices, the sum of the scalar products of these ratios taken two at a time is unity. (Each ratio may be associated with a vector in the direction of an altitude of T.)

5. The cevian tetrahedron. As an example of the use of volume coordinates, let us find the locus of a point $P(z^\alpha)$ if the volume of its cevian tetrahedron* relative to T bears a constant ratio (k) to the volume of T. The line joining $P(z^\alpha)$ to a vertex P_σ of T intersects the face of T opposite P_σ in the point P_σ'. The tetrahedron with vertices P_σ' is the cevian tetrahedron of P relative to T.

By use of the parametric equations of a line

$$x^\alpha = x_1^\alpha + rL^\alpha,$$

where x_1^α are the coordinates of a fixed point on the line, L^α are constants for the line, and r is a parameter, the volume coordinates of P_σ' are readily found to be given by

$$P_1'\left(\frac{0, z^2, z^3, z^4}{1-z^1}\right),\quad P_2'\left(\frac{z^1, 0, z^3, z^4}{1-z^2}\right),\quad P_3'\left(\frac{z^1, z^2, 0, z^4}{1-z^3}\right),\quad P_4'\left(\frac{z^1, z^2, z^3, 0}{1-z^4}\right).$$

From (5) the volume of the tetrahedron with vertices P_σ' is

$$\frac{\Delta}{6}\begin{vmatrix} 0 & z^2 & z^3 & z^4 \\ z^1 & 0 & z^3 & z^4 \\ z^1 & z^2 & 0 & z^4 \\ z^1 & z^2 & z^3 & 0 \end{vmatrix} \div (1-z^1)(1-z^2)(1-z^3)(1-z^4),$$

so that

$$-\frac{3z^1z^2z^3z^4}{(1-z^1)(1-z^2)(1-z^3)(1-z^4)} = k.$$

Therefore, the point $P(z^\alpha)$ must lie on the quartic surface represented by

$$3x^1x^2x^3x^4 + k(1-x^1)(1-x^2)(1-x^3)(1-x^4) = 0.$$

6. A sequence of cevian tetrahedra. The cevian tetrahedron S_1 of the centroid G of T relative to T has its vertices at the centroids ${}_1G_\sigma$ of T, the vertex ${}_1G_\sigma$ being in the face opposite P_σ. The cevian tetrahedron S_2 of G relative to S_1 has vertices ${}_2G_\sigma$ at the centroids of the faces of S_1. On continuing in this manner, we may speak of the vertices ${}_nG_\sigma$ of the nth cevian tetrahedron S_n of G relative to T. It is proposed to find the volume coordinates of the vertices ${}_nG_\sigma$.

The coordinates y_1^α of a point P referred to the tetrahedron S_1 are first obtained. For example, by the definition of volume coordinates,

$$y^1_{(1)} = \frac{(P\ {}_1G_2\ {}_1G_3\ {}_1G_4)}{({}_1G_1\ {}_1G_2\ {}_1G_3\ {}_1G_4)}, \tag{14}$$

* N. A. Court, On the Cevian Tetrahedron, this MONTHLY, Vol. 43, No. 2, 1936, p. 89.

where the symbol in parentheses indicates the volume of the tetrahedron with the four points as vertices. When the coordinates $y^{\sigma}_{(0)}$ of P referred to T (regarded as the zeroth cevian tetrahedron of P) and the coordinates of the vertices of S_1 referred to T are substituted into (14), one finds

$$y^{1}_{(1)} = -2y^{1}_{(0)} + y^{2}_{(0)} + y^{3}_{(0)} + y^{4}_{(0)},$$

which, by relation (3), can be written as

$$y^{1}_{(1)} = 1 - 3y^{1}_{(0)}.$$

Thus, we have $y^{\sigma}_{(1)} = 1 - 3y^{\sigma}_{(0)}$. Now the coordinates of the vertices of S_2 relative to S_1 are of the same form as those of S_1 relative to T. That is,

$$y^{\sigma}_{(2)} = 1 - 3y^{\sigma}_{(1)} = 3^0 - 3^1 + 3^2 y^{\sigma}_{(0)},$$

and, for the general case,

$$y^{\sigma}_{(n)} = \sum_{r=0}^{n-1} (-1)^r 3^r + (-1)^n 3^n y^{\sigma}_{(0)}. \tag{15}$$

Hence, the coordinates $x^{\sigma}_{(n)}$ of the vertices ${}_nG_{\sigma}$ of the nth cevian tetrahedron S_n of G relative to T are given by

$$x^{\sigma}_{(n)} = \frac{y^{\sigma}_{(n)} - \sum_{r=0}^{n-1} (-3)^r}{(-3)^n} = \frac{4y^{\sigma}_{(n)} - 1 + (-3)^n}{4(-3)^n}, \tag{16}$$

where $y^{\sigma}_{(n)}$ are the coordinates of ${}_nG_{\sigma}$ relative to S_n. If we use (1, 0, 0, 0) for $y^{\sigma}_{(1)}$, (0, 1, 0, 0) for $y^{\sigma}_{(2)}$, *etc.*, we find the coordinates of ${}_nG_{\sigma}$ given by

$${}_nG_1: \frac{(-3)^n + 3, (-3)^n - 1, (-3)^n - 1, (-3)^n - 1}{4(-3)^n},$$

$${}_nG_2: \frac{(-3)^n - 1, (-3)^n + 3, (-3)^n - 1, (-3)^n - 1}{4(-3)^n},$$

$${}_nG_3: \frac{(-3)^n - 1, (-3)^n - 1, (-3)^n + 3, (-3)^n - 1}{4(-3)^n},$$

$${}_nG_4: \frac{(-3)^n - 1, (-3)^n - 1, (-3)^n - 1, (-3)^n + 3}{4(-3)^n}.$$

It is to be observed that the points ${}_nG_{\sigma}$ approach G with coordinates $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ as n becomes infinite. Further, it is readily verified that the vertices ${}_nG_{\sigma}$ of the nth cevian tetrahedron of G relative to T lie on the line joining P_{σ} to G for all values of n. Finally, by means of formula (5), the numerical value of the ratio of the volume of the nth cevian tetrahedron to that of T is $(27)^{-n}$.

7. A pencil of quadric surfaces relative to a tetrahedron. The four lines $GP_\sigma(\sigma=1, 2, 3, 4)$ determine a pencil of hyperbolic cones. In order to find the equation of this pencil, it is observed that if the quadric surface represented by $M_{\alpha\beta}x^\alpha x^\beta=0$ passes through the four vertices P_σ of T, then $M_{\alpha\alpha}=0(\alpha=1, 2, 3, 4)$. If the four additional points ${}_1G_\sigma$ lie on this quadric, its equation takes the form

$$(1+\lambda)(x^1x^4+x^2x^3)-\lambda(x^1x^3+x^2x^4)-(x^1x^2+x^3x^4)=0, \tag{17}$$

where λ is a parameter. The hyperbolic cones (17) are degenerate in the cases $\lambda=0, \lambda=-1$.

The tangent plane to a cone of the pencil (17) at the vertex P_4 of T intersects the face of T opposite P_4 in a line l_4, which is represented by $x^4=0$ together with

$$(1+\lambda)x^1-\lambda x^2-x^3=0. \tag{18}$$

In an analogous manner, one finds lines l_1, l_2, l_3 in the other three faces of T. To each hyperbolic cone in (17), that is, to each value of λ, there correspond four space lines l_1, l_2, l_3, l_4, one in each face of T. These four lines are generators of one regulus of the quadric surface with the equation

$$\begin{aligned}\lambda(\lambda+1)[(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2]-(2\lambda^2+2\lambda+1)(x^1x^2+x^3x^4)\\ -\lambda(\lambda^2+2\lambda+2)(x^1x^3+x^2x^4)+(\lambda+1)(\lambda^2+1)(x^1x^4+x^2x^3)=0.\end{aligned} \tag{19}$$

The four faces of T intersect this quadric surface in four other lines l'_σ, each of which intersects the four lines l_σ. The lines l'_σ belong to the second regulus of the quadric surface (19).

DISCUSSIONS AND NOTES

EDITED BY E. F. BECKENBACH, University of California, Los Angeles 24, Calif.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE RATIONAL ROOTS OF POLYNOMIAL EQUATIONS

J. L. DORROH, Illinois Institute of Technology, and
LYNDEN B. HOWELL, Louisiana State University

A simple consideration of the solutions of conditional congruences gives tests which sometimes can be used to limit the search for the rational roots of a given polynomial equation with integral coefficients.

Let

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n, \tag{1}$$

where $a_0\neq 0$, and a_j is an integer for $j=0, 1, \cdots, n$. Then if the equation

$$f(x) = 0 \tag{2}$$

has an integral root, it follows that for any prime p the congruence

$$f(x) \equiv 0 \pmod{p} \tag{3}$$

has a solution in the set of integers $0, 1, \cdots, p-1$. The following theorem is thus established.

THEOREM 1. *If there is a prime p such that for the polynomial $f(x)$ defined in* (1) *we have*

$$f(0)f(1) \cdots f(p-1) \not\equiv 0 \pmod{p},$$

then the equation $f(x)=0$ has no integral root.

If (2) has a rational root, let such a root in its lowest terms be denoted by s/t. Then

$$f(x) = (tx - s)q(x), \tag{4}$$

where $q(x)$ is a polynomial with integral coefficients. Further, t divides a_0, so that if p is a prime not dividing a_0 then t is prime to p and the congruence

$$tx - s \equiv 0 \pmod{p} \tag{5}$$

has a solution. From (4) it follows that a solution of (5) is also a solution of (3). A second theorem may now be stated.

THEOREM 2. *If for a prime p and the polynomial $f(x)$ defined by* (1) *we have*

$$a_0 f(0)f(1) \cdots f(p-1) \not\equiv 0 \pmod{p},$$

then the equation $f(x)=0$ has no rational root.

Variations of the above theorems for $p=2$ and for $p=3$ are of interest for their simplicity. For $p=2$ we have:

(1.1) *If both a_n and $f(1)$ are odd, then the equation $f(x)=0$ has no integral root.*

(2.1) *If a_0, a_n, and $f(1)$ are all odd, then the equation $f(x)=0$ has no rational root.*

For $p=3$, the products in the conditions of the theorems may be replaced by $f(0)f(1)f(-1)$ and $a_0f(0)f(1)f(-1)$, respectively.

If for some prime p, the numbers $f(0), f(1), \cdots, f(p-1)$ or their equivalents modulo p have been computed, even if the condition of Theorem 1 is not satisfied, it is possible that information which will limit the search for a rational root of (2) has been gained. For it is to be noted that an integral root of (2) is a divisor of a_n in the form $kp+r$, where k is an integer and r is one of the solutions of (3) already determined by the presumed computations. If p is prime to a_0, then any rational root of (2) is associated through (5) with an r in the set of solutions of (3), and has the form $(rt+mp)/t$, where t is a divisor of a_0, and $rt+mp$ is a divisor of a_n.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1945-46

Mathematics Club, Boston University

During the year 1945–46 the Mathematics Club of Boston University held biweekly meetings at which faculty and student speakers gave talks which were both educational and interesting, as follows.

The mathematics of lion hunting, by June Westgate

A unique derivation of an ellipsoid, by Harry Kouyoumjian

The conquest of the "impossible," by Professor R. E. Bruce

Mathematical logic, by David Wellinger

Fun with mathematics, by Evelyn Nutile

The automatic sequence controlled calculator, presented by Lieutenant Robert Campbell at a meeting of the *Harvard Mathematical Club* at which a group of our members were guests

Spherical triangles, by Professor L. A. Brigham

Mrs. Miniver's problem, or *How to be happy though married*, by Professor E. B. Mode.

A group of members attended an exhibit of spiral shells at the Wellesley College Art Museum. Several informal supper parties were held by the club, during the year.

Officers for the year 1945–46 were: President, Marion King; Vice-President, June Westgate; Secretary, Evelyn Nutile; Treasurer, Robert Goodwin; Faculty Adviser, Professor R. N. Johanson. Officers elected for the year 1946–47 are: President, Robert Goodwin; Vice-President, Norma Algeri; Secretary, Bessie Shuris; Treasurer, Betty Beyea.

Junior Mathematics Club, Iowa State College

Quarterly meetings were held during the school year, at which talks were given by undergraduates, graduates, and faculty. The general theme of each meeting was

Nonsense night

Mathematics of convoys and meteorology

Probability and chance.

The club was in charge of the mathematics open house during the Iowa State College open house which is called *Veishea.*

Miss Adelaide Madsen was club director until graduation at the end of the second quarter. Then Miss Mary Beth Lieberknecht assumed these duties. The Faculty Adviser was Professor Fred Robertson.

Pi Mu Epsilon, Iowa State College

The Iowa Alpha Chapter held quarterly program meetings during the school year. The topics discussed were:

Sample universes

Teaching experiences

The analogy between circular and hyperbolic functions.

The annual *Pi Mu Epsilon* prize to the person who has achieved the highest scholastic mathematical average in his freshman and sophomore years was awarded this year to Mr. Paul F. Radics.

Officers for the year 1945–46 were: Co-Directors, Dick Hales, V12, and Amanda Christensen; Vice-Director, Adelaide Madsen; Treasurer, Carl Langenhop; Secretary, Carol Yetter. The officers for the year 1946–47 are: Director, Mary Beth Lieberknecht; Vice-Director, Mary Alice Barber; Secretary, Juavanta Young; Treasurer, Ralph Robinson. For both years the Faculty Adviser is Fred Robertson.

Mathematics-Physics Club, College of Saint Teresa

The meetings of this year were devoted chiefly to the discussion of

Audio-visual aids in the teaching of mathematics, including films, slide films, microfilms, slides, opaque projectors, recordings, and a review of the Eighteenth Yearbook on *Multi-sensory aids in the teaching of mathematics*, illustrated by *Curve-stitching.*

There were discussions on the atomic bomb and radar. A guest speaker and members of the faculty reported on the mathematical conventions held during the year. A picnic furnished diversion for the members.

Officers were: President, Leota Ruppert; Vice-President, Ramona Blaschke; Secretary, Gertrude Curran; Treasurer, Delphine Meisch.

Pi Mu Epsilon, Duke University

North Carolina Alpha held two initiations during the year 1945–46. At the May meeting an illustrated lecture was presented entitled

Instruments for mathematical computation, by Ensign W. F. Gabriel.

At a regular business meeting the following officers were elected to serve in 1946–47: Director, George Epps; Vice-Director, Ruth Neuhoff; Secretary, Jo Anne Walker; Treasurer, Wayne Bainbridge. The Faculty Adviser is Professor W. W. Elliot.

Pi Mu Epsilon, Kansas State College, Manhattan (1944–45)

Our student enrollment in advanced work was so low because of the war that only one or two members from the previous year returned. Although regular meetings were not held, we had a Mathematics Club, and in May we had a dinner program and initiation, elected eight students to membership in *Pi Mu Epsilon*, and selected the following officers for 1945–46: Director, Professor A. E. White; Vice-Director, Robert Remking; Secretary, Ethel Rogers; Treasurer, Thirza Mossman; Corresponding Secretary, Professor W. T. Stratton.

Mathematics Club, Connecticut College

Loran was the title of a lecture by Mr. John M. Ide, Executive Director of the Navy Underwater Sound Laboratory at Fort Trumbull, New London. This talk, sponsored by the Mathematics Club, was one of the very first on the *Loran* (long range navigation) System, which until recently has been a carefully guarded war secret. Developed in the winter of 1941–42, *Loran* was effectively used for long range bombing over Europe and Japan. It also has a worthwhile peacetime application in facilitating long range navigation at sea, especially since *Loran* stations have already been established at points covering one fourth of the world and all the most important sea lanes. Mr. Ide said that *Loran* was not only important because of its long range effectiveness, but also because its precision is greater than that of any other range finders. Its error is one per cent of its distance from the key station. A skillful operator can get a fix (or position) in less than a minute by *Loran*.

The final meeting of the year was a social one—a picnic at which two new books were presented to the Club by its members: *Mathematical Recreations*, by Kraitchik and *Non-Euclidean Geometry* by Wolfe.

Officers for 1946 were: President, Jean Compton; Secretary Treasurer, Mary Corning; Chairman of Refreshments, Mary Bolz; Publicity and Social Chairman, Rosalie Creamer. Officers elected for 1946–47 are: President, Shirley Bodie; Secretary-Treasurer, Edith Lechner; Chairman of Refreshments, Jacqueline Greenblatt; Chairman of Social Activities, Roberta Richards; Chairman of Publicity, Virginia Doyle; Faculty Adviser, Miss Julia Wells Bower.

Mathematics Club, New York State College for Teachers at Albany

Seven regular meetings were held during the year 1945–46 at which the following topics were presented:

Cryptography, by Dr. Caroline A. Lester
Geometry for a closed convex portion of the Euclidean plane, by Adele Kasper
Geometry of paper folding, by Jane Farmer
The game of Nim, by Dr. R. A. Beaver
Celestial navigation, by Mr. C. J. Haughey
Singular points, by Herbert Ford
Mathematical puzzles, by Theresa Jones
Asymptotes, by Carmella Russo
Perfect squares, by Doris Ives
Women in mathematics, by Martha Neighbour.

In addition to these meetings a picnic was held in May and a Christmas party was held in collaboration with the Chemistry Club.

The officers for the year 1945–46 were as follows: President, Pauline Myers; Vice-President, Doris Ives; Treasurer, Marie Balfoort; Secretary, Ruth Seelbach. The newly elected officers for the year 1946–47 are as follows: President, Ruth Seelbach; Vice-President, Betty Whitney; Treasurer, Elsa Moberg; Secretary, Florence Mace.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Select Topics of Plane Analytic Geometry for Scientific and Technical Workers. By M. A. Rosanoff. Brooklyn, Long Island University Press, 1944. 8 +76 pages. $1.25.

This booklet is intended mainly for workers in scientific and technical laboratories and, according to the author, aims to give "a clearer grasp of the Cartesian method than that usually retained from the orthodox college course in Analytic Geometry." Equations and their peculiarities are emphasized rather than the geometric properties of the curves represented by them. Three distances —between two points, between two parallel lines, and between a point and a line—are introduced early and are fundamental in the discussion. The equation $y=a+bx$, called the "slope" equation of the straight line, and $y=a+bx+cx^2$ receive special attention due to their importance in representing scientific data. By far the greater part of the monograph is concerned with the interpretation of the general equation of the second degree in two unknowns; in particular, pairs of straight lines and the parabola are treated more fully than the other conics. Instead of transforming the reference axes, as is frequently done, the author studies certain combinations of the coefficients and puts special stress on the significance of the constant term. The booklet closes with a brief discussion of certain characteristic invariants of the general equation of the second degree.

The reader is urged not to proceed without pencil and paper; frequent numerical examples are given and, as an encouragement to the student, their solutions are outlined. The booklet is attractively printed and only a few unimportant typographical errors were noted. The extent to which the author has succeeded in his effort to clarify certain topics may be questioned. Too often proofs lack rigor or are not given at all and frequently it is not clear what assumptions have been made. A combination of statements such as "an asymptote is defined in general as a tangent to a curve touching it at infinity" (page 10) and "the line OC will meet the hyperbola only at plus and minus infinity and is therefore, by definition, an asymptote" (page 21) should, in the opinion of the reviewer, be avoided.

As the title indicates, the booklet is not intended to be a text for the beginner and no claim is made for completeness. The thoughtful reader will find here much that is novel and stimulating.

C. H. YEATON

Industrial Algebra and Trigonometry with Geometrical Applications. By J. H. Wolfe, W. F. Mueller and S. D. Mullikin. New York and London, McGraw-Hill Book Co., Inc., 1945. 13+389 pages. $2.20.

This book should come to the attention of engineers and technicians, particularly those in the mechanical industries. They will find it a useful reference in the solution of specific problems. Such use will probably lead to a review of related topics and should promote a systematization of computing methods. The problems are well classified and graded, and the range of topics is broad. Included, for example, are: reciprocating motion, camshafts, gearing of various types, radians, compound angles, and a great variety of dimensioning problems. The chapter on higher degree equations is particularly good, with an adequate discussion of Horner's Method.

The book merits serious consideration as a text because of its strong motivation. The standard topics of algebra and trigonometry are thoroughly covered and selections can be made from the supplementary material. Prospective draftsmen and designers will find it well adapted to independent study. Since illustrative examples are worked in detail, I can heartily approve the authors' policy of omitting answers. The pure mathematician would doubtless find several flaws in the text, none likely to lead to any serious misunderstanding.

A five-place table of logarithms of numbers and trigonometric functions (to minutes) is included. All six natural functions (to minutes) are given to five figures or more.

The authors are to be complimented for producing a stimulating book, practical and thorough.

W. W. BIGELOW

The Development of Mathematics. Second Edition. By E. T. Bell. New York, McGraw-Hill Book Co., Inc., 1945. 13+637 pages. $5.00.

This magnificent, inclusive, and provocative survey of the origin and adventures of mathematical ideas has now appeared in a second edition. Various material has been added; an extensive survey of recent developments in lattice theory, together with notes on recent advances in such disparate subjects as Diophantine Analysis (Mordell, Segré), Waring's problem (Niven), unified field theory (Einstein), surface area (Youngs), three-valued logic and quantum mechanics (Reichenbach), the inconsistency of Quine's system of logistic (Rosser), the advances in completeness theorems in logic (Kleene), and the use of mathematics during the second world war. Various other statements have been brought up to date by the simple device of replacing 1940 by 1945; thus the somewhat dubious statement "Only the arithmetics of rings with commutative multiplication has been discussed with anything approaching completeness up to 1945" (p. 256).

The great virtue of this book is that it does not merely record facts, but it arranges ideas and passes judgment as to their importance. This aim, combined with the tremendous scope of the work, makes it inevitable that there should be

errors both of fact and of judgment. There are simple slips (p. 220, last line, if α is irreducible, then in any factorization one factor is a unit and the other is an *associate* of α—not α itself). There are misstatements; thus (p. 260), A. E. Noether did *not* determine all commutative rings in which there is a unique prime ideal factorization, but merely found a set of postulates for such rings. There are gross misapprehensions. For example, Chapter IX treats the recent development of algebra and is entitled, "Toward an analysis of Mathematical structure," yet a perusal of the chapter reveals that the author changes his definition of structure on every page. On page 214, "*all* instances of a field have the same structure," while on page 216, "these groups are said to be simply isomorphic or to have the same structure." Finally, on page 217, "If it is possible to establish a one-to-one correspondence between the postulates of two systems such that correlated postulates have the same structure, then the systems are said to have the same structure." It may well be true that modern algebra tends toward an analysis of structure, but the author should tell us whether the statement "two systems have the same structure," means (a) the systems are isomorphic; (b) the systems satisfy equivalent postulates, or (c) the postulates for the systems are essentially identical.

But enough of carping criticism. It's great fun to read this book, just because there are so many chances profitably to disagree with its provocative author. The wealth of possible topics of difference must be read to be appreciated. Is Plato as vicious as Bell's everywhere dense cracks would indicate? Does Bell overemphasize the importance of lattice theory and miss some of the significant developments in modern topology? Has this hard-headed author been duped by the advocates of Brouwerian logic and many-valued logics? Is Frêchet's work as significant as Bell claims? Might some mathematical war workers disagree with Bell's dismissal of spherical trigonometry as useless?

The book is of great value for many classes of readers. The specialist will find a new perspective and enjoyable provocation in his own field. The historian will find a writer with the courage to attempt a rational assessment of modern mathematics. The student will find a wealth of suggested new vistas. The teacher will find the historical origin and (possibly) the lack of significance in the material appearing in courses. The writer of textbooks will find new suggestions, well barbed (elementary books still fail to prove Taylor's theorem in the same way in which Taylor failed). The philosopher will disagree with the jabs at Kant, but will profit from the view of living mathematics. The young mathematician will gain background and will learn of the ebb and flow of fashion in the specialties of research. To all these and others one might say: don't wonder about it, but go, read, and disagree for yourself.

SAUNDERS MACLANE

Mathematical Cuneiform Texts. Edited by O. Neugebauer and A. Sachs. With a chapter by A. Goetze. New Haven, Connecticut, the American Oriental Society and the American School of Oriental Research, 1945. 177 p. +49 plates. Price $5.00.

Many of us who have had the privilege of developing courses in the history of mathematics live over again, with the appearance of this volume, the thrilling experiences that were ours through the revelations of Babylonian achievements in mathematics made by Professor Neugebauer and set forth in his *Mathematische Keilschrift—Texte*, 3 v. *Quellen u. Studien zur Geschichte der Mathematik . . . A. Quellen*, v. 3, 1935–37, and several articles. What delight we had in the solution of the quadratic equation with the positive sign before the radical as taken from a tablet of about 2000 B. C., and from other seemingly incredible items! We have here, then, the continuation of that work in one of inestimable value to the history of mathematics.

The preface tells us that "this volume is devoted to the edition of hitherto unpublished mathematical texts, chiefly from American collections. Not only does this new group of documents supplement the previously published material in many respects, but is itself sufficiently extensive to offer a fair impression of the main types of Babylonian mathematical texts. New frontiers, the existence of which had been no more than suspected are actually reached by a tablet (Plimpton 322) involving 'Pythagorean' Number Theory." What more exciting adventure could an explorer embark upon than this one! And the authors made discoveries which have greatly enriched the world in a scholarly and intellectual sense.

From this same preface comes the gratifying knowledge concerning "The Yale Babylonian Collection the source of the largest part of the material presented here" that: "More mathematical problem-texts have now been published from this collection than from any other museum in the world." All of this shows to what extent this country has come to the front in the matter of cuneiform texts.

An additional feature of great value is the contribution to this volume by Dr. A. Goetze "of his important chapter on the dialects of the Old-Babylonian Akkadian mathematical texts." The preface states that: "In addition to numerous table-texts, most of the problem-texts published in this book were discovered by Dr. A. Goetze."

Without making any hollow pretense to reading the whole book, much may be gleaned by reading all that is set forth so well in general form. The chapters are five in number. Chapter I INTRODUCTION gives The Texts; The Sexagesimal Number System; Translation and Transcription; Metrology; Examples of Metrological Calculations. Chapter II TABLE-TEXTS gives Introduction (which tells us that: "The Mathematical tables by means of which all numerical calculations were carried out, played a basic role in the high development of Old-Babylonian Mathematics and of Babylonian astronomy in the Seleucid period"). Reciprocals; Multiplication Tables; Squares, Cubes and Varia; (under

the last, surprisingly enough, occurs "Logarithms" since Tablets which contain tables of exponents a^n where n is an integer between 2 and 10 and a is one of the numbers 9, 16, 1,40, 3, 3,45 · · · are known.") Chapter III PROBLEM-TEXTS: This chapter contains a wealth of material for study and reference under the headings: Introduction; Pythagorean Numbers; Cube Root; Geometrical Problems. The second of these (Plimpton 322) is of outstanding interest. It tabulates the answers to a problem containing Pythagorean numbers (or Pythagorean triangles). "*It is the oldest preserved document in ancient number theory*" (italics mine). We learn that this tablet falls between 1900 and 1600 B.C. A review is no place for a complete treatment of a text but the elucidation of this one makes a fascinating study. Under "Historical Consequences," it is pointed out that: "We now have a text of purely number theoretical character treating a problem organically developed from other problems already well known . . . " Chapter IV THE AKKADIAN DIALECTS OF THE OLD-BABYLONIAN MATHEMATICAL TEXTS, by A. Goetze. Chapter V INDICES covers Bibliography and Abbreviations; Concordance of Museum Numbers; Vocabulary; Subject Index; Map showing Ancient Sites which are mentioned. The utter completeness and meticulous care of the entire work must be noted. As a climax and crowning feature, there are forty-nine Plates which present all the texts whose descriptions are so perfectly presented.

We learn that the: "Greatest amount of new information about practical questions is yielded by texts dealing with bricks. Here we learn for the first time the dimensions of several standard types of bricks and the metrological system used in counting bricks. It is clear that such information is not only of use for the better understanding of similar mathematical texts, but will also influence our interpretation of economic documents, and archeological data." And most interesting is the fact that: "For the first time we have texts which have the character of pages from a general handbook."

"It lifts the heart," to use a phrase coined by J. B. Priestley, to live with a piece of scholarship like this one even though one brings to the living very little ability to follow its details.

L. G. SIMONS

NEW BOOKS RECEIVED

Elementary Applied Aerodynamics. By P. E. Hemke. New York, Prentice-Hall, Inc., 1946. 8+231 pages. $3.25.

Amortizacion y Seguro de Vida. By Eric Michalup. Caracas, 1944. 26 pages.

Analytic Geometry. By F. D. Murnaghan. New York, Prentice-Hall, Inc., 1946. 8+402 pages. $3.25.

Analytische Geometrie der Ebene und des Raumes. By Rudolf Fueter. (Lebrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, No. 4; Mathematische Reihe, Band II.) Basel, Verlag Birkhäuser, 1945. 180 pages, pamphlet bound, 18.50 s. fr.; cloth bound, 22.50 s. fr.

A History of the Conic Sections and Quadric Surfaces. By J. L. Coolidge. Oxford University Press, 1945. 11+214 pages. $6.00.

Science in a Changing World. Revised Edition. By E. J. Cable, R. W. Getchell, and W. H. Kadesch. New York, Prentice-Hall, Inc., 1946. 18+622 pages. $5.00.

Mathematical Tables. Vol. 1. Second Edition. British Association for the Advancement of Science. Cambridge, University Press, 1946. 11+72 pages. $2.50.

Analytic Geometry and Calculus. Second Edition. By H. B. Phillips. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1946. 13 +504 pages. $4.50.

College Mathematics; A General Introduction. By C. H. Sisam. New York, Henry Holt and Co., 1946. 13 +561 pages. $3.50.

The Common Sense of the Exact Sciences. By W. K. Clifford. (Prefaces by Karl Pearson and Bertrand Russell; Introduction by J. R. Newman.) New York, Alfred A. Knopf, Inc., 1946. 66 +249 pages. $4.00.

Developing Number Readiness. By Maurice Hartung and Anita Riess. Chicago, Scott, Foresman and Co., 1946. 36 pages.

Essentials of Plane and Spherical Trigonometry. Revised Edition. By Clifford Bell and T. Y. Thomas. New York, Henry Holt and Co., 1946. 9 +165 pages. Without tables $2.00; with tables $2.30.

Mathematician's Delight. By W. W. Sawyer. New York, Penguin Books, 1946. 224 pages. $0.25.

Mathematics of Investment. Third Edition. By W. L. Hart. Boston, D. C. Heath and Co., 1946. 7 +306 +126 pages. $3.60.

Scientific, Medical, and Technical Books Published in the United States of America 1930–1944; A Selected List of Titles in Print with Annotations. Edited by R. R. Hawkins. Washington, National Research Council, 1946. 15 +1114 pages. $20.00.

Theory of Lie Groups. I. (Princeton Mathematical Series, No. 8.) Princeton University Press, 1946. 12 +217 pages. $3.00.

A Manual of Operation for the Automatic Sequence Controlled Calculator. (Annals of the Computation Laboratory of Harvard University, Vol. I.) By the Staff of the Computation Laboratory, Harvard University. Cambridge, Harvard University Press, 1946. 561 pages. $10.00.

Tables of the Modified Hankel Functions of Order One-Third and their Derivatives. (Annals of the Computation Laboratory of Harvard University, Vol. II.) By the Staff of the Computation Laboratory, Harvard University. Cambridge, Harvard University Press, 1945. 235 pages. $10.00.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 731. *Proposed by Howard Grossman, New York City*

At any point P on the hypocycloid $x^{2/3}+y^{2/3}=a^{2/3}$, the tangent to the curve is the shortest line through P lying between the axes.

E 732. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find, in the system of base 9, a number of three digits which, when transformed to the system of base 13, is composed of the same three digits.

E 733. *Proposed by E. D. Schell, Arlington, Va.*

Show that a square matrix of order n, whose elements form a magic square, has a characteristic root equal to $n(n^2+1)/2$.

E 734. *Proposed by Henry Scheffé, Princeton University*

A body in the form of a plate is supported by a smooth horizontal surface, which we take as the x,y-plane. A vertical peg is fastened to the body through its center of gravity. Two similar springs of natural length l run from this peg to pegs fixed in the plane, one at $(0, l)$, and the other at $(0, -l)$, the springs being parallel to the plane. The body is released from rest in a position where its center of gravity is on the x-axis. Assuming the forces exerted by the springs proportional to their elongations, and the masses of the springs negligible, show that for small oscillations the frequency is proportional to the amplitude.

E 735. *Proposed by Paul Erdös, Stanford University*

Six points can be arranged in the plane so that all triangles formed by triples of these points are isosceles. Show that seven points in the plane cannot be so arranged. What is the least number of points in space which cannot be so arranged?

SOLUTIONS

Magic Squares with Zero Determinant

E 674 [1946, 99]. *Editorial Note*

Do all fifth order pandiagonal magic squares have zero determinants?

Reply by E. D. Schell, Arlington, Va. Represent the general square by

$$\begin{array}{ccccc} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{array}$$

and let Z be the common sum of the rows, columns, and diagonals, including broken diagonals such as B, F, O, S, W. These twenty sums yield seventeen independent relations, making it possible to eliminate all but eight of the elements, yielding

$$\begin{array}{ccccc} A & B & C & D & -A-B-C-D+Z \\ F & G & H & I & -F-G-H-I+Z \\ -A+H+I & -B-F-G-H+Z & -C-G-H-I+Z & -D+F+G & A+B+C+D+G+H-Z \\ -C-D-H-I+Z & A+B+C+F+G+H-Z & B+C+D+G+H+I-Z & -A-B-F-G+Z & -B-C-G-H+Z \\ C+D-F & -A-B-C-G+Z & -C-D-H-B+Z & A+B-I & B+C+F+G+H+I-Z \end{array}$$

as the general pandiagonal square. Unfortunately, the fact that this determinant is not identically zero is not simple to establish. The general form, however, may be of some interest.

After failing to establish that the above determinant is not identically zero, I tried, as did the proposer, a specific instance. By chance, my first choice proved to have a non-zero value:

$$\begin{vmatrix} 10 & 18 & 1 & 14 & 22 \\ 4 & 12 & 25 & 8 & 16 \\ 23 & 6 & 19 & 2 & 15 \\ 17 & 5 & 13 & 21 & 9 \\ 11 & 24 & 7 & 20 & 3 \end{vmatrix} = -4{,}680{,}000.$$

Hence not all pandiagonal magic squares of order five have zero determinants.

Editorial Note. Schell's canonical form for fifth order pandiagonal magic squares given above is quite neat. It is easy to verify by actual expansion that his determinant does not vanish identically. For example, the term in A^2Z^3 has coefficient one. When $A=Z=1$, and the other letters are all zero, the determinant has the value one.

Power Points

E 705 [1946, 36]. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

A point in the interior of a simple closed curve, which is nowhere concave, will be called a *power point* if the product of the segments of a variable chord through the point is constant. The existence of how many distinct power points insures that the curve is a circle?

Solution by J. B. Kelly, Hampton, Va. We shall show that two power points are sufficient. Let O and O' be the two power points and let them have the

respective power k and k'. Let OO' intersect the curve in R and R' in such a way that O is between O' and R. Each member of the one-parameter family of circles passing through R and R' will have O and O' as power points with the powers k and k', respectively.

Let S_1 be a point on the curve distinct from R and R'. Draw the straight line $S_1O'S_1'$, where $(S_1O')(O'S_1') = k'$. Then draw the straight line $S_1'OS_2$, where $(S_1'O)(OS_2) = k$. Next draw the straight line $S_2O'S_2'$, where $(S_2O')(O'S_2') = k'$. Continue in this way, getting two infinite sequences of points $\{S_n\}$ and $\{S_n'\}$ which lie on the curve and also on the circle determined by S_1, R, and R'. It is fairly easy to see that the sequence $\{S_n\}$ has R as a limit point. The slope of the curve at R is thus the same as that of the circle through S_1, R, and R'. Suppose the curve contains a point not on this circle. We can perform the same sequence of constructions and obtain a sequence of points lying on the curve and on *another* circle in the one-parameter family of circles through R and R'. This new sequence of points will also have R as a limit point. This leads to a different value of the slope at R. We assume that the curve has a unique tangent at each of its points. Hence we have a contradiction, and all points of the curve lie on the same circle.

Also solved, in the same way, by the proposer. W. A. Rees showed that any everywhere dense set of points on a chord is sufficient.

The proposer has given a number of interesting results connected with his investigation of closed curves possessing a power point. First of all, it is clear that two power points are also *necessary* if the curve is to be a circle, for it is easy to construct non-circular nowhere concave closed curves having only one power point. Such closed curves, although possessing some interesting properties, seem not to be found in the literature. A simple property of such curves, and one reminiscent of the circle, is that in general a chord through the power point intersects the curve in equal angles. To this the proposer adds the immediate corollary: If two intersecting curves each have a power point on their common chord, then the angles of intersection are equal. He also states that if we take the power point as origin of cartesian coordinates and let the equation of, say, the upper arc of the closed curve be $y = f(x)$, then the equation of the lower arc is

$$k^2 y + (x^2 + y^2) f\left(\frac{-k^2 x}{x^2 + y^2}\right) = 0,$$

where k^2 is the power of the origin. In particular he has shown that if through the Gergonne point P of a triangle we draw lines parallel to the sides, and then inscribe circles in the triangles cut off by each of these parallels, the original triangle with its vertices rounded off by arcs of these inscribed circles, constitutes a closed curve having P as a power point. Finally, the proposer has also shown that in a closed surface, the existence of three non-collinear power points within is necessary and sufficient for the surface to be a sphere. It is easy to conjecture the corresponding theorem for euclidean n-space.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4210. *Proposed by R. Goormaghtigh, Bruges, Belgium*

If the parallels to the sides of triangle ABC drawn through a point P on the circumcircle meet that circle again at A', B', C', the orthocenter H of ABC and those α, β, γ of $A'BC$, $B'CA$, $C'AB$ are on a straight line perpendicular to the Simson line Δ of P as to ABC; and the center of gravity of α, β, γ divides into the ratio 2:1 the distance from H to the circumdiameter parallel to Δ.

4211. *Proposed by K. L. Chung, Princeton, N. J.*

Prove that

$$\binom{nd}{n} = d(d-1)\sum_{k=1}^{n} \frac{(dk-2)!}{(k-1)!(dk-k)!}\binom{nd-kd}{n-k}.$$

4212. *Proposed by H. F. Sandham, Trinity College*

Evaluate

$$\int_0^\infty \frac{e^{-x^2}dx}{(x^2+1/2)^2}.$$

4213. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a tetrahedron $ABCD$ and an arbitrarily chosen point M: (1) The sum of the powers of the vertices, respectively, with respect to three spheres on (MB, MC, MD), (MC, MD, MA), (MD, MA, MB), (MA, MB, MC), as diameters, is equal to the sum of the squares of the edges. (2) Construct the point M when the sums of the powers of the vertices A, B, C, D, relative to the corresponding set of three spheres, are proportional to given numbers α, β, γ, δ. Consider the case where these last four numbers are equal.

4214. *Proposed by Victor Thébault, Tennie, Sarthe, France*

On the sides AB, CD of an arbitrary quadrangle $ABCD$ isosceles triangles are constructed with the same sense $A'AB$, $C'CD$, with the base angle θ; and on the sides BC, DA the isosceles triangles $B'BC$, $D'DA$ with the base angle $\pi/2-\theta$ and in the same sense as the first. Prove that (1) The lines $A'C'$, $B'D'$ are perpendicular and that the lengths of the segments are in the ratio $\tan\theta$.

(2) The centroid of the vertices of the quadrangle is on the straight line joining the midpoints of $A'C'$, $B'D'$, which it divides in the ratio $\cot^2\theta$. (3) Find the locus of the midpoints of the sides and diagonals of the quadrangle $A'B'C'D'$, and the envelope of all these lines.

SOLUTIONS

Orthopolar Triangles

4060 [1942, 619]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If a point P is the orthopole of the three sides of a triangle $A_1B_1C_1$ with respect to another triangle $A_2B_2C_2$ inscribed in the same circle as the first, the product of its distances from the sides of the first triangle is equal to the similar product for the second.

Solution by the Proposer. (1) If the sides of a triangle $A_1B_1C_1$ inscribed in a circle (O) have the same orthopole P, as to a triangle $A_2B_2C_2$ inscribed in the same circle, the sides of this latter triangle have also the same orthopole P as to $A_1B_1C_1$, and the point P is the midpoint of the segment H_1H_2 with its ends at the respective orthocenters of the two triangles, which are then said to be *orthopolar*. This theorem, due to T. Lalescu, *Gazeta Matematica*, Bucarest, 1915, p. 213, has been refound by other geometers, and has been generalized in the following manner by S. Kantor, *Mathesis*, 1906, p. 144; and M. J. Capoulade, *Journal de Vuibert*, 1917, p. 162; and our own geometric proof appeared in *Mathesis*, 1945, nos. 4, 5, pp. 157–161.

Kantor's Theorem. Two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are inscribed in the same circle (O); the orthopoles α_2, β_2, γ_2 of the sides B_2C_2, C_2A_2, A_2B_2 as to $A_1B_1C_1$, and the orthopoles α_1, β_1, γ_1 of sides B_1C_1, C_1A_1, A_1B_1 as to $A_2B_2C_2$, are on the same circle whose center is the midpoint of the segment H_1H_2 with its ends at the orthocenters of the respective triangles.

(2) A generalization of the proposition of 4060 follows.

THEOREM. *The product of the distances of the points α_1, β_1, γ_1 to the respective sides B_1C_1, C_1A_1, A_1B_1 is equal in absolute value to the similar product of the distances of the points α_2, β_2, γ_2 to the sides B_2C_2, C_2A_2, A_2B_2.* See V. Thébault, *Mathesis, l.c.*

In fact, if the sides of $A_1B_1C_1$ meet the sides of $A_2B_2C_2$ at the following angles in the same sense

$$\begin{aligned} x_1 &= (B_1C_1, B_2C_2), & y_1 &= (B_1C_1, C_2A_2), & z_1 &= B_1C_1, A_2B_2), \\ x_2 &= (C_1A_1, B_2C_2), & y_2 &= (C_1A_1, C_2A_2), & z_2 &= (C_1A_1, A_2B_2), \\ x_3 &= (A_1B_1, B_2C_2), & y_3 &= (A_1B_1, C_2A_2), & z_3 &= (A_1B_1, A_2B_2); \end{aligned}$$

the sides B_2C_2, C_2A_2, A_2B_2 meet the sides of $A_1B_1C_1$ at angles which are the supplements of the above angles. Denote by d_a, d_b, d_c the algebraic distances of the points α_1, β_1, γ_1 to the respective straight lines of B_1C_1, C_1A_1, A_1B_1; and similarly by d_a', d_b', d_c' for α_2, β_2, γ_2 and the respective sides of $A_2B_2C_2$; then

from the expression for the distance of a point to its orthopolar,* there result the following equations

$$d_a = |\,2R\cos x_1\cos y_1\cos z_1\,|, \qquad d_a' = -|\,2R\cos x_1\cos y_3\cos z_2\,|,$$
$$d_b = |\,2R\cos x_2\cos y_2\cos z_2\,|, \qquad d_b' = -|\,2R\cos x_2\cos y_1\cos z_3\,|,$$
$$d_c = |\,2R\cos x_3\cos y_3\cos z_3\,|, \qquad d_c' = -|\,2R\cos x_3\cos y_2\cos z_1\,|.$$

From these follow that $|d_a d_b d_c| = |d_a' d_b' d_c'|$, and this completes the proof.

If the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are orthopolar as in 4060, the products of the distances of their sides to the common orthopole P are equal in absolute value.

Note. All these theorems have been again generalized in a communication to the *Académie des Sciences de Paris* with the title *Sur les triangles isopolar.*†

Consecutive Primes

4143 [1944, 593]. *Proposed by Paul Erdös, Purdue University*

Let $p_1<p_2<\cdots<p_n<\cdots$ be the consecutive primes. Prove that

$$\frac{p_n!}{p_n(p_n+1)\cdots(p_{n+1}-1)}$$

is always an integer except when $p_n=3$.

I. *Solution by Fritz Herzog, Michigan State College.* We shall use the following lemma: Let a, b and n be positive integers with

$$a < b < 3a/2 \tag{1}$$

and let $A(n)=2[a/n]-[b/n]$. Then

$$A(n) = 0 \quad \text{for} \quad n > b, \tag{2}$$
$$A(n) = -1 \quad \text{for} \quad a < n \leqq b, \tag{3}$$
$$A(n) \geqq 0 \quad \text{for} \quad a/3 < n \leqq a, \tag{4}$$
$$A(n) \geqq 1 \quad \text{for} \quad 1 \leqq n \leqq a/3. \tag{5}$$

We put $[a/n]=u$ and $[b/n]=v$, so that $u>(a/n)-1$, $v\leqq b/n$ and $A(n)=2u-v$. If $n>b$ then $u=v=0$, proving (2). If $a<n\leqq b$ then $u=0$ and, by (1), $v=1$, proving (3). In general, $A(n)=2u-v=u/2+3u/2-v>u/2+(3a/2-b)/n-3/2>(u-3)/2$. Thus if $a/3<n\leqq a$ then $u\geqq 1$ and $A(n)>-1$, proving (4), and if $1\leqq n\leqq a/3$ then $u\geqq 3$ and $A(n)>0$, proving (5).

Let the fraction in the proposed problem be denoted by P_n. For $p_n=2$, 5 and 7 we obtain $P_n=1$, 4 and 1, respectively. We assume from now on that $p_n\geqq 11$. By Tchebysheff's Theorem,‡ $p_{n+1}<5p_n/4$ for $p_n\geqq 29$ and a simple com-

* J. Neuberg, Bull. de l'Académie royale de Belgique, 1910, July–August.

† R. Bouvaist et V. Thébault, Comptes Rendus, vol. 217, pp. 223–225, 1943.

‡ See, for instance, E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, vol. I.

putation shows that $p_{n+1}<3p_n/2$ for $p_n\geqq 11$. Let p be any prime. If $p>p_n$ then neither numerator nor denominator of P_n contains p as prime factor. If $p=p_n$ then p is contained in both numerator and denominator of P_n in exactly the first power, since $p_{n+1}-1<2p_n$. We assume from now on $p<p_n$.

We put $p_n=a$, $p_{n+1}-1=b$, so that (1) is satisfied, and write

$$P_n = \frac{(p_n!)^2}{p_n(p_{n+1}-1)!} = \frac{(a!)^2}{b!\cdot a}. \tag{6}$$

Since $p<p_n$ the numerator and denominator of (6) contain p in exactly the following powers, respectively:*

$$2\sum_{r=1}^{\infty}[a/p^r], \qquad \sum_{r=1}^{\infty}[b/p^r].$$

Thus in the notation of the lemma, stated above, we have to show that

$$\sum_{k=1}^{\infty} A(p^k) \geqq 0 \tag{7}$$

for all primes $p<p_n=a$. If no power of p satisfies the inequality $a<p^r\leqq b$ then (7) follows immediately from (2), (4) and (5).

Thus there remains the case in which one power of p, say p^m, is such that $a<p^m\leqq b$, that is, $p_n<p^m<p_{n+1}$. From (1) it is obvious that there can be no more than one such power p^m. Since, by (3), $A(p^m)=-1$ (7) will be proved in this case also if we can show that for another power of p, say p^k, $A(p^k)\geqq 1$. We define this k to be equal to $m-1$, $m-2$ or $m-3$, according to whether $p\geqq 5$, $p=3$ or $p=2$. From $p_n<p^m<p_{n+1}$ and from $p_n\geqq 11$ it is easily concluded that in any case $k\geqq 1$; also in any case $p^{m-k}\geqq 5$. Hence, $p^k=p^m/p^{m-k}\leqq b/5<3a/10<a/3$. Thus, by (5), $A(p^k)\geqq 1$, which concludes the proof.

II. *Solution by Alfred Brauer, University of North Carolina.* Instead of the problem, I prove the following sharper result:

Let x be any positive integer, and

$$N = \frac{x!}{x(x+1)\cdots([3x/2]-1)}.$$

If we reduce N to its lowest terms, then the denominator is the product of the primes t_ν for which $x<t_\nu\leqq[3x/2]-1$.

It was proved by I. Schur, *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Phys.-Math. Klasse* 1929, p. 128 that $p_{n+1}\leqq 5p_n/4$ for $p_n\geqq 29$, hence $p_{n+1}-1\leqq 5p_n/4-1\leqq[3p_n/2]-1$. It follows directly that $p_{n+1}-1\leqq[3p_n/2]-1$ for $11\leqq p_n\leqq 23$, and the problem is obvious for $p_n\leqq 7$. Therefore the problem follows from the sharper result.

* See, for instance, E. Landau, Vorlesungen über die Zahlentheorie, vol. I, p. 13, Satz 27.

For $x \leqq 8$, this result is obtained directly. Therefore we may assume that $x \geqq 9$. Let q^b be the highest power of the prime q which is less than or equal to x. If k is the exponent of the highest power in which q enters in $x!$, then $k = [x/q] + [x/q^2] + \cdots + [x/q^b]$. We have to prove that $x(x+1)\cdots([3x/2]-1)$ is not divisible by q^{k+1}. We set

$$(1) \qquad x = m_\beta q^\beta + r_\beta, \quad 0 \leqq r_\beta < q^\beta, \quad m_\beta = [x/q^\beta] \geqq 1, \qquad \beta = 1, 2, \cdots, b$$

and have to distinguish between two cases.

I. $$q^{b+1} \geqq [3x/2].$$

The $[x/2]$ consecutive integers of the denominator contain at most $[1m^\beta/2]+1$ multiples of q^β by (1). Since $m \geqq [m/2]+1$ for $m=1, 2, \cdots$, the numeration contains at least as many factors divisible by q^β as the denominator for each β. Hence these primes q are not divisors of the denominator of N in reduced form.

II. $$q^{b+1} < [3x/2].$$

It follows here as above that the numerator contains at least as many factors divisible by q^β as the denominator for $\beta=1, 2, \cdots, b$; but one factor of the denominator divisible by q^b is divisible by q^{b+1}. Therefore, if we have

$$m_\beta = [m_\beta/2] + 1 \quad \text{for} \quad \beta = 1, 2, \cdots, b$$

it could be possible that the denominator contains one factor q more than the numerator. Hence we have to prove that for each such q

$$(2) \qquad m_\beta \geqq [m_\beta/2] + 2 \quad \text{for at least one value of } \beta.$$

Since $q^b \leqq x < q^{b+1} \leqq 3x/2$, it follows that $x \geqq 2q/3 \cdot q^b$, hence $m_b \geqq [2q/3]$ by (1).

If therefore $q \geqq 5$, then $m_b \geqq 3$ and (2) holds for $\beta = b$. If $q=3$, then $b \geqq 2$ since $x \geqq 9$. We have $x \geqq 2q^2/3 \cdot q^{b-1}$, hence $m_{b-1} \geqq [2q^2/3] \geqq 6$, and (2) is satisfied for $\beta = b-1$. If finally $q=2$, then $b \geqq 3$ and $x \geqq 2q^3/3 \cdot q^{b-2}$, hence $m_{b-2} \geqq [2q^3/3] \geqq 5$, and (2) holds for $\beta = b-2$.

Solved also by Murray Barbour, Leo Moser, and W. J. Robinson.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

The Mathematical Tables Project of the National Bureau of Standards is continuing under the administrative direction of Dr. J. H. Curtiss. Members of the Mathematical Association are cordially invited to visit the Project (150 Nassau Street, New York City) and to confer with the Project Director, Dr. Arnold Lowan, concerning their computational problems.

On May 16, 1946, the Board of Examiners of the Board of Education of New York City granted licenses to the following teachers to serve as chairmen of departments in day High Schools: Irving Adler, Jack Deutsch, David Gordon, George Grossman, Peter Lopiparo, Max Peters, David Sole, William Wernick.

The following have received fellowships from the John Simon Guggenheim Memorial Foundation: Assistant Professor R. H. Bruck of the University of Wisconsin, Professor G. B. Price of the University of Kansas, Assistant Professor P. C. Rosenbloom of Brown University, and Associate Professor Henry Scheffé of Syracuse University.

Professor Emeritus L. L. Dines of Carnegie Institute of Technology has been awarded the honorary degree of doctor of laws by the University of Saskatchewan.

Assistant Professor Jesse Douglas of Brooklyn College has been elected a member of the National Academy of Sciences.

Associate Professor H. T. Engstrom of Yale University has been awarded the Distinguished Service Medal "for exceptionally meritorious services to the Government of the United States in duty of great responsibility while attached to the Division of Naval Communications."

Assistant Professor R. F. Rinehart of Case School of Applied Science has been awarded the Medal of Merit by the Navy Department and the Medal of Freedom by the United States Army for mathematical research in connection with submarine warfare.

The University of Georgia announces the following: Assistant Professor T. A. Bancroft of Iowa State College, Associate Professor G. B. Huff of Southern Methodist University, and Assistant Professor J. A. Ward of the United States Naval Academy have been appointed to associate professorships; C. A. Cope of the United States Navy, Dr. Erik Hemmingsen of the University of Pennsylvania, and Dr. R. J. Levit of the University of California have been appointed to assistant professorships.

At Rutgers University the following promotions to associate professorships are announced: Assistant Professors L. H. Bunyan, H. S. Grant, and M. S. Robertson. Dr. E. R. Ott has been appointed to an associate professorship.

The University of Nevada announces the following appointments to assistant professorships: Dr. O. G. Owens of the University of California and Dr. J. V. Lewis of Aberdeen Proving Ground.

Associate Professor C. B. Allendoerfer of Haverford College has been promoted to a professorship.

Dr. B. H. Arnold of Purdue University has been appointed to an assistant professorship at Montana State College.

Associate Professor M. T. Bird of Utah State Agricultural College has been appointed to an assistant professorship at Allegheny College.

Dr. T. A. Botts has been appointed to an assistant professorship at the University of Delaware.

Assistant Professor Harold Chatland of Montana State University has been promoted to an associate professorship.

Dr. L. E. Cunningham has been appointed assistant professor of astronomy at the University of California.

Associate Professor J. L. Dorroh of Louisiana State University has been appointed to an associate professorship at Illinois Institute of Technology.

Dr. C. H. Dowker has been appointed to an assistant professorship at Tufts College.

D. H. Erkiletian, Jr. of the University of Missouri, School of Mines and Metallurgy, has been promoted to an assistant professorship.

Dr. A. B. Farnell of the United States Military Academy has been appointed to an assistant professorship at the University of Colorado.

Dr. J. A. Fleming, Director of the Department of Terrestrial Magnetism of the Carnegie Institution of Washington, has retired. He is succeeded by Dr. M. A. Tuve, Chief Physicist.

Dr. J. H. Giese has been appointed mathematician at the Ballistic Research Laboratory at Aberdeen Proving Ground.

E. L. Godfrey has been appointed to an assistant professorship at Defiance College, Defiance, Ohio.

Edison Greer of the University of Kansas has been appointed to an associate professorship at Kansas State College.

Assistant Professor H. J. Hamilton of Pomona College has been promoted to an associate professorship.

Dr. M. H. Heins of Harvard University has been appointed to an associate professorship at Brown University.

Dr. J. F. Heyda of the University of Nebraska has been appointed to an assistant professorship at Franklin and Marshall College.

Assistant Professor Witold Hurewicz of the University of North Carolina has been appointed to an associate professorship at the Massachusetts Institute of Technology.

Dr. G. K. Kalisch of Cornell University has been appointed to an assistant professorship at the University of Minnesota.

Associate Professor E. C. Kennedy of Texas College of Arts and Industries has accepted the position of Research Engineer with the Consolidated Vultee Aircraft Corporation, Daingerfield, Texas.

Dr. D. M. Krabill of the United States Naval Academy has been appointed to an associate professorship at Bowling Green State University, Bowling Green, Ohio.

Dr. J. A. Larrivee has been appointed to an assistant professorship at the University of Vermont.

Mary Ann Lee has been appointed to an assistant professorship at Sweet Briar College.

Dr. Joseph Lehner has been appointed mathematician with Hydrocarbon Research, Inc., New York.

Dr. J. F. Locke of the United States Naval Academy has been appointed to a professorship at Birmingham-Southern College.

Assistant Professor W. L. Massey of the University of Chattanooga has been promoted to an associate professorship.

Professor Karl Menger of Notre Dame University has been appointed to a professorship at Illinois Institute of Technology.

Dr. D. D. Miller has been appointed to an associate professorship at the University of Tennessee.

Dr. W. M. Miller has been appointed professor of mathematics and head of the department at Robert College, Istanbul, Turkey.

Associate Professor Gordon Pall of McGill University has been appointed to a professorship at Illinois Institute of Technology.

Margaret Ramsey of Linfield College, McMinnville, Oregon, has been promoted to an associate professorship.

Dr. L. A. Ringenberg of Ohio State University has been appointed to an assistant professorship at the University of Maryland.

Assistant Professor Arthur Rosenthal of the University of New Mexico has been promoted to an associate professorship.

Dr. Max Shiffman of Princeton University has been appointed to an associate professorship at New York University.

E. L. Stanley of Clemson College, South Carolina, has been promoted to an assistant professorship.

Professor M. H. Stone of Harvard University has been appointed to a pro-

fessorship at the University of Chicago, and to the chairmanship of the department of mathematics.

C. S. Sutton of the University of Delaware has been appointed to an assistant professorship at The Citadel, Charleston, South Carolina.

C. W. Topp has returned to Fenn College, Cleveland, as an assistant professor.

Assistant Professor H. L. Turrittin of the University of Minnesota has been promoted to an associate professorship in the department of mathematics and mechanics.

Dr. L. F. Walton has been appointed lecturer at Santa Barbara College of the University of California.

Associate Professor Hassler Whitney of Harvard University has been promoted to a professorship.

The following appointments to instructorships are announced:
The College of the City of New York: Dr. Lee Lorch
Dartmouth College: Dr. William H. Durfee, Dr. W. C. G. Fraser
Harvard University: Dr. Lowell Schoenfeld
Hofstra College: E. Marie Hove
Illinois Institute of Technology: R. R. Bentley, Dr. E. G. H. Comfort, Dr. Martinus Esser, Dr. Anatol Rapoport, Mary C. Rapp, M. M. Resnikoff, Beulah I. Shoesmith, T. T. Tanimoto
Montana State University: Walter Hook, Dr. Andrewa Noble
Temple University: Irma Moses
University of Georgia: Eugene Park
University of Illinois: Dr. E. J. Scott
University of Washington: Dr. Fumio Yagi
Yale University: P. T. Bateman

Professor Emeritus A. R. Crathorne of the University of Illinois died March 7, 1946. He was a charter member of the Association.

Associate Professor A. E. Staniland of the University of Pittsburgh died June 28, 1946.

Professor Emeritus E. E. Whitford of the College of the City of New York died May 3, 1946. He was a charter member of the Association.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE STANFORD UNIVERSITY MATHEMATICS EXAMINATION

DEPARTMENT OF MATHEMATICS, Stanford University

In the spring of 1945 the following ideas were discussed at Stanford University.

Problem. Every year several thousand boys and girls graduate from the high schools of California. Upon the basis of experience we can assume that among these students are some who are capable of outstanding achievements in college. It is less obvious to expect, but there is no reason *not* to expect, that at least in the course of several years some singularly capable students graduate from high school. Stanford can offer them superior training in mathematics and allied fields. And likewise the University will profit by having them in attendance. The problem is how to find such students and how to attract them to Stanford.

The need for a search of this kind is immediately clear. In the future, training in every field and for every purpose is going to be more and more complex. This in itself calls for higher achievement. But high achievement alone is hardly sufficient if we think in terms of future leaders in the fields of research, production, and administration. For such tasks creative ability is needed. We need students who possess this quality not only for the study of mathematics, but also for the study of the exact sciences and various branches of engineering. Although exceptional mathematical ability is not an absolute indication of future achievement in these other fields, it strongly points to high qualification.

Contest in Mathematics. We know from experience that mathematical ability can be tested and discovered with rather safe conclusiveness at a comparatively early age. The reason for this is that such ability is indicated not so much by the amount of accumulated knowledge as by the originality of mind displayed in the game of grappling with difficult though elementary problems. Frequently mathematical ability is shown by the reaction of a young person to simple questions of euclidean geometry. Also, as matters now stand in the high schools, mathematics is one of the few subjects taught over a sufficiently long period of time to furnish a reliable testing subject for traits of an analytical mind and originality of approach, factors so essential to the advancement of science. Thus a contest in mathematics would be in the interest of all fields of science.

Plan. As a first step, the Department of Mathematics should contact several high school principals and teachers of mathematics in regard to the proposed contest in mathematics. Various persons on the Stanford campus should also be consulted. It seemed especially desirable to interest the large high schools in

the cities which are the natural reservoirs of future university students. The existing contact of the Department with its previous students holding teaching positions in high schools might be valuable. The support of alumni whose interest in the matter might be anticipated could also be mobilized.

By the end of August, letters announcing the contest should be sent out to the principals of the large high schools in the state. We would ask them to inform both the teachers of mathematics and the members of the graduating class of the next year about the contest. The principal would be asked to notify the Department by December 31 whether his school wished to participate. The contest would also be announced in the publications of the School Board.

The contest itself would take place some time in the spring, preferably not later than May 1. Any member of the graduating classes would be admitted. For each participating school the questions would be sent in a sealed envelope to a committee consisting of the principal, the teachers of mathematics named by the principal, and a representative of Stanford (a local alumnus) who is not a teacher of mathematics. At the preassigned day, which will be published by the principal of the school well ahead of time, the contest will take place in a school room designated by the principal. The contest will be held preferably on a Saturday afternoon and for three hours. The committee would then seal the papers in an envelope, and return them without delay to the Department of Mathematics at Stanford.

The examination would be composed of three problems. It is very essential that these problems should be elementary, not requiring any knowledge higher than that covered in the usual secondary courses in algebra, plane geometry, trigonometry and solid geometry. On the other hand each problem should be sufficiently difficult that it requires definite originality in seeing the essential point in the question and finding the right way to the solution.

Incentive. A fundamental question existed in regard to the proper incentive for such a contest. We could give publicity to the result of the contest. But since it was our purpose to attract the winner to Stanford, it would be more desirable to offer him a scholarship for a year, covering at least his tuition. This would call for setting aside a scholarship of about $500 for this purpose. It is conceivable that some year no contestant would display the degree of ability which we expect. In this case no prize should be awarded; this point of view is based on the consideration that not necessarily the best among the contestants should be the winner but rather the best among those who achieve a certain level. On the other hand it may happen some years that two contestants appear to be eligible for the prize.

The award of a scholarship is frequently connected with certain stipulations such as need or general scholastic and character qualifications. It would be difficult to combine such requirements with the winning of the contest. Due to the winner's exceptional ability in one field, a certain allowance must be made for his possible shortcomings in other fields. One-sided ability is probably not desirable from the point of view of general education. But it is almost a necessity for later leadership.

The student would of course have to fulfil the general admission requirements of Stanford University.

Conclusion. We believed that such a contest would be comparatively inexpensive, and could be carried out with little clerical help. It seemed desirable to try out the plan first on a small scale, and in case of a satisfactory response the area could be expanded to include other western states.

The Stanford University Mathematics Examination was given simultaneously at 60 California high schools on Saturday, April 6, 1946, and was taken by 322 students. The following problems were proposed:

Problem (1) In a tennis tournament there are $2n$ participants. In the first round of the tournament each participant plays just once, so there are n games, each occupying a pair of players. Show that the pairing for the first round can be arranged in exactly

$$1 \times 3 \times 5 \times 7 \times 9 \cdots \times (2n - 1)$$

different ways.

Problem (2) In a tetrahedron (which is not necessarily regular) two opposite edges have the same length a and they are perpendicular to each other. Moreover they are each perpendicular to a line of length b which joins their midpoints. Express the volume of the tetrahedron in terms of a and b, and prove your answer.

Problem (3) Consider the following four propositions, which are not necessarily true.

I. If a polygon inscribed in a circle is equilateral it is also equiangular.

II. If a polygon inscribed in a circle is equiangular it is also equilateral.

III. If a polygon circumscribed about a circle is equilateral it is also equiangular.

IV. If a polygon circumscribed about a circle is equiangular it is also equilateral.

Part A. State which of the four propositions are true and which are false, giving a proof of your statement in each case.

Part B. If, instead of general polygons, we should consider only quadrilaterals which of the four propositions are true and which are false? And if we consider only pentagons? In answering Part B you may state conjectures, but prove as much as you can and separate clearly what is proved and what is not.

Members of the Department of Mathematics of Stanford University examined the papers. In their opinion, the best paper submitted was that of Alan K. Jennings, Alhambra City High School, Alhambra, California. Mr. Jennings was awarded a one-year scholarship of $500 by Stanford University. Three other participants were found worthy of honorable mention; their names in alphabetical order are:

Dan Alfred Cowan, Santa Monica High School, Santa Monica, California.
Boyd James Preble, Madera Union High School, Madera, California.
Jack L. Shehi, Tamalpais High School, Mill Valley, California.

Each of these three persons received a copy of the book *How to Solve It* by Professor G. Polya, Stanford University.

The problems proposed were intended to be more difficult than those usually treated in California high schools in order to test effectively the participants' mathematical ability and understanding. Since presumably this was the first examination of its kind in California, it was not expected that any participant would be able to solve all three problems completely in the allotted time of three hours; and it turned out as was expected. Nevertheless the performance of some of the participants, and especially that of the four receiving awards, must be regarded as creditable. Our hope is that this examination will induce those high schools which have superior students in mathematics not only to intensify the mathematical preparation of prospective college students but also to encourage their creative activity in solving problems.

Stanford University intends to repeat this examination in mathematics each year under similar conditions. We believe that it would be in the public interest to give more and earlier attention to outstanding mathematical and scientific talent, and our examination was designed to contribute to the effort that the high schools may undertake in this direction.

NEW SELECTIVE SERVICE POLICIES*

Following the passage of an act extending Selective Service until April 1, 1947, a moratorium was declared on inductions until September 1, 1947, to see if raising the pay of military personnel might result in sufficient voluntary enlistments to meet military needs. In view of the fact that the number of voluntary enlistments appears to be short of requirements, a serious situation involving scientific personnel is developing. The following three paragraphs are excerpts from Bulletin No. 33 of the Office of Scientific Personnel, July 16, 1946.

"On July 11 Colonel G. A. Irvin of National Selective Service Headquarters spoke to a Conference on Emergency Problems in Education called by the American Council on Education for July 11–13. He stated that voluntary enlistments will probably fail by 225,000 to reach the number established by Congress for the size of the Army by July, 1947. This appears to follow a statement by a War Department spokesman, based on an analysis of June enlistments. Quotas for September have been set at 25,000.

"Colonel Irvin predicted that this situation would result in the most stringent deferment policy yet experienced. Deferments would be extended only in the rarest cases. It was his opinion that the induction of college faculty personnel and even research personnel is strongly indicated. This follows from the figures

* Selective Service Memorandum No. 115 (amended), released on August 23, met many problems of this article by providing a basis for the deferment of college teachers and scientific personnel.

quoted by Col. G. T. Garnett, of Selective Service, at the Round-Table discussion on Friday, July 12, 1946. He stated that only 92,000 draft eligible non-fathers remain in the age group 19–34. Of these, 60,000 are in the Merchant Marine. If these are to be regarded as veterans, and not draft vulnerable then only 32,000 remain. Practically all of these are scientific or technical personnel.

"The recent Magnuson amendment, which would exempt scientific and technical students and other personnel from Selective Service, was referred to by the speaker in response to a question. He stated that the elimination of the amendment by the Conference Committee has been considered by some attorneys in the Selective Service System as indicating the intent of the conferees not to encourage deferment of this group. It was called to his attention that the Conference report specifically states that the reason for the elimination of the amendment was that adequate authority already exists in the Act for deferments in this category. The fact that the Congress saw fit to exempt specifically such groups as agricultural workers, where similarly adequate authority for deferment might have been deemed to exist in the Act, will undoubtedly be used to show that Congress did elect certain categories for exemption. Their refusal to include scientific and technical personnel may be deemed a specific act of Congress in the light of the existence of specific excluded categories. There seems little doubt that if the Selective Service spokesman's prediction with regard to enlistment is true, something approaching a real crisis confronts scientific personnel."

In preparation for resuming the draft in September, President Truman directed on July 16 that only men aged 19 through 29 be called up. Acting immediately after the issuance of this directive, General L. B. Hershey sent the following telegram to State Selective Service directors.

"The War Department has made requisition on the Selective Service System for the procurement of 25,000 registrants during September, 1946. Call on your State will be forwarded to you in the near future. In accordance with the War Department's request this call will be filled with registrants 19 through 29 years of age.

"In order to provide the number of men required to fulfill the anticipated requests on the Selective Service System and to comply with the provisions of Public Law 473, 79th Congress, extending the Selective Training and Service Act of 1940, as amended, to March 31, 1947, regulations are being revised to (1) provide for the classification, examination and forwarding for induction of registrants 19 through 29 years of age; (2) limit occupational deferment to those few registrants who are determined by the local boards to be indispensable and irreplaceable to the national existence, except that registrants in agriculture will continue to be considered for deferment under the provisions of the Tydings Amendment; (3) provide for the postponement of induction of only those registrants in high school specifically authorized by Section 5(F) of the Selective Training and Service Act of 1940, as amended; (4) provide for the elimination of postponement of induction during quarter or semester for under-graduate

college students; (5) provide for the review of the classification of 26 through 29-year-old registrants previously found unfit for general military service; and (6) reopen and reconsider the classification of all registrants in Class 1-C who have not served on active duty in the land or naval forces of the United States outside the Continental limits of United States or Alaska or who have not served on active duty for at least six months after September 16, 1940, exclusive of time served while pursuing a course of instruction in a university, college or other similar institution of learning."

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

The following thirty-nine persons have been elected to membership on applications duly certified:

N. N. ARASHIRO, B.S. in M.E.(North Dakota) Stress Analyst, A. O. Smith Corp., Milwaukee, Wis.

WINIFRED A. ASPREY, Ph.D.(Iowa) Instr., Vassar Coll., Poughkeepsie, N. Y.

R. G. AYOUB, M.Sc.(McGill Univ.) 6848 Drolet St., Montreal, P. Q., Canada

JOSHUA BARLAZ, Ph.D.(Cincinnati) Instr., Ohio State Univ., Columbus, Ohio

F. MARION CLARKE, A.M.(Smith Coll.) Visiting Instr., Univ. of Southern California, Los Angeles, Calif.

J. L. CONNORS, Pvt., U. S. Army. C-4-3, A.R.T.C. Sect. 2, Ft. Knox, Ky.

F. E. COTHRAN, B.S.(California) 709 S. Norton Ave., Los Angeles 5, Calif.

L. E. CUNNINGHAM, Ph.D.(Harvard) Univ. of California, Berkeley, Calif.

M. S. DAVIS, A.B.(Brooklyn) Grad. Asst., Univ. of Missouri, Columbia, Mo.

G. C. FRASER, A.M.(Pennsylvania) Teacher, George School, Bucks Co., Pa.

W. W. GANDY, M.S.(A. and M. Coll. of Texas) Instr., School of Mines and Met., Rolla, Mo.

A. A. GRAU, Ph.D.(Michigan) Instr., Drake Univ., Des Moines, Iowa

M. N. HALLER, B.S. in E.E.(Louisville) Equipment Engr., Western Union, Montgomery, W. Va.

MORRIS HERTZIG, M.S.(C.C.N.Y.) Chm. of Dept., High School, Forest Hills, N. Y.

J. J. L. HINRICHSEN, Ph.D.(Harvard) Iowa State Coll., Ames, Iowa

RUFUS ISAACS, Ph.D.(Columbia) Asst. Prof., Univ. of Notre Dame, Notre Dame, Ind.

W. J. JAFFE, A.M., M.S. in I.E.(Columbia) Instr., Newark Coll. of Engineering, Newark, N. J.

D. A. JOHNSON, A.M.(Minnesota) Instr., Coll. of Educ., Univ. of Minnesota, Minneapolis, Minn.

J. M. KINGSTON, Ph.D.(Toronto) Asst. Prof., Univ. of Washington, Seattle, Wash.

JOACHIM LAMBEK, B.S.(McGill Univ.) 403 Sherbrooke St. W., Montreal, P. Q., Canada

H. R. LAWRENCE, A.M.(Michigan) Engr., Aerodynamics, Northrop Aircraft Co., Los Angeles, Calif.

A. O. LINDSTRUM, JR., Ph.D.(Illinois) Instr., Univ. of Wisconsin, Madison, Wis.

S. W. MCCUSKEY, Ph.D.(Harvard) Prof., Chm. of Dept., Case School of Appl. Sci., Cleveland, Ohio

JOSEPHINE M. MITCHELL, Ph.D.(Bryn Mawr) Asso. Prof., Winthrop Coll., Rock Hill, S.C.

MARTIN MOLIVER, Ed.M.(Temple Univ.) Teacher, Public School, Philadelphia, Pa.

R. V. MUGUERCIA, B.S.L.(Inst. de Stgo. de Cuba) Teacher, Academia XY, Colegio Moncada, Santiago de Cuba

G. M. MULLER. Student, Bowdoin Coll., Brunswick, Me.

HELEN K. NICKERSON (Mrs. W. J.), A.M. (Radcliffe) Instr., Math. and Physics, Wheaton Coll., Norton, Mass.

K. L. NOBLE, A.M.(Colo. St. Coll. of Educ.) Asst. Prof., Univ. of Denver, Denver, Colo.

L. R. NORWOOD, A.B.(Stanford) Instr., Yale Univ., New Haven, Conn.

P. A. PIZÁ. P.O. Box 627, San Juan, Puerto Rico

ELLEN F. RASOR, A.M.(Duke) Asst. Prof., Winthrop Coll., Rock Hill, S.C.

JOHN SALERNO, A.B.(Brooklyn) Jr. Mathematician, U. S. Coast and Geodetic Survey. 530 Lincoln Ave., Brooklyn 8, N. Y.

W. P. SHARP, JR., A.M.(Syracuse) Instr., Newark Coll. of Engineering, Newark, N. J.

D. R. SUDBOROUGH, M.S.(Michigan) Prof., Detroit Inst. of Tech., Detroit, Mich.

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W. B. CARVER, *Secretary-Treasurer*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-sixth regular meeting of the Southern California Section of the Mathematical Association of America was held at the California Institute of Technology, Pasadena, California, on Saturday, March 9, 1946. Professor R. P. Dilworth, Chairman of the Section, presided at the morning and afternoon sessions.

The attendance was seventy-seven, including the following forty-eight members of the Association: L. J. Adams, O. W. Albert, E. F. Beckenbach, Clifford Bell, L. T. Black, W. D. Cairns, Frances L. Campbell, L. M. Coffin, D. R. Curtiss, L. C. Damsgard, P. H. Daus, R. P. Dilworth, J. D. Donaldson, D. C. Duncan, W. H. Glenn, J. W. Green, H. J. Hamilton, P. C. Hammer, P. G. Hoel, D. H. Hyers, C. G. Jaeger, G. R. Kaelin, L. C. Lay, Margaret B. Lehman, Ada A. McClellan, G. F. McEwen, A. D. Michal, P. M. Niersbach, W. B. Orange, W. T. Puckett, Jr., H. R. Pyle, L. T. Ratner, E. C. Rex, A. P. Rhodes, J. M. Robb, G. E. F. Sherwood, R. H. Sorgenfrey, D. V. Steed, A. E. Taylor, V. C. Throckmorton, S. E. Urner, F. A. Valentine, H. C. Van Buskirk, Morgan Ward, L. E. Wear, Mabel C. Whiting, B. R. Wicker, M. A. Zorn.

The following officers were elected for the next year: Chairman, H. J. Hamilton, Pomona College; Vice-Chairman, D. V. Steed, University of Southern California; Program Committee, A. E. Taylor, H. R. Pyle, Frances L. Campbell; Secretary, P. H. Daus.

A memorial resolution honoring the late Professor Bateman was read by Professor A. D. Michal.

The following papers were presented:

1. *Geographic maps and their scales*, by Professor D. R. Curtiss, Emeritus, Northwestern University.

Professor Cairns considered transformations from sphere to plane which characterize some of the better known geographic maps. Scale was defined as the

value of the directional derivative $d\sigma/ds$, where $d\sigma$ denotes the element of arc on the sphere, and ds the corresponding element of arc in the plane. The scale function is thus a function of both point and direction; if it depends on the point only, the map is said to be conformal. The paper included an elementary proof that no geographic map, even though conformal, can have a scale function constant throughout any region. Scale functions for various maps were discussed as a means of studying distortion.

2. *Moments of inertia of fluid filled tanks*, by Dr. Edmund Penney, Oregon State College, introduced by Professor R. P. Dilworth.

In the theory of airplane wing flutter, account must be taken of the moment of inertia of the fuel tanks about the torsional axis of the wing. From the theory of incompressible, non-viscous fluid flow, expressions for the moment of inertia are derived in the two cases corresponding to partially and completely filled tanks. The former case, which depends on the frequency of oscillation, is specialized to the high frequencies which are of interest in flutter theory.

3. *Symposium on mathematics for the veteran*, led by Professor C. G. Jaeger, Pomona College.

Participating in this discussion were Professor D. V. Steed, University of Southern California, Professor W. B. Orange, Los Angeles City College, and Professor Clifford Bell, University of California at Los Angeles. Professor Jaeger set the stage by citing some of the problems involved. Other speakers presented data and experience from their own institutions. All agreed that there was no unique solution, and that more effective counselling and a little encouragement at the right time were badly needed.

4. *Some properties of the breadth of curves*, by Professor J. W. Green, University of California at Los Angeles.

Professor Green proved that if a plane convex body has a diameter not greater than 2, then its area is not greater than π. This was done by the method of symmetrization and comparison with a circle. It was shown that if the equal signs hold, the body is a circle.

5. *An illustration of the correspondence between integral equations and the algebra of an infinite number of variables*, by Professor W. D. Cairns, Emeritus, Oberlin College.

6. *Intrinsic theory of lambda matrices*, by Professor Ernst Snapper, University of Southern California, introduced by Professor D. H. Hyers.

Let $A=(\alpha_{ij})$, $i=1, \cdots, m$ and $j=1, \cdots, n$, be an $m\times n$ matrix whose elements $\alpha_{ij}\in F[\lambda]$ are polynomials in one variable λ with coefficients in a field F. The classical theory of these matrices is based on the notion of the elementary divisors $\epsilon_r=\Delta_r/\Delta_{r-1}$, $\epsilon_1=\Delta_1$, where r is the rank of A, and Δ_i the highest common factor of the $i\times i$ sub-determinants of A. Several applications of this theory exist, for instance the discussion of a system S of ordinary, homogeneous, linear differential equations with constant coefficients in the case that F is the field of complex numbers.

Against this classical theory, two objections can be raised. In the first place, it is well known that if two matrices of the above type have the same column spaces, they have the same elementary divisors. Although this fact shows that the elementary divisors are completely determined by the column space of A, the classical theory fails to give the intrinsic meaning of the elementary divisors for the column space. Secondly, the way in which the classical theory is used to discuss the above system S of differential equations gives the false impression that all the elementary divisors $\epsilon_r, \cdots, \epsilon_1$ are important for S, while a proper study of S reveals that only ϵ_r and Δ_r matter.

It is possible to develop a theory of the matrix A against which these objections cannot be raised. In this theory the elementary divisors are defined intrinsically in terms of the column space, while sub-determinants do not occur. For example, in this intrinsic theory of λ-matrices, ϵ_r is defined as $M:Cl(M)$ where M is the column space of A, $Cl(M)$ is the closure of the column space, and $M:Cl(M)$ the quotient of M by $Cl(M)$. These same intrinsic column space methods can be used to develop the theory of the system S of differential equations properly, *i.e.*, by using only ϵ_r and Δ_r.

This intrinsic theory of λ-matrices contains the ideas on which the theory of matrices whose elements are polynomials in several variables is based. The above theories will be published in two papers, called *Polynomial Matrices in One Variable, Differential Equations and Module Theory*, and *Polynomial Matrices in Several Variables.*

7. *An application of integral equations to the problem of torsional divergence and aileron reversal*, by H. R. Lawrence, Northrop Aircraft, Inc., introduced by Professor Clifford Bell.

The condition of equilibrium between the elastic and aero-dynamic forces acting on a wing section is set up in integral equation form. The aerodynamic forces are taken to be the solution of the integral equation of the Prandt lifting line theory. The first eigenvalue of the above equation is shown to be the dynamic pressure at torsional divergence. Upon imposing the restriction of zero rolling acceleration, the first eigenvalue becomes the dynamic pressure at aileron reversal. A method of obtaining torsional divergence speed, aileron reversal speed, and aileron effectiveness is indicated. The utility of this procedure is discussed with the aid of a numerical example.

P. H. DAUS, *Secretary*

MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the annual convention of the Oklahoma Education Association in Oklahoma City on Friday morning, February 15, 1946. Professor Dora McFarland, Chairman of the Section, presided.

Seventy-one persons attended the meeting, including the following eleven members of the Association: J. C. Brixey, N. A. Court, O. H. Hamilton, E. E.

Heimann, J. E. LaFon, Dora McFarland, G. E. Meador, W. T. Short, D. R. Shreve, C. E. Springer, J. H. Zant.

At the business session the following officers were elected: Chairman, O. H. Hamilton, Oklahoma A. and M. College; Vice-Chairman, G. E. Meador, Oklahoma City University; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following five papers:

1. *Discussion of the sign test of significance*, by Professor W. J. Dixon, University of Oklahoma, introduced by Professor J. C. Brixey.

The definition and uses of the sign test were discussed. This test compares the medians of a two variate sample by using only the signs of the differences of the paired variates. Departure from equality of numbers of plus and minus signs which are as great or greater than departures which would occur p per cent of the time by chance, when equal numbers are expected, were given as a table of significance values for examples of sizes 1 to 100 and $p=1, 5, 10, 25$.

2. *Some theorems on homotopic transformations*, by Professor O. H. Hamilton, Oklahoma A. and M. College.

This paper dealt with the definitions of continuous transformations, homeomorphisms, and homotopic transformations with emphasis upon the relations between the following types of theorems: (1) The Brouwer fixed point theorem for an n-cell; (2) Theorems concerning fixed point free transformations of an n-sphere into itself; (3) Theorems concerning transformations of an $(n+1)$ cell into an n-sphere; (4) Separation theorems in an n-dimensional euclidean space.

3. *Volume Coördinates*, by Professor C. E. Springer, University of Oklahoma.

Tensor notation was used to obtain the transformation from rectangular cartesian coördinates to volume coördinates. After developing some analytical formulas in volume coördinates, certain theorems concerning the geometry of a tetrahedron which can be arrived at conveniently by the use of volume coördinates were announed. One of these results was described as follows. A pencil of cones is determined by the four lines joining the centroid to the vertices of the tetrahedron T. The four tangent planes to any member of the pencil of cones at the four vertices of T intersect the opposite faces of T in four skew lines. These four lines belong to one regulus of a quadric surface. The other four skew lines in which the faces of T intersect the quadric belong to the second regulus of the quadric surface.

4. *Simplified use of half-angles in teaching*, by Professor A. Bernhart, University of Oklahoma, introduced by the Secretary.

Easy, rapid methods for simplifying the cartesian equation of the general conic, and for obtaining the slopes of the bisectors of the angles between two lines, were demonstrated.

5. *Zeno's Arrows*, by Professor N. A. Court, University of Oklahoma.

A historical and logical discussion of Zeno's paradoxes and certain related paradoxes was presented.

J. C. BRIXEY, *Secretary*

MEETING OF THE NORTHERN CALIFORNIA SECTION

The eighth annual meeting of the Northern California Section was held at the University of California in Berkeley, on Saturday, January 26, 1946. The Chairman of the Section, Professor Pauline Sperry, presided at both morning and afternoon sessions.

The attendance was seventy-five, including the following thirty members of the Association: H. M. Bacon, G. A. Baker, T. J. Bass, Jr., B. A. Bernstein, F. A. Butter, Jr., Louise Chin, G. C. Evans, E. L. Fitzgerald, S.J., S. A. Francis, M. A. Heaslet, Emma V. Hesse, R. H. Hoskins, Free Jamison, L. C. Leithold, Sophia L. McDonald, D. E. Marrs, E. D. Miller, W. H. Myers, C. D. Olds, George Polya, Edris P. Rahn, R. M. Robinson, E. B. Roessler, Kathryn B. Rolfe, Ethel Spearman, Pauline Sperry, R. K. Wakerling, Harriet A. Welch, A. R. Williams, František Wolf.

Professor Bacon, having served as Secretary-Treasurer of the Section since its formation in 1939, asked to be relieved of the duties of office. The following officers were then elected for the coming year: Chairman, Professor W. H. Myers, San Jose State College; Vice-Chairman, Professor George Polya, Stanford University; Secretary-Treasurer, Professor E. B. Roessler, University of California at Davis; Representative on the *California Journal of Secondary Education*, Mrs. Ruth G. Sumner, Oakland High School.

By invitation of the Section, Professor George Polya of Stanford University gave an hour's address during the morning session.

The following papers were read:

1. *Embedding a Desarguesian plane in projective 3-space*, by Dr. A. Seidenberg, University of California, introduced by the Chairman.

Dr. Seidenberg remarked that the classical theorem establishing a scale on a projective line depends essentially only on the theorem of Desargues. An immediate consequence is the possibility of embedding a Desarguesian plane in a projective 3-space. This result is obtained in a more direct and intuitive fashion from the observation that the set of collineations of a Desarguesian plane having a given line of fixed points is projective isomorphic to a 3-space minus two planes.

2. *Capacity and space curves*, by Professor G. C. Evans, University of California.

Professor Evans showed how, in spite of its being a tautology, a necessary and sufficient condition possesses the novelty of a discovery. He applied Wiener's necessary and sufficient condition for a regular boundary point, in the Dirichlet problem, to the determination of the values on the bounding curve of the potential function which corresponds to the cap of the curve which has minimum capacity.

3. *Some observations on the mathematical preparation of college students*, by Professor Sophia Levy McDonald, University of California.

Professor McDonald described examinations given at the University of Cali-

fornia to students wishing to qualify for registration in the first course in analytic geometry. The examination serves two ends; not only does it free the analytic geometry classes of their weakest students, but it guides those same students to a course in algebra set up just for them, and for which they are given credit. Replies to a questionnaire given to students indicated that a number of students who have prerequisites for university courses on paper, do not have the actual prerequisites. It was noted that the teaching of mathematics in the schools is good, but that often there is not enough of it, for one reason or another. Professor McDonald referred to a recent report of the Subcommittee on Mathematics of the California Committee for the Study of Education. This report, it was noted, did not approach the study of mathematics from the social studies point of view, and it was not accepted by the California Committee. Considerable discussion followed this paper.

4. *Mathematical reasoning and the teacher*, by Professor George Polya, Stanford University, an address presented on invitation of the Section.

Professor Polya presented a diagrammatic representation of the progress of the solution of an elementary mathematical problem. This representation involves two series of diagrams. The first series shows how the conception of the geometric figure considered in the problem changes as the work progresses; new parts are added at each step or the parts appear in a new combination and differently emphasized. The second series of diagrams symbolizes the logical evolution; newly introduced elements are symbolized as newly plotted points, the logical connections as connecting lines. (See the speaker's paper in *Acta Psychologica*, vol. 4, 1938, pp. 113–170.) This diagrammatic representation provides a sort of "slow motion picture" of the course of the solution which may be useful to the teacher who wishes to visualize the purpose and the motives of each step and the connection of the successive steps. The representation may also help one to understand the role of certain "standard" questions and suggestions, useful to the teacher who wishes to develop the independent work of his students. (The speaker attempted to state generally and list systematically such questions and suggestions in his recent booklet, *How to Solve It*, Princeton University Press, 1945.)

5. *The application of symbolic logic to the teaching of calculus*, by Professor František Wolf, University of California.

Professor Wolf noted that mathematics is built on logic. In mathematics classes more attention should be paid to the logical background involved in the proofs. The easiest and the most natural way to do it is to use symbols of formal logic: of equivalence, of implication, of conjunction and disjunction. At the root should be a good discussion of a proposition and a propositional function. Using quantifiers we can perform all the logical reasonings involved in the calculus in a formal way. It offers the student a graphic picture, it gives the logical reasoning a new concrete reality. It can prevent the negative attitude of the majority of students towards the principles of a mathematical proof.

6. *A general method of variation in conformal mapping*, by Professor A. C. Schaeffer, Stanford University, introduced by the Secretary.

Professor Schaeffer discussed a general variational method in conformal mapping. He outlined a method of obtaining the first variation when a mapping function is perturbed slightly.

7. *An asymmetric diophantine inequality*, by Professor C. D. Olds, San Jose State College.

Professor Olds gave an arithmetical proof of a recent theorem, due to B. Segre, on the rational approximation to irrational numbers.

8. *On sets of distances of n points*, by Dr. Paul Erdös, Guggenheim Fellow, Stanford University, introduced by the Secretary.

In the unavoidable absence of Dr. Erdös, his paper was summarized by Professor Polya as follows: Let there be given n points in the plane. Consider all the distances between them. Denote by $f(n)$ the minimum number of different distances. The author proves that

$$\sqrt{n-1} - 1 < f(n) < cn(\log n)^{-1/2}.$$

At present he is unable to obtain an asymptotic formula for $f(n)$. If the n points form a convex polygon, he conjectures that $f(n) = [n/2]$. Several related problems and conjectures were also mentioned.

H. M. BACON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirtieth Annual Meeting, Swarthmore, Pa., December 26–27, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., October, 1946
ILLINOIS, Peoria, May 9–10, 1947
INDIANA, Terre Haute, October 18, 1946
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., December 7, 1946
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Claremont, March 8, 1947
SOUTHWESTERN
TEXAS
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OCTOBER 1946

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THE TEACHING OF COLLEGE MATHEMATICS*

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1. Introduction. The subject I wish to discuss this afternoon is one with which we are all vitally concerned. Indeed now that I think about it, I realize that when I selected it I was guided by the demon of rashness rather than by the angel of caution. For in a roomful of teachers there are likely to be as many opinions as there are heads, and these opinions are not readily relinquished. Any proposals which I may make towards a change in our methods will appear to some of you tainted with the restless radicalism that is everywhere in the world around us; and the tolerant liberals amongst you will say with that kindly cynicism that is so terrible: The more you change it the more it remains ever the same. Nevertheless the matter seems to me of such fundamental importance for the welfare of our country that I shall proceed with my declared intention. If my remarks seem addressed to the young and beginning teachers among you, it is because I feel that in your hands lies the mathematical future of our students. To the older and more experienced teachers among you I simply remark that I too have the advantage of experience; for I am finishing this year my thirtieth year of teaching college mathematics.

The subject of my talk has two main aspects: What should we teach and how should we teach it? Both of these are important, but I am firmly convinced that the importance of the second is overwhelmingly greater than the importance of the first. I shall, then, say just a few words about what we should teach and shall devote myself mainly to the question of how our teaching should be done. You probably feel that the content of the Freshman course is largely determined by the high-school preparation of the students who come to college. In some colleges the first year is devoted to algebra and trigonometry, and in others to trigonometry and analytic geometry.

At Johns Hopkins, where I teach, we have two classes of entering students: those who enter with trigonometry and those who have not had trigonometry. Students of the first category devote the first sixteen-week term to analytic geometry and the second sixteen-week term to differential calculus, the classes meeting four hours weekly. Students who have not had trigonometry meet five hours weekly and cover trigonometry, analytic geometry, and differential calculus in the two sixteen-week terms of the Freshman year. The Sophomore year, consisting of two sixteen-week terms with classes meeting four hours weekly, is devoted to integral calculus and advanced parts of differential calculus and of analytic geometry. Beginning next fall a third year of mathematics is to be required for all our engineering students. In this third year the classes meet three hours weekly, and the subjects treated are differential equations, matrices and complex variable theory. For majors in mathematics, semester courses in modern algebra, differential equations, topology, and statistics are offered.

* Address delivered at the Spring Meeting of the Ohio Section of the Mathematical Association of America in Columbus, Ohio, April 4, 1946.

However, my interest today centers on the first two years, and I shall not discuss the more advanced courses.

2. The Freshman course. What should be the content of the Freshman course? In this friendly family group I may be permitted to speak my mind frankly without thought or fear of giving offense. Where did this worship of algebra (I care not whether you call it college algebra or algebra unadorned) and of trigonometry arise? When some painstaking historian tries to assess the contribution of teachers of mathematics to the war effort he will have to say: They certainly made a gallant effort to teach algebra and trigonometry. The thousands of classes in ESMWT and ASTP courses which were devoted to algebra and trigonometry stagger the imagination. What did it all amount to? Forget the ordinary run-of-the-mine student who just managed to pass the course and think for a moment only of the two or three excellent students in each class who got 95 or 100. What does it profit them, from the point of view of mathematics, that they can manipulate identities, juggle half-angle formulas and solve triangles, even mastering the ambiguous case?

The one thing that is of mathematical value and interest in a trigonometry class is simply this: The cosine of an angle is a simpler concept than the angle itself. I have to confess that I have never been fortunate enough to meet a student who has passed a trigonometry class and who has grasped this one essential fact. When I ask such students what a right angle is they reply without exception that it is 90 degrees; and when I ask what a degree is they are always pleased to be able to tell me that it is one-ninetieth part of a right angle. There is something malevolently wrong and hypnotic about a course which permits a bright, alert young American who can detect sham in many matters to hoodwink himself in this way.

My first suggestion, then, is that you forget your complaints about the poor preparation in algebra and the lack of trigonometry in the high school. Start teaching analytic geometry in the first term of the Freshman year, but teach it by the methods of vector analysis. In this way you will not only be able to make the teaching of analytic geometry easier, but you will at the same time be teaching those concepts of algebra which are at once the simplest, the most fundamental and the most significant. Start with vectors on a line and your students will learn just what negative numbers are, and they will be pleased to find out just what is going on when they manipulate their slide rules. When you come to plane vectors you can teach, through the concept of direction cosines, the essential elements of trigonometry. For instance the central formula of trigonometry.

$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$

is nothing but a variant of the formula furnishing the distance between two points. When one remembers the complicated constructions involved in the usual proofs of this formula, one can only wonder at the genius for making simple things complicated. Before dealing with second degree curves in the plane,

such as the circle, ellipse, hyperbola, and parabola, take up the simple first degree problems in space. Introduce two-row and three-row determinants in a simple natural way as vector products.

When a student who has been taught analytic geometry by means of vectors looks at an equation such as $2x-3y+7=0$, his first reaction is that this is an equation of a line perpendicular to the coefficient vector $\mathbf{v}(2, -3)$. He knows that this coefficient vector serves to direct the line and that the positive side of the directed line is that towards which the coefficient vector points. The formula for the distance from a point (x, y) to the line $2x-3y+7=0$ gives him no trouble; all he has to do is to divide the expression $2x-3y+7$ by the magnitude $\sqrt{13}$ of the coefficient vector. Similarly when he looks at an equation such as $3x-2y+4z-11=0$ he knows at once that this is an equation of a plane perpendicular to the coefficient vector $\mathbf{v}(3, -2, 4)$ and that this coefficient vector serves to orient the plane (the positive side of the oriented plane being that towards which the coefficient vector points). He thus gets a clear understanding of the fact that it is without meaning to speak of the positive side of a line or of a plane; it is only when the line is directed, or the plane oriented, that it makes sense to speak of its positive side.

We have taught analytic geometry in this way at Hopkins for several years and we find that it goes over well. Our text has been published and so I shall merely refer you to this text for further details and shall pass on to the problem of teaching calculus. Before doing so, I throw out the suggestion that the elementary (or linear) part of analytic geometry, taught by the method of vectors, might well be given in the last year of high school. If, in order to do this, you have to jettison courses in trigonometry, algebra B or solid geometry, you may do this without loss. It is more important for a high school student, who may not go on to college, to know what a vector is, and how cosines facilitate the resolution of vectors, than it is for him to manipulate trigonometric identities, or to solve quadratic equations or to memorize the complicated constructions of solid geometry. A strong argument in favor of teaching the elementary parts of analytic geometry in high school is that the student who goes to college can begin his calculus in the second term of the Freshman year. The demands of modern physics, chemistry, and engineering for the elements of calculus are so pressing that it is no longer practicable to postpone the teaching of calculus to the Sophomore year.

3. The calculus. How should calculus be taught? Here I am very definitely in the minority, but I have the conviction of the true believer: I *know* that I am right. Most of my friends, in a kindly endeavor to prevent me from wasting my time, assure me that calculus simply cannot be taught correctly. The only thing to do is to teach manipulative technique; the student who can differentiate and integrate complicated expressions is at least not scared when he sees symbols for derivatives and integrals. The few students who can clearly understand what a derivative or an integral is will find this out later either in more advanced courses or by independent study. For many years I have at every opportunity

asked college teachers what text they use in their calculus courses and, as nearly as I can estimate, eighty percent of our teaching of calculus is done from a text which does not tell the student what a number is. Certainly ninety percent of the students who come to my graduate course in applied mathematics (for which courses in calculus and differential equations are prerequisite) have never heard of the upper and lower bounds of a bounded function and do not even seem to have heard the term bounded function. I have difficulty in finding words strong enough to express my disagreement with this opinion of my friends and my dissatisfaction with the resulting situation. The times are moving so fast that this situation simply cannot stand, and now is the moment to change it. Have courage, you young men of my audience who wish that things were different. They can be, and you will make them different. When I speak to a successful engineer he almost invariably says to me: For Heaven's sake, teach your young students understanding of the fundamental principles rather than manipulative detail. The latter they can gain for themselves, but they will never learn the former unless you teach it to them. It is my firm conviction, confirmed by several years of experience, that one can teach calculus correctly from the very beginning without losing the ordinary run-of-the-mine student and without sacrificing manipulative technique. We have been teaching calculus in this way at Hopkins for some four years from mimeographed notes, and we hope to have these published by next fall.* I trust that you will examine this text when it appears.

Why is it that so few students grasp the fundamental concepts of calculus in a first course? It is simply because they are deluged with terms such as limit, derivative, differential, integral, and they are never told what a number is. No student can hope to know what a limit is until he understands that every number is defined by a nested sequence of intervals. It takes energy on the part of the teacher to explain this and to make clear, for example, what $3^{1/2}-2^{1/2}$ means. But you are simply not worth your salary, and are certainly not earning it, if you have not the energy and the will to do this. It is quite unnecessary to use high-sounding phrases such as Dedekind cuts, but even poor students can be taught such a theorem as the one that says that a continuous function over a closed interval is bounded. They will make mistakes in repeating the proof, but this is no reason for lying awake at night worrying. They will have some idea of what calculus is about and will have at least a chance of understanding the Theorem of the Mean which is the central theorem of differential calculus. After all, calculus is the science of calculation, and the Theorem of the Mean is the theorem on which calculations are based. We may devoutly hope that students of calculus will in the future understand the paradox which is resolved by the Theorem of the Mean. All calculations involve a fateful leap from a safe initial point a to a final point b, and differentiation is a local process involving no such leaps. How, then, can calculations be made by means of derivatives?

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This is not the place to discuss in great detail isolated points in the teaching of calculus. I shall content myself with a few remarks which will serve to indicate how calculus should, in my opinion, be taught. In the first place I must protest against the universal use of the non-mathematical word small. Every present day student thinks that dx is a "small" increment of x. It is in fact unfair to a student to define the differential of a dependent variable or function and then to use, without explanation, a symbol for the differential of the independent variable. At this point the student who has learnt his analytic geometry by the method of vectors has a decided advantage. He knows that the tangent to the graph of a differentiable function $y=y(x)$ is simply the straight line through (x, y) which is parallel to the vector (dx, dy), where dx is arbitrary. This advantage is more marked when he passes to a study of functions of two or more variables. He sees at once that the tangent plane to the (surface) graph of the differentiable function $z=z(x, y)$ is simply the plane through (x, y, z) which is parallel to the vector $\mathbf{v}(dx, dy, dz)$, where dx and dy are arbitrary.

Most of the teaching of calculus is devoted to functions of a single variable, but most of the interesting applications to engineering, physics and chemistry have to do with functions of several variables. This makes it important that functions of a single variable should be treated in a manner that is readily extensible to functions of several variables. Instead of plunging at once into a discussion of the derivative, I find it advantageous to introduce the concept of a differentiable function; a differentiable function is one which may be linearized or straightened out in the neighborhood of a given point. A linear function $y=y(x)$ is one for which $\Delta y=c\Delta x$, where c is a constant, and a differentiable function is one for which $\Delta y=c\Delta x+\nu|\Delta x|$ where ν is a null-function of Δx, *i.e.*, a function whose limit at $\Delta x=0$ is zero. The constant function c of Δx is then simply the derivative of $y=y(x)$. Similarly a function $z=z(x, y)$ of two variables is differentiable if $\Delta z=c_1\Delta x+c_2\Delta y+\nu r$, where ν is a null-function of Δx and Δy and $r=\{(\Delta x)^2+(\Delta y)^2\}^{1/2}$. The differentials $dy=c\Delta x$ and $dz=c_1\Delta x+c_2\Delta y$ appear then in a natural way and the student understands that we always take differentials of functions or dependent variables (the differential dx of the independent variable x being merely a name for the differential of the identity function $y=x$ of x). This emphasis on Δy rather than on the derivative $y'=\lim \Delta y/\Delta x$ makes the proof of the important chain-rule theorem much simpler.

You all know that a critical point in the teaching of calculus occurs at the place where one proves that the limit at $x=0$ of $(\sin x)/x$ is 1; in other words, we prove that the derivative of $y=\sin x$ at $x=0$ is 1. This is a preliminary theorem to the proof of the fact that $y=\sin x$ is differentiable at any point x with derivative $\cos x$ and the transfer from $x=0$ to an arbitrary x involves a formula from trigonometry that most of your students have forgotten. The proof of the preliminary theorem proceeds as follows: It is clear that the length of a circular arc is greater than the chord and is less than the sum of the tangents at its end points. Once this is admitted, it is easy to prove the theorem. But are the two statements clear? if so, why not prove them? This proof cannot be furnished if

the term "length of an arc" is not defined. It seems to be the accepted opinion that this concept is too difficult for an intelligent young man of eighteen who is quite capable of piloting a B-29. Accepted or not, I hold the opinion absurd: the length of an arc is simply the upper bound of all chord sums. The only thing you have to prove is that the chord sums constitute a bounded variable; and since the length of each chord $\leqq |\Delta x| + |\Delta y|$ this is trivially evident for an arc for which y is a monotone function of x. It is more difficult to prove the additive property of arc length, but most students can follow the proof, even if they find difficulty in repeating it. Once the additive property has been proved, the proof of the relation $s_x^2 = 1 + y_x^2$ for a smooth arc by means of the Theorem of the Mean is immediate. Applying this to the circular arc we obtain at once the derivative of $y = \text{Arc cos } x$ and this gives (on using the theorem furnishing the derivative of the inverse of a given function whose derivative is known) the derivative of $y = \cos x$ and, hence of $y = \sin x$ at any point x. There is more to this than just a mere matter of taste. Once you permit yourself to shrink from a definition of the concept of arc length you have no secure foundation on which to build your treatment of such fundamental concepts as velocity, acceleration and curvature. The fact that some, at least, are disturbed by the claim that an arc of a circle is less than the sum of the tangents at its end-points is evidenced by the occasional use of inequalities between areas to obtain the bracketing inequalities which are necessary to prove that the limit at $x = 0$ of $(\sin x)/x$ is 1. You will agree with me, I believe, that it is a reversal of the direct order of procedure to use results on areas (area being naturally an undefined concept at this stage) to obtain results on arc lengths. It seems to me, then, indefensible to use shoddy arguments to derive a partial result (namely that $y = \sin x$ is differentiable at $x = 0$ with derivative 1) when the general result (namely that $y = \sin x$ is differentiable at any value of x with derivative $\cos x$) may be easily proved through the concept of the inverse function, particularly since this correct proof furnishes a solid basis for the applications of calculus to mechanics and engineering.

When you ask a student taught as I would have him taught the basic questions: What is a derivative? What is an integral? he will not reply: A derivative is the slope of a curve and an integral is the area under a curve. For he knows that the next questions will be: What is the slope of a curve and what is the area under a curve? No; he will define properly the basic terms derivative and integral and he will explain the terms slope and area by means of these.

The usual definition of an integral as $\lim_{\Delta x=0} \sum_{i=1}^{n} y_i \Delta_i x$ is open to serious objection. Remember that your defenseless student has been exposed only to limits of functions of a single variable; of what single variable is $\sum_{i=1}^{n} y_i \Delta_i x$ a function? Certainly not of Δx, which is merely a symbol for the greatest of the numbers $\Delta_i x$. The whole matter is simple if you have courageously grasped the nettle of upper and lower bounds. You simply introduce the familiar upper and lower sums $S = \sum_{i=1}^{n} M_i \Delta_i x$, $s = \sum_{i=1}^{n} m_i \Delta_i x$ and show that the variable S is bounded below, with lower bound B, and that the variable s is bounded above

with upper bound b. If $B=b$ the function y is integrable over the interval in question and the common value of B and b is its integral over this interval. The proof of the fact that any monotone function is integrable is so easy that even the poorest students learn to repeat the proof. One of the easiest ways to introduce the important function $y=e^x$ and its inverse function $y=\log x$ is to define the latter as the integral of the monotone function $1/x$ over the interval $[1, x]$, where $x>0$.

The first challenge I may expect from you is the following: In your effort to provide understanding, do you propose to sacrifice technique? My answer is no. The good teacher judiciously intermingles theory and practice. There is no doubt that some of the time at present devoted to elaborate differentiations and integrations must be given over to teaching correctly the underlying theory, but we find that even the poor students who find the theory difficult end up about as well trained in manipulation as those taught under the old system.

We stand at a critical juncture. Whether we like it or not, the torch of leadership in mathematical culture has been thrust into our hands. Hold it courageously aloft and do not let it drop to the ground to be picked up by some other nation that knows its value and the honor attached to it.

TWO THEOREMS ON FIBONACCI'S SEQUENCE

DOV JARDEN (JUZUK), Jerusalem

Let u_n denote the nth member of the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, \cdots \qquad (u_{n+1} = u_n + u_{n-1})$$

then the following theorem follows easily from known properties* of this sequence:

THEOREM 1. *Let the greatest common divisor (m, n) of m and n be* 1, 2, *or* 5, *then u_{mn} is divisible by $u_m u_n$.*

The object of the first part of this note is to show that the converse is also true:

THEOREM 2. *If u_{mn} is divisible by $u_m u_n$, then $(m, n)=1$, 2, or 5.*

The proof of Theorem 2 is based on the following well known results on Fibonacci's sequence.

Let $u_\alpha(\alpha>0)$ be the first member of the Fibonacci sequence divisible by a given prime p. Then $\alpha=\alpha(p)$ is called the rank of apparition of p and is some divisor of $p-(5/p)$.† Any term of the sequence u_r is divisible by p if and only if α divides r. Let p^λ be the highest power of p dividing u_α and let $r=p^\lambda k$, where p

* For proofs of the known results quoted in what follows see, for example, R. D. Carmichael, Annals of Math. (2), v. 15, 1913–14, pp. 30–70.

† The symbol $(5/p)$ is Legendre's symbol.

does not divide k, then Lucas' "law of repetition" for the Fibonacci sequence states that the highest power of p dividing u_r is $p^{\pi+\lambda+\gamma(r)}$ where

$$\gamma(r) = \begin{cases} 1 & \text{if } p^\lambda = 2 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that for a fixed p

$$\gamma(rs) \leqq \gamma(r) + \gamma(s). \tag{1}$$

To prove Theorem 2 suppose that d, the greatest common divisor of m and n is different from 1, 2 and 5. Now u_d is not a power of 5, since otherwise by the above paragraph with $p=5$, α would be 5, d would be divisible by 5 and would contain the same power of 5 as u_α, whereas $u_d > d$ for $d > 5$. Therefore there exists a prime $p \neq 5$ dividing u_d since $u_d > 1$ for $d > 2$. Then $\alpha(p)$, being a divisor of $p \pm 1$, is prime to p and is a factor of d. Let us write

$$d = \alpha h, \qquad m = m'd = m'\alpha h, \qquad n = n'd = n'\alpha h, \qquad mn = m'n'\alpha^2 h^2.$$

Finally let p^μ and p^ν be the highest power of p dividing m and n respectively, then the highest powers of p dividing u_m, u_n, u_{mn} and $u_m u_n$ are

$$p^{\mu+\pi+\gamma(m)}, \qquad p^{\nu+\pi+\gamma(n)}, \qquad p^{\mu+\nu+\pi+\gamma(mn)}, \qquad p^{\mu+\nu+2\pi+\gamma(m)+\gamma(n)}.$$

Since by assumption $u_m u_n$ divides u_{mn}we must have

$$\mu + \nu + \pi + \gamma(mn) \geqq \mu + \nu + 2\pi + \gamma(m) + \gamma(n),$$

or

$$\pi \leqq \gamma(mn) - \gamma(m) - \gamma(n).$$

But by (1) this is $\leqq 0$, which is absurd. Thus d must be 1, 2 or 5, which proves the theorem.

The second part of this note is devoted to the following theorem.

THEOREM 3. *Let $\alpha(p)$ be the rank of apparition of p. Then $[p-(5/p)]/\alpha(p)$ is an unbounded function of p.*

To prove this we need the following lemma.

LEMMA. *For every set $k_1, k_2, \cdots, k_s$ of integers >0 there exists a prime $q>5$ such that all the numbers*

$$k_1 q \pm 1, k_2 q \pm 1, \cdots, k_s q \pm 1$$

are composite.

Proof: By Dirichlet's theorem, the arithmetical progression

$$\{(3k_1 - 1)(3k_1 + 1) \cdots (3k_s - 1)(3k_s + 1)\}x + 3, \qquad x = 0, 1, 2, \cdots$$

contains at least one prime $q>5$ for let us say, $x = x_q$. But then

$$k_i q \pm 1 = \{(3k_1 - 1)(3k_1 + 1) \cdots (3k_s - 1)(3k_s + 1)x_q + 3\}k_i \pm 1$$

is divisible by $3k_i \pm 1$, a factor >1 and $<k_iq+1$, since $q \geqq 7$. Hence for all i, $k_i \pm 1$ are composite.

Proof of Theorem 3: Let q be a prime >5, and let p be a prime factor of u_q. Then $p = kq + (5/p) = kq \pm 1$. Since $\alpha(p) = q$, the function $[p-(5/p)]/\alpha(p) = k$. Suppose, if possible, that this function is bounded so that k can take on only the values

$$k_1, k_2, \cdots, k_s.$$

Then by the lemma the prime q can be chosen so that all $k_iq \pm 1$ are composite. This contradicts the fact that $kq \pm 1$ is a prime p, which proves the theorem.

DETERMINANTS WHOSE ELEMENTS ARE 0 AND 1

JOHN WILLIAMSON, Queens College

1. Introduction. It was proved by Hadamard [1] that the maximum value of a determinant of order n, each element of which is not greater in absolute value than unity, is $n^{n/2}$. Further it was shown that this maximum value can be attained only, if each element of the determinant has the value ± 1 and if $n = 1$, 2, or $n \equiv 0 \bmod 4$. Here we investigate determinants of matrices K_n, each element of which has the value ± 1, when $n \not\equiv 0 \bmod 4$. In particular we show that, when $n = 7$, the maximum value for such a determinant is $2^6 9$ and that there is essentially only one type of determinant with this value. In the last two sections we construct special matrices K_n first, when $n = 4m$, and then, when $n = 9$, 10 and 11. Unfortunately it is impossible to tell by the methods used here whether or not these matrices K_n have determinants of maximum value. However, we do show that, when $n = 9$, 10 or 11, $|K_n|$ is greater in absolute value than any nth order minor of a determinant $|K_{12}|$ of maximum value 12^6.

2. Preliminary definitions. We shall say that two matrices K_n and K_n' are equivalent, $K_n \approx K_n'$, if K_n' can be obtained from K_n by a finite number of transformations of the following types: multiplication of any row by -1, interchange of any two rows, and transposition of the matrix. It follows that, if $K_n \approx K_n'$, the absolute values of $|K_n|$ and $|K_n'|$ are the same. Further there exists a matrix $K_n' \approx K_n$ where each element of the first column has the value $+1$ and each element of the first row, except the first, the value -1. We therefore assume that K_n has this special form. By adding the first column in succession to the other $n-1$ columns and expanding in terms of the first row, we find that $|K_n| = |P_{n-1}|$, where P_{n-1} is a matrix of order $n-1$, each element of which has the value 2 or 0. Therefore

$$|K_n| = |P_{n-1}| = 2^{n-1}|A_{n-1}|, \tag{1}$$

where each element of A_{n-1} has the value 1 or 0. Since A_{n-1} can be obtained from the first minor of the first element of $|K_n|$ by replacing each element -1 by 0, it is easy to reverse the process and, given A_{n-1}, write the corresponding K_n.

If A'_{n-1} can be obtained from A_{n-1} by a rearrangement of the rows of A_{n-1} or by transposing A_{n-1} or by a succession of such changes, we shall say that A_{n-1} and A'_{n-1} are equivalent, $A_{n-1} \approx A'_{n-1}$. It is evident that, if A_{n-1} and A'_{n-1} are equivalent, so are the corresponding matrices K_n and K'_n but that the converse is not true. In K_n the first row and the first column have special forms. If we wish the $(i+1)$th row of K_n to take the place of the first, each element of the $(i+1)$th row, except the first element, must be -1. This can be accomplished by multiplying each column, in which the element in the $(i+1)$th row is $+1$, by -1. The $(i+1)$th row is now the same as the original first row and the first row the same as the original $(i+1)$th row. If we now interchange the first and the $(i+1)$th rows, we have a matrix $K'_n \approx K_n$. The matrix A'_{n-1} corresponding to K'_n may therefore be obtained directly from the matrix A_{n-1} by keeping the ith row and column unchanged, by keeping any column with a 0 in the ith row unchanged, and by replacing all other elements 1 by 0 and all other elements 0 by 1. From this we obtain the following simple rule for finding A'_{n-1}. The ith row of A'_{n-1} is the same as the ith row of A_{n-1}; to find the jth row of A'_{n-1}, $j \neq i$, we add the ith row of A_{n-1} to the jth row and reduce each element modulo 2.

When A'_{n-1} can be obtained from A_{n-1} by a succession of such processes we shall say that A_{n-1} and A'_{n-1} are semi-equivalent, $A_{n-1} \sim A'_{n-1}$. While it is evident that, if A_{n-1} and A'_{n-1} are semi-equivalent, the corresponding matrices K_n and K'_n are equivalent, the following lemma is also true:

LEMMA 1. *If A_{n-2} is a submatrix of A_{n-1} and $A_{n-2} \sim A'_{n-2}$, there exists a matrix $A'_{n-1} \sim A_{n-1}$ of which A'_{n-2} is a sub-matrix.*

To find the matrix A'_{n-1} we simply apply the transformations, by which A'_{n-2} is obtained from A_{n-2}, to the matrix A_{n-1}.

By Δ_{n-1} we shall mean the determinant of a matrix A_{n-1} and by D_{n-1} the determinant of any A_{n-1} which has a maximum possible value. Our main purpose is now to prove that D_6 is 9 and that all matrices K_7 whose determinants have the value $2^6 9$ are equivalent.

3. A fundamental process. Let $A_{n-1} = (a_{ij})$ and let $D_{n-1} = |A_{n-1}|$. Then, if $a_{ij} = 1$, its cofactor $A_{ij} \geqq 0$, and, if $a_{ij} = 0$, $A_{ij} \leqq 0$, as otherwise D_{n-1} would not have a maximum value. If

$$\Delta_n = \begin{vmatrix} q & \epsilon \\ \delta' & A_{n-1} \end{vmatrix},$$

where ϵ is the row vector $(e_1, e_2, \cdots, e_{n-1})$ and δ' the transpose of the row vector $(d_1, d_2, \cdots, d_{n-1})$,

$$\begin{aligned} \Delta_n &= q\,|A_{n-1}| - d_1 \sum_{j=1}^{n-1} e_j A_{1j} - d_2 \sum_{j=1}^{n-1} e_j A_{2j} - \cdots - d_{n-1} \sum_{j=1}^{n-1} e_j A_{n-1,j} \\ &= qD_{n-1} - \sum_{i,j=1}^{n-1} d_i e_j A_{ij}. \end{aligned}$$

Since each d_i or e_j has the value 1 or 0,

$$\Delta_n = qD_{n-1} - \sum' A_{ij}, \tag{2}$$

where $\sum' A_{ij}$ is the summation of the cofactors A_{ij} of all the elements of a submatrix of A. This submatrix is formed from the rows of A, for which the corresponding d_i have the value 1, and from the columns, for which the corresponding e_j have the value 1. If $q=1$, Δ_n will have its largest absolute value when $\sum' A_{ij}$ has its minimum negative value. If $q=0$, Δ_n will have its maximum absolute value when $\sum' A_{ij}$ is a maximum.

For small values of n it is easy to determine what submatrices of A to choose to make the absolute value of Δ_n a maximum. We shall now apply this process several times, starting with $n=3$.

4. Determination of D_5. It is obvious that $D_2=1$ and that there are two nonequivalent matrices A_2 and A_2' whose determinants have the value 1. They are the matrices

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Since $A_2 \sim A_2'$, there is only one type of matrix K_3 with the maximum value 4 for its determinant. Any $\Delta_3 > 0$ must contain a D_2 as a first minor and in finding D_3, as a consequence of Lemma 1, we need only consider the case when $D_2 = |A_2|$. Since the cofactor of any element of D_2 is positive or zero, in (2) $q=0$, and we obtain

$$A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Therefore $D_3=2$ and there is a single type of matrix K_4 with the maximum value 16 for its determinant.

If D_4 contains a D_3 as a first minor, A_4 must contain a submatrix

$$A_3' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

with a matrix of cofactors

$$B_3 = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

The sum of the elements of any submatrix of B_3 is a minimum for the matrix b_{13} and a maximum for B_3. Therefore by (2) we obtain two matrices

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_4' = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

the absolute value of whose determinants is 3. Although A_4 is not $\approx A_4'$, $A_4 \sim A_4'$ and there is a single type of matrix K_5. If $\Delta_4 \geqq 3$ and does not contain a D_3 as a first minor, it can have no row with less than three 1's in it, and this is easily seen to be impossible. Therefore $D_4 = 3$ and there is a single type of matrix K_5 whose determinant has the maximum value 48.

To find D_5, if D_5 contains a D_4 as a first minor, by Lemma 1 we need only consider the possibility that A_5 contains a submatrix A_4'' equivalent to A_4 or A_4'. We choose

$$A_4'' = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

which has as its matrix of cofactors the matrix

$$B_4 = \begin{bmatrix} 1 & 1 & 1 & -2 \\ -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix}.$$

The sum of the elements of any submatrix of B_4 has a minimum value -2 for the submatrices of which b_{14} and

$$\begin{bmatrix} b_{11} & b_{14} \\ b_{21} & b_{24} \end{bmatrix}$$

are typical, and this sum has a maximum value 4. By (2) we obtain matrices

$$A_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_5' = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix},$$

whose determinants have a maximum value 5. Though A_5 is not $\approx A_5'$, $A_5 \sim A_5'$, and there is only one type of matrix K_6. (Actually there are three types of non-equivalent A_5 whose determinants have the value 5. This can be shown by considering both the matrices A_4 and A_4').

If $\Delta_5 \geqq 5$ and does not contain a first minor of absolute value 3, Δ_5 can have no row (1 1 0 0 0). Nor can it have two rows of types

$$(\alpha)\begin{bmatrix}1&1&1&0&0\\0&1&1&0&1\end{bmatrix} \text{ or } (\beta)\begin{bmatrix}1&1&1&1&0\\0&1&1&1&1\end{bmatrix},$$

since in either case it would equal a determinant with the row (1 0 0 0 -1). Accordingly such a determinant must contain at least four rows with three 1's in each row, two of which must be of type (α). Therefore $D_5 = 5$ and there is a single type of matrix K_6 with the maximum value 160 for its determinant.

5. Determination of D_6. Let $\Delta_6 = |A_6|$ contain a first minor which is a D_5. To obtain all non-equivalent A_6 we should have to consider all non-equivalent A_5. To find all non-equivalent K_7 we need consider only, as a consequence of Lemma 1, one A_5. We choose

$$A_5 = \begin{bmatrix}0&1&1&1&1\\1&1&0&0&1\\1&0&1&0&1\\1&0&0&1&1\\1&1&1&1&0\end{bmatrix}$$

and find that the matrix of cofactors is

$$B_5 = \begin{bmatrix}-3&1&1&1&2\\1&3&-2&-2&1\\1&-2&3&-2&1\\1&-2&-2&3&1\\2&1&1&1&-3\end{bmatrix}.$$

The sum of the elements of any submatrix of B_5 has a minimum value -4 for the typical submatrices

$$(b_{23}\ \ b_{24}) \text{ and } \begin{bmatrix}b_{11}&b_{13}&b_{14}\\b_{21}&b_{23}&b_{24}\end{bmatrix},$$

and a maximum value 8. Hence Δ_6 has a maximum value 9 and is obtained from

$$A_6 = \begin{bmatrix}1&0&0&1&1&0\\0&0&1&1&1&1\\1&1&1&0&0&1\\0&1&0&1&0&1\\0&1&0&0&1&1\\0&1&1&1&1&0\end{bmatrix} \text{ and } A_6' = \begin{bmatrix}1&1&0&1&1&0\\1&0&1&1&1&1\\1&1&1&0&0&1\\0&1&0&1&0&1\\0&1&0&0&1&1\\0&1&1&1&1&0\end{bmatrix}.$$

Once again A_6 is not $\approx A_6'$ but $A_6 \sim A_6'$, so that there is only one type of matrix K_7. (These are, however, actually five non-equivalent A_6 whose determinants have the maximum value 9.)

If Δ_6 does not contain a D_5 as a minor and if $\Delta_6 \geqq 9$, Δ_6 cannot contain a row with only two 1's or two rows of the types

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Moreover Δ_6 cannot contain two rows of the type

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

or three rows containing three 1's in each row with a column of three 1's.

If Δ_6 contains a row with five 1's and the last element 0, each other element in the last column must be 1. Therefore Δ_6 can only have two rows with three 1's and two rows with four 1's, and it is impossible to obtain six rows.

If Δ_6 does not contain a row with five 1's, Δ_6 may have at most three rows with four 1's of the type

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and four rows with three 1's of the type

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

There are accordingly two possibilities for Δ_6;

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{vmatrix}$$

both of which have the value ± 8. Therefore $D_6 = 9$ and *there is only one type of matrix* K_7 *whose determinant has the maximum value* $2^6 9$.

The fact that $D_6=9$ is interesting for two reasons. First it falls far short of the possible value given by Hadamard's Theorem. For, by this theorem, $|K_7| = m2^6 < 7^{7/2}$ and m could be as large as 14. Second, an Hadamard matrix H_8, *i.e.* a matrix K_8 with the maximum value 8^4 for its determinant, exists [2]. Since H_8, apart from a factor common to all its elements, is orthogonal, each first minor of $|H_8|$ has the value $\pm 8^4/8 = \pm 8^3$. Therefore any first minor of $|H_8|$ determines by (1) a Δ_6 of value $8^3/2^6 = 8$. Accordingly *no first minor of an Hadamard determinant* $|H_8|$ *has the greatest possible value.*

6. A value for Δ_{4n}. If A is the matrix obtained by (1) from an Hadamard matrix H_{4n},

$$\Delta_{4n-1} = |A| = (4n)^{2n}/2^{4n-1}.$$

In each row of Δ_{4n-1} there are $2n$ elements 1 and $2n-1$ elements 0, while the cofactor of each element 1 is n^{2n-1} and of each element 0 is $-n^{2n-1}$. Therefore, if B is the adjugate matrix of A, the sum of all the elements of B is $(4n-1)n^{2n-1}$ and by (2) there exists a Δ_{4n} of value $(4n-1)n^{2n-1}$. Further, in any two rows of A there are exactly $n-1$ columns with both elements 0 and $2n$ columns with one element 0 and the other element 1. Therefore there exists a submatrix of B, consisting of $4n-3$ rows and $3n-1$ columns, the sum of whose elements is $[3(n-1)+2n]n^{2n-1}$. Accordingly there exists a Δ_{4n} such that

$$\Delta_{4n} = (5n-3)n^{2n-1}. \tag{3}$$

Examination of the matrix B, when $n=2$, 3, or 4, did not lead to any value of Δ_{4n} greater than that given by (3), but, when $n=5$, a Δ_{20} of value 25×5^9 was obtained.

7. Values for Δ_8, Δ_9 and Δ_{10}.
When $n=2$, we find

$$\Delta_8 = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{vmatrix},$$

which has the value 56. A calculation shows that the sum of the elements in the submatrix

$$\begin{bmatrix} b_{13} & b_{14} & b_{16} \\ b_{23} & b_{24} & b_{26} \end{bmatrix}$$

of the corresponding matrix B is -60. Therefore, if this submatrix is used to find by (2) a matrix A of order 9, $|A| = 116$. The element $a_{33} = 1$ while its cofactor has the value -4. If C is the matrix obtained from A by replacing the element a_{33} by 0, $|C| = 120$ and the sum of all of the elements in the adjugate of C is 264. Therefore the determinant

$$\Delta_{10} = \begin{vmatrix} 0 & x \\ x' & C \end{vmatrix},$$

where x denotes a row of 9 1's and x' a column of 9 1's, has the value 264. We have thus obtained determinants Δ_8, Δ_9 and Δ_{10} with the values 56, 120 and 264 respectively. These numbers fall far short of the corresponding maximum ones given by Hadamard's Theorem, which are 76, 195 and 521, but as D_5 is 5 and not 6 and D_6 is 9 and not 14, we already know that the maxima given by Hadamard's Theorem are not always attained.

Since $(4n)^{-1/2}H_{4n}$ is an orthogonal matrix and any minor of an orthogonal matrix is equal in absolute value to its complementary minor,

$$\Delta_{4n-m-1} = n^{2n-m}\Delta_{m-1}, \tag{4}$$

where Δ_j is obtained by (1) from a minor of order $j+1$ of H_{4n}. The maximum values of Δ_j obtained from minors of $|H_{12}|$ are therefore $\Delta_{10} = 243$, $\Delta_9 = 81$, $\Delta_8 = 27$, $\Delta_7 = 18$ and $\Delta_6 = 9$ and these values, except the last, are all less than those previously obtained. Accordingly $|H_{12}|$ *contains a minor of order* 7 *with the maximum value* $2^6 9$ *but does not contain a minor of order greater than* 7 *with the maximum value.*

In fact, if $n > 2$, the value n^{2n-3} for Δ_{4n-4} in (4) is less than the value $(5n-8)(n-1)^{2n-3}$ given by (3) and the value $2n^{2n-4}$ for Δ_{4n-5} much less than $D_{4n-5} = 2(n-1)^{2n-2}$. These last comparisons are based on the assumption that Hadamard matrices of orders $4n$ and $4n-4$ both exist. Unfortunately, it is not known at present whether or not this is true [3].

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Theft. Charlie was crazy about mathematics. He could do almost any example in his head. He said he would like to go to college, if he could get in, so he could learn more mathematics. But he didn't get in any college. He got into a bank in Dover Plains, and after he'd worked there twenty or thirty years he got in jail. Well, I might have ended in jail, too, like Charlie, but I could never be sure if it was πr^2 or $r\pi^2$, and now I've forgotten what π is—or maybe its r, and I guess I'm just as well off. Percival Wilde, *Inquest.*—*E. D. Schell.*

HELMERT'S DISTRIBUTION

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1. Purpose. The joint distribution of sample mean and standard deviation for samples of independent random observations, drawn from a normal population, may be called *Helmert's distribution.** The purpose of this note is to derive Helmert's distribution by a straightforward inductive procedure.

2. Background. Helmert's distribution is of central importance in the theory of sampling from a normal (*i.e.* Gaussian) population because from it follow:

1. The statistical independence of sample mean and sample standard deviation.
2. The separate single distributions of sample mean and sample standard deviation.
3. The distribution of the Student t.

Helmert's distribution was apparently first established (in effect) by Helmert in reference [1]; perhaps a more readily accessible version of his proof appears on pages 151 to 163 of reference [2]. Helmert's procedure was to set up a pair of linear transformations which transformed the joint distribution of the individual errors of observation into a joint distribution of sample mean error and standard deviation, plus dummy variables which could be "integrated out" over all possible values.

Helmert's distribution may also be established by an appeal to the geometry of hyper-space, as in reference [3] and similarly in reference [4], pages 129 to 133. A third method of proof is to first establish a theorem about the independent chi-square distributions of quadratic forms, under suitable restrictions. Reference [5], pages 105 to 110, follows this procedure, first proving Cochran's theorem and then showing that Helmert's distribution is an immediate consequence.

It seems of pedagogical and conceptual interest that Helmert's distribution may be simply and directly established by mathematical induction. Although Helmert himself used an inductive method in reference [6] to establish a somewhat simpler distribution, it is apparently not generally recognized that the joint distribution with which we are concerned may be inductively derived.

3. Notation. Notation will be similar to that of reference (3). We suppose that n independent random observations, x_i, have been drawn from a normal population with mean μ and standard deviation σ.

Sample mean: $\bar{x} = (\sum x_i)/n$

Error of sample mean: $u = \bar{x} - \mu$

Error of individual observation: $\epsilon_i = x_i - \mu$

Sample standard deviation squared (sample variance):

$$s^2 = \frac{1}{n}[\sum (x_i - \bar{x})^2] = \frac{1}{n}[(\sum \epsilon_i^2) - nu^2].$$

* This name is sometimes given to the single distribution of sample standard deviation.

4. Statement of Helmert's distribution. Helmert's distribution is given by the distribution element:

$$(1) \qquad K_1 s^{n-2} \exp\left[-\frac{n}{2\sigma^2}(s^2+u^2)\right] du\,ds$$

$$K_1 = \frac{n^{n/2}}{2^{(n-2)/2}\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)\sigma^n}.$$

That is, the probability that u and s will simultaneously lie within given limits is the double integral, over those limits, of (1).

5. Proof. Suppose that (1) is the correct probability element for a sample of size n. Increase the sample size to $n+1$ by drawing a new independent observation, x_{n+1}, from the parent normal population. The corresponding ϵ_{n+1} will of course be distributed normally with zero mean and standard deviation σ. Hence the probability element for the joint distribution of u, s, and ϵ_{n+1} will be:

$$(2) \qquad \frac{K_1}{\sigma\sqrt{2\pi}} s^{n-2} \exp\left[-\frac{n}{2\sigma^2}(s^2+u^2+\epsilon_{n+1}^2/n)\right] du\,ds\,d\epsilon_{n+1}.$$

Let $\bar{u}$ and $\bar{s}$ be, respectively, the error of the new sample's mean and the new sample's standard deviation. The plan of proof is as follows:

1. Express $\bar{u}$ and $\bar{s}$ in terms of u, s, and ϵ_{n+1}.
2. Transform the distribution element (2) into the distribution element for $\bar{u}$, $\bar{s}$, and ϵ_{n+1}.
3. Integrate the resulting transformed distribution element over all possible values of ϵ_{n+1}.

The distribution element thus obtained for $\bar{u}$ and $\bar{s}$ will have the same form as (1) except that "n" will be replaced by "$n+1$." Finally Helmert's distribution will be established for the special case, $n=2$.

The relevant relationships between the five variables, u, s, $\bar{u}$, $\bar{s}$, and ϵ_{n+1}, may readily be obtained from their definitions:

$$\begin{cases} u = \dfrac{1}{n}\left[(n+1)\bar{u} - \epsilon_{n+1}\right] \\ s^2 = \dfrac{n+1}{n^2}\left[n\bar{s}^2 - (\bar{u}-\epsilon_{n+1})^2\right] \end{cases}$$

$$\begin{cases} \bar{u} = \dfrac{1}{n+1}\left[nu + \epsilon_{n+1}\right] \\ \bar{s}^2 = \dfrac{n}{(n+1)^2}\left[(n+1)s^2 + (u-\epsilon_{n+1})^2\right] \end{cases}$$

$$n(s^2+u^2+\epsilon_{n+1}^2/n) = (n+1)(\bar{s}^2+\bar{u}^2).$$

Henceforth the subscript will be dropped from "ϵ_{n+1}". The Jacobian of the above transformation is:

$$\frac{\partial u}{\partial \bar{u}} \frac{\partial s}{\partial \bar{s}} = \frac{(n+1)^{3/2}}{n} \bar{s}[n\bar{s}^2 - (\bar{u} - \epsilon)^2]^{-1/2}.$$

Hence the joint distribution element for $\bar{u}$, $\bar{s}$, and ϵ is:

$$K_2\bar{s}[n\bar{s}^2 - (\bar{u} - \epsilon)^2]^{(n-3)/2} \exp\left[-\frac{n+1}{2\sigma^2}(\bar{s}^2 + \bar{u}^2)\right] d\bar{u}d\bar{s}d\epsilon$$

$$K_2 = \frac{K_1}{\sigma\sqrt{2\pi}} \frac{(n+1)^{(n+1)/2}}{n^{n-1}}.$$

We now wish to integrate this distribution element over all (real) values of ϵ. Since $s^2 \geqq 0$, ϵ is limited by the inequality:

$$\epsilon^2 - 2\bar{u}\epsilon - n\bar{s}^2 + \bar{u}^2 \leqq 0. \tag{3}$$

(3) is satisfied if and only if ϵ lies between $\bar{u} - \bar{s}\sqrt{n}$ and $\bar{u} + \bar{s}\sqrt{n}$. Hence it is necessary to evaluate:

$$\int_{\bar{u}-\bar{s}\sqrt{n}}^{\bar{u}+\bar{s}\sqrt{n}} [n\bar{s}^2 - (\bar{u} - \epsilon)^2]^{(n-3)/2} d\epsilon = 2n^{(n-2)/2}\bar{s}^{n-2} \int_0^1 (1 - y^2)^{(n-3)/2} dy$$

by symmetry of the integrand, putting $\epsilon = \bar{u} - \sqrt{n}\bar{s}y$. Now let $y = \sin\theta$. The integral becomes:

$$2n^{(n-2)/2}\bar{s}^{n-2} \int_0^{\pi/2} \cos^{n-2}\theta d\theta.$$

But this is a well known special case of the complete Beta integral. (See, for example, formula 483 of reference [7].) The integral is equal to:

$$n^{(n-2)/2}\bar{s}^{n-2}\sqrt{\pi} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

Therefore the final distribution element is:

$$K_3\bar{s}^{n-1} \exp\left[-\frac{n+1}{2\sigma^2}(\bar{s}^2 + \bar{u}^2)\right] d\bar{u}d\bar{s}$$

$$K_3 = \frac{(n+1)^{(n+1)/2}}{2^{(n-1)/2}\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)\sigma^{n+1}}$$

and this is just (1) with "n" replaced by "$n+1$."

It remains to show that Helmert's distribution holds for the case $n=2$. Here the joint distribution element for ϵ_1 and ϵ_2 is:

$$\frac{1}{2\pi\sigma^2}\exp\left(-\frac{\epsilon_1^2+\epsilon_2^2}{2\sigma^2}\right)d\epsilon_1 d\epsilon_2.$$

Make the transformation:

$$\left\{\begin{aligned}\epsilon_1 &= u+l\\ \epsilon_2 &= u-l\end{aligned}\right\} \qquad \left\{\begin{aligned}u &= \tfrac{1}{2}(\epsilon_1+\epsilon_2)\\ l &= \tfrac{1}{2}(\epsilon_1-\epsilon_2)\end{aligned}\right\}$$

where u is the sample mean error and l is a variable whose absolute value is the sample standard deviation. The absolute value of the Jacobian is 2, and:

$$\epsilon_1^2+\epsilon_2^2 = 2u^2+2l^2.$$

Hence the joint distribution element for u and l is:

$$\frac{1}{\pi\sigma^2}\exp\left[-\frac{u^2+l^2}{\sigma^2}\right]dudl.$$

And since the sample standard deviation, s, is the absolute value of l, the joint distribution element for u and s is:

$$\frac{2}{\pi\sigma^2}\exp\left[-\frac{u^2+s^2}{\sigma^2}\right]duds$$

which is just (1) with $n=2$.

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Woolgathering. While studying the *pons asinorum* in Euclid, he suffered every cuddie upon the common to trespass upon a large field belonging to the Laird. Walter Scott, *Heart of Mid-Lothian.—E. D. Schell*

PARTIALLY ORDERED FIELDS AND GEOMETRIES*

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1. Introduction. Despite its simplicity, *order* is one of the most important and pervasive ideas in modern mathematics. It is everpresent in analysis from the elementary process of counting to the refined techniques of modern function theory and plays an important, if not quite so obvious a role in algebra and geometry.

The abstract theory of order usually is based on a 2-term relation $a<b$ (read "a precedes b") called *precedence*, which satisfies the following postulates, where a, b, c denote arbitrary elements of a given set S.

P1. *$a<b$ implies that a, b are distinct.*

P2. *$a<b$, $b<c$ imply $a<c$.*

P3. *If a, b are distinct, one of the relations $a<b$, $b<a$ holds.*†

On the other hand an equivalent theory of order may be developed in terms of a 3-term relation *betweenness*, denoted (abc), (read "b is between a and c" or "a, b, c are in the order abc") which satisfies the following postulates:

B1. *(abc) implies that a, b, c are distinct.*

B2. *(abc) implies (cba).*

B3. *(abc), (bcd) imply (abd), (acd).*

B4. *(abc), (acd) imply (abd), (bcd).*

B5. *If a, b, c are distinct, one of the relations (abc), (bca), (cab) holds.*‡

Observe that P3 and B5 play similar roles in the respective theories, requiring that two or three elements be in a sense "comparable."

To show the two theories of order equivalent, we introduce in a system satisfying P1, P2, P3

DEFINITION A. If elements a, b, c satisfy the conditions $a<b$, $b<c$ or $c<b$, $b<a$ we say (abc).

It is easily seen that the 3-term relation so defined satisfies B1, · · ·, B5. Thus any "precedence" system satisfying P1, P2, P3 is converted by Definition A into a "betweenness" system satisfying B1, · · ·, B5. Conversely every abstract betweenness system satisfying B1, · · ·, B5 arises from some precedence system satisfying P1, P2, P3 in just this way.§ Thus the two theories of order are logically equivalent and it is immaterial in theory which we choose to employ. In algebra precedence is preferred because of its simplicity, while in geometry it seems more natural to use betweenness relations.

* The author wishes to express his thanks to his former student Frank Harary for working out details of several of the proofs.

† See J. W. Young, Lectures on Fundamental Concepts of Algebra and Geometry, New York, 1911, p. 68.

‡ See E. V. Huntington and J. R. Kline, Sets of independent postulates for betweenness, Trans. Amer. Math. Soc. 18, 1917, pp. 301–325 for a very detailed study of several postulate systems for betweenness and their independence. Our choice of postulates was made for simplicity and convenience rather than irredundance.

§ See Huntington and Kline, *loc. cit.* p. 324 for the definition of the associated precedence relation.

Order is important in algebra not merely because so many algebraic systems are ordered, often in a unique or so-called "natural" manner, but also because ordinal notions and algebraic ideas often are closely interlinked. Thus consider a field F in which is defined a relation $<$ satisfying P1, P2, P3. Suppose the order relation $<$ is linked to the field operations $+$, $\times$ by the monotonic laws

M1. $a<b$ *implies* $a+x<b+x$,

M2. $a<b$, $0<x$ *imply* $ax<bx$.

These conditions characterize F as an *ordered field* of which the rational number system is the simplest example and the real number system the most important.*

In geometry the most important example of order is the relation betweenness for points on a line. We observe that this is closely related to the theory of ordered fields since if we set up an analytic geometry in an ordered field in the usual manner and define between in the natural way, postulates B1, · · ·, B5 are satisfied for the points of a line.

Recently there has been developed a flourishing generalization of the classical theory of order, based merely on P1, P2, called the theory of *partial* order. Since P3, the postulate of "comparability" has been omitted, this theory is applicable to what might be termed non-linear situations, and has many interesting and important applications to relations such as set inclusion, divisibility, implication arising in algebra, geometry, analysis and logic.†

As we have noted, B5 plays the same role in the theory of betweenness as P3 in the theory of precedence. This suggests a theory of 3-term partial order based merely on B1, · · ·, B4. One would expect such a theory to be equivalent to the theory of 2-term partial order in the sense in which the original theories of precedence and betweenness are equivalent—namely with Definition A forming the connecting link.

This is not so. For there exist 3-term relations satisfying B1, · · ·, B4 which can not be generated by Definition A from 2-term relations satisfying P1, P2. As an illustration consider (pqr) the betweenness relation for points in the euclidean *plane*, which requires that q be an interior point of segment pr. Clearly the relation satisfies B1, · · ·, B4 but not B5. Suppose this relation definable in terms of some relation $<$ satisfying P1, P2 by Definition A. Let a, b be distinct points. There exists a point x such that (axb). Hence by Definition A $a<x$, $x<b$ or $b<x$, $x<a$ so that P2 implies $a<b$ or $b<a$. Thus P3 holds. Hence $<$ is a "precedence" relation and the given relation *between* must satisfy B5. This contradiction justifies our assertion.

* For equivalent characterizations in terms of the notion "positive element" see (1) A. A. Albert, Modern Higher Algebra, Chicago, 1937, p. 110, (2) G. Birkhoff and S. MacLane, A Survey of Modern Algebra, New York, 1941, p. 54, The monotonic laws are discussed in Young, *loc. cit.*, pp. 100, 110–111.

† See Garrett Birkhoff, Lattice Theory, New York, 1940, Chap. I, and Lattice theory, this Monthly, 50, 1943, pp. 484–487. Our postulates P1, P2, are equivalent to P1, P2, P3 of Birkhoff, see Lemma 2, p. 7 of the first reference.

Since this theory of 3-term partial order is not equivalent to the theory of 2-term partial order in the simple and natural sense described above, it seems worthy of development on its own account. In this note we construct two elementary applications of this idea, first to a generalization of ordered field which we term *partially ordered field*, second to the notion *partially ordered geometry*, a generalization of ordinary linear geometry in which no one of three points on a line need be between the other two.*

2. Partially ordered fields. We begin by stating the precise sense in which we shall employ the term partial ordering.

Consider a set S of elements $a, b, c, \cdots$ in which is defined a 3-term relation called *order*, indicated by the notation (abc). We say S is a *partially ordered* set, or S is *partially ordered* by the relation, or the relation is a *partial ordering* of S if postulates B1, $\cdots$, B4 and the following existential postulate B6 are satisfied:

B6. *There exist elements a, b, c such that (abc).*

If in addition postulate B5 is satisfied (classic theory of betweenness), we say S is *fully* or *simply ordered*, or the relation is a *simple ordering* of S.

We observe that B6 is an implicate of B5 if S contains three or more elements; it is introduced to exclude cases in which the order relation is vacuous. Although we can easily weaken the postulate system B1, $\cdots$, B4, B6 we shall not do so, since it is not our object to study the logical interrelations of the postulates but rather to present a simple and natural generalization of the familiar theory of order.

Before continuing with partially ordered *fields* we give one very elementary property of partially ordered sets.

THEOREM. *In a partially ordered set, (abc) implies (bca) false.*

Proof. By B3, (abc) and (bca) imply (aba) contrary to B1.

Let $\mathfrak{F}$ be a field which is partially (simply) ordered by a relation (abc). If $\mathfrak{F}$ satisfies the following *invariance principles* I1, I2, we say it is a *partially (simply) ordered field*:

I1. *(abc) implies $(a+x\ b+x\ c+x)$;*

I2. *(abc) implies $(ax\ bx\ cx)$ provided $x \neq 0$.*

These principles correspond to the monotonic laws M1, M2 for 2-term order but are neater and more symmetrical in form. The notion partially ordered field is a generalization of ordered field as characterized in §1, since the latter can be shown to be equivalent to simply ordered field as we have defined it.

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* That is B5 need not hold for points of a line. The present paper had its inception in the attempt to show the independence of one of the postulates (equivalent to assuming B5 for collinear points) in a new type of characterization of ordered linear geometries given in Prenowitz, Descriptive geometries as multigroups, Trans. Amer. Math. Soc., vol. 59, 1946, pp. 333–380.

3. Elementary properties. We now deduce some elementary formal properties of partially ordered fields which we shall employ later.

A. $(0a1)$ *implies* $(0\ 1-a\ 1)$.

Proof. $(0a1)$ implies $(0\ -a\ -1)$ by I2. From this by I1 we have $(1\ 1-a\ 0)$ so that $(0\ 1-a\ 1)$ follows by B2.

B. $(0a1)$, $(0b1)$ *imply* $(0\ ab\ 1)$.

Proof. By I2, $(0a1)$ implies $(0\ ab\ b)$. This with $(0b1)$ implies $(0\ ab\ 1)$ by B4.

C. $(0a1)$ *implies* $(0\ a\ 1+a)$, $(0\ 1\ 1+a)$.

Proof. By Property A, $(0a1)$ implies $(0\ 1-a\ 1)$. By I1 this yields $(a\ 1\ 1+a)$ which with $(0a1)$ implies the desired conclusions by B3.

D. $(0a1)$, $(0b1)$ *imply* $(0\ a\ a+b)$.

Proof. Adding a to each term in the relation $(0b1)$ and applying B2 we get

$$(1 + a\ a + b\ a). \tag{1}$$

By Property C and B2 $(0a1)$ implies

$$(1 + a\ a\ 0). \tag{2}$$

Applying B4 to (1), (2) we get $(a+b\ a\ 0)$, and $(0\ a\ a+b)$ follows by B2.

E. (acb) *is equivalent to the existence of elements* λ, μ *satisfying*

$$c = \lambda a + \mu b, \qquad \lambda + \mu = 1, \qquad (0\lambda 1), \qquad (0\mu 1). \tag{3}$$

Proof. Assume (acb). We have the algebraic identities

$$c = \lambda a + \mu b, \qquad \lambda + \mu = 1$$

where

$$\lambda = (c - b)/(a - b), \qquad \mu = (c - a)/(b - a).$$

Subtracting b from the terms in (acb) we get $(a-b\ c-b\ 0)$. Dividing in this relation by $a-b$ and applying B2 we get $(0\lambda 1)$. Similarly we show $(0\mu 1)$ and the necessity is established.

Conversely assume (3). Solving for λ we get $\lambda=(c-b)/(a-b)$. Hence multiplying in $(0\lambda 1)$ by $a-b$ we get $(0\ c-b\ a-b)$. Adding b to the terms in this relation and applying B2 we get (acb), which completes the proof.

4. The characteristic. We show now that the characteristic of a partially ordered field, as in the familiar theory of (simply) ordered fields, is infinite.

Let $\mathfrak{F}$ be a partially ordered field. By B6 there exist elements a, b, c, in $\mathfrak{F}$ satisfying (abc). Subtracting a from the terms in this relation and dividing by $c-a$ we get

$$(0x1) \tag{1}$$

where $x=(b-a)/(c-a)$. By Property C, (1) implies

$$(0\ 1\ 1 + x). \tag{2}$$

Adding 1 to the terms in (1) we get

$$(1\ 1 + x\ 2) \tag{3}$$

where $2=1+1$. By B3 relations (2), (3) imply

$$(012).$$

Repeatedly adding 1 to the terms in this relation we get

$$(123), (234), \cdots, (n\ n+1\ n+2)$$

where $3=2+1$, $4=3+1$ and n is an arbitrary element of $\mathfrak{J}$ obtained by summation of 1. From this it follows by B3 that the elements 1, 2, 3, $\cdots$, n are ordered in the *natural* way and so are distinct. Thus $\mathfrak{J}$ has infinite characteristic. It readily follows that the rational subfield of $\mathfrak{J}$ is ordered in the natural manner. Thus we may assert the

THEOREM. *Any partially ordered field has characteristic infinity. Its rational subfield has the natural (simple) ordering.**

Applying this to the field of rational numbers we have the

COROLLARY. *The only partial ordering of the field of rational numbers is the natural ordering.*†

5. A partially ordered field. Although there is no partial ordering of the field of rational numbers which is non-simple, we can easily define non-simple orderings for more complicated fields. Let $\mathfrak{R}$ denote the real number field and $\mathfrak{R}'$ the field of rational functions in a real variable x with coefficients in $\mathfrak{R}$. We can consider the elements of $\mathfrak{R}'$ to be fractions with polynomial numerator and denominator, assuming the usual agreements for equality or equivalence of fractions. Elements of $\mathfrak{R}'$ are denoted A, B, C, $\cdots$ and their respective functional values for argument x are denoted $A(x)$, $B(x)$, $C(x)$ $\cdots$. We assume the familiar simple ordering of field $\mathfrak{R}$ and indicate it by the notation (abc), which is equivalent to $a<b<c$ or $a>b>c$. Using it we induce a partial ordering of field $\mathfrak{R}'$ in the following manner.

We define (ABC) to mean that

$$(A(x)\ B(x)\ C(x))$$

holds for all x in $\mathfrak{R}$ except a finite number, or as we shall say, for *almost all* x.

Observe that this property is independent of the form in which the elements of $\mathfrak{R}'$ are expressed. Thus if we replace A, B or C by equal or equivalent elements of $\mathfrak{R}'$ the relation still holds.

It is easy to verify that $\mathfrak{R}'$ with order as defined is a partially ordered field. For example consider B3. Suppose (ABC), (BCD).‡ By definition $(A(x)\ B(x)\ C(x))$ and $(B(x)\ C(x)\ D(x))$ hold separately for almost all x. Hence they hold *simultaneously* for almost all x. Since B3 holds in $\mathfrak{R}$ we get

* Compare Albert, *loc. cit.*, pp. 110–112, Birkhoff and MacLane, *loc. cit.*, p. 56 Theorem 13.

† Compare Albert, *loc. cit.*, p. 112 Theorem 12.

‡ More strictly we should assume (ABC), $(B'C'D)$ where B', C', are equivalent to B, C respectively. But this reduces to the case given, since we can substitute B, C for B', C' by the observation in the preceding paragraph.

$$(A(x)\ B(x)\ D(x)), \qquad (A(x)\ C(x)\ D(x))$$

for almost all x and B3 is verified in $\mathfrak{R}'$.

Observe that the elements of $\mathfrak{R}$, which are included in $\mathfrak{R}'$ as "constant" functions, maintain in $\mathfrak{R}'$ their familiar order in $\mathfrak{R}$. Thus we may say that the ordering in $\mathfrak{R}'$ is an *extension* of that in $\mathfrak{R}$. Moreover $\mathfrak{R}'$ is not simply ordered. For there subsists no order relation involving the function x and the constant functions 0, 1, so that B5 fails. Thus we have constructed a non-simple ordering in $\mathfrak{R}'$ which is an extension of the familiar simple ordering in $\mathfrak{R}$. This construction can be applied to any partially ordered field $\mathfrak{F}$ to yield a non-simply ordered field $\mathfrak{F}'$, the ordering in which is an extension of that in $\mathfrak{F}$.

6. Linearity. In this section we clarify the notion *linear* or *one-dimensional* ordering and show that $\mathfrak{R}'$ is linearly ordered.

All simply ordered sets are in a natural sense linear or one-dimensional. This is not so for partially ordered sets. For example a euclidean plane geometry is a partially ordered set and we would say it has a two-dimensional rather than a linear type of order. Thus it is important to clarify the notion *linear ordering* as one of the simplest types of partial ordering.

We define linear ordering as a natural generalization of simple ordering. We say that partially ordered set S is *linearly ordered* or is *linear* provided each element of S is related to any two given elements of S by a finite chain of order relations. We express this in more formal language. Let us say in a partially ordered set S that c is *linearly related* to a, b where $a \neq b$, provided there exists in S a sequence

$$x_1 = a, \qquad x_2 = b, \cdots, x_n = c$$

such that for each i, $3 \leqq i \leqq n$, there exist j, $k < i$ for which one of the relations

$$(x_i x_j x_k), \qquad (x_j x_i x_k)$$

holds. Then, if each element of S is linearly related to each pair of elements of S, we say S is *linearly ordered* or is *linear*. A simple ordering is the most elementary example of a linear ordering since in this case n has the minimum possible value 3, if $c \neq a, b$.

We can now state the

THEOREM. *The field $\mathfrak{R}'$ is linearly ordered.*

Proof. Let A, B, C be elements of $\mathfrak{R}'$, where $A \neq B$. We shall show that C is linearly related to A, B.

First we consider the case $A = 0$, $B = 1$. Assume that C is a polynomial and express $C(x)$ in the form

$$C(x) = c_0 + c_1 x + \cdots + c_n x^n$$

where n is even, so that $c_n = 0$ is possible. We determine C' by the relation

$$C'(x) = 2 + \sum_{i=0}^{n} |c_i| + (|c_1| + |c_2|)x^2 + (|c_3| + |c_4|)x^4 + \cdots + (|c_{n-1}| + |c_n|)x^n.$$

Since

$$1 < C'(x), \qquad C(x) < C'(x)$$

for all x, we have

$$(0\ 1\ C'(x)), \qquad (0\ 1\ C'(x) + 1), \qquad (C(x)\ C'(x)\ C'(x) + 1)$$

for all x. Thus any polynomial C is linearly related to 0, 1, the sequence 0, 1, C', $C'+1$, C establishing the desired relation.

Now we show that the reciprocal of a non-zero polynomial is linearly related to 0, 1. Suppose polynomial $C \neq 0$. We have

$$(1/C(x)\ 0\ -C(x))$$

for almost all x. Thus we have

$$(1/C\ 0\ -C). \tag{1}$$

Since C is linearly related to 0,1 as we have just shown, $-C$ is linearly related to 0, -1. Hence in view of $(-1\ 0\ 1)$, $-C$ is linearly related to 0,1. Hence (1) implies that $1/C$ is linearly related to 0,1.

Now let C be an unrestricted element of $\mathfrak{R}'$ and express C as $C_1 \cdot C_2$ where C_1 is a polynomial and C_2 is the reciprocal of a polynomial. C_1 is linearly related to 0, 1. Hence by I2, $C = C_1 \cdot C_2$ is linearly related to 0, C_2. Hence, since C_2 is linearly related to 0, 1, C is likewise. Thus the theorem is established for $A = 0$, $B = 1$.

To complete the proof we observe that the assertion C is linearly related to A, B is equivalent by I1, I2 to the assertion $(C-A)/(B-A)$ is linearly related to 0, 1.

7. Partially ordered geometries. We now wish to clarify the idea *partially ordered geometry, i.e.* a generalization of the familiar idea of ordered linear geometry (*e.g.* euclidean, hyperbolic or ordered affine geometry) in which the points on a line are not simply ordered. We are confronted by a difficulty at the outset: How shall we define *line*?

In the familiar theory line ab is defined as the set of points consisting of the distinct points a, b and all points which bear an order relation to a, b, *i.e.* all points x satisfying one of the relations (xab), (axb), (abx). This definition is unsatisfactory for our purpose having been designed for a theory in which a line is a *simply* ordered set. However a line in the familiar theory does have the property of containing all points bearing an order relation to any two of its points. Thus it contains with the distinct points a, b all points *linearly related* to a, b. This suggests the formal definition:

In a partially ordered system, the set of all elements linearly related to the distinct elements a, b is called a *line* and is denoted *line* ab or simply ab.

Now we define a *partially ordered geometry* as a partially ordered set S of elements called *points* which satisfies the following postulates:

G1. (Closure) *If* $a \neq b$ *there exist* x, y *in* S *satisfying* (axb), (aby).

G2. (Triangle Property) *If* a, b, c, *do not colline and* (bcx), (cya) *there exists* z *in* S *satisfying* (azb), (xyz).

G3. (Linearity) *There exists not more than one line containing two given points.*

S is called a *simply ordered geometry* if in addition it satisfies postulate

G4. *A line is simply ordered.*

This is our formulation of the familiar concept of an *ordered linear geometry* which is referred to in the foundations of geometry as a *descriptive geometry.** It is worthy of note that in the familiar formulation of this idea only G1, G2, G3 (or their equivalent) are assumed, but our definition of line is replaced by the more restrictive definition of line mentioned at the beginning of this section.

We now deduce two elementary properties of partially ordered geometries. First we supplement G3 by the

THEOREM. *There is at least one line containing two points* a, b *namely, line* ab.

Proof. If $a \neq b$, G1 implies (axb). Hence the existence of the sequence a, b, x, a implies that a is linearly related to a, b. Thus line ab contains a, and by symmetry it contains b.†

Now we consider the nature of the ordering of a line in a partially ordered geometry.

THEOREM. *A line is linearly ordered.*

Proof. Let p, q, r be points of ab, where $p \neq q$. We must show that r is linearly related to p, q. By the last theorem and G3 we have $ab = pq$. Hence r is contained in pq. Thus r is linearly related to p, q and the proof is complete.

This property is easily shown to be equivalent to G3 and justifies our description of G3 as a *linearity* property.

8. A partially ordered geometry. We now make the ideas of the last section concrete by constructing a partially ordered geometry which is not a simply ordered geometry.

Consider the partially ordered field $\mathfrak{R}'$ of §5. We construct an analytic geometry in $\mathfrak{R}'$, which for simplicity we make two dimensional. Let S be the set of ordered pairs (x_1, x_2) of elements of $\mathfrak{R}'$. We denote (x_1, x_2) by x and use the Greek letters $\alpha, \beta, \lambda, \mu$ for elements of $\mathfrak{R}'$. It is convenient to introduce the vectorial notation for addition of elements of S and multiplication of elements of S by those of $\mathfrak{R}'$:

* See O. Veblen, Foundations of Geometry, in Monographs on Topics of Modern Mathematics, edited by J. W. A. Young, New York, 1915; also see H. S. M. Coxeter, Non-Euclidean Geometry, Toronto, 1942, Chap. 8.

† From this it follows that line ab is the smallest set containing the distinct points a, b and all points bearing an order relation to any two of its points. This is an interesting type of closure property well known in modern algebra. Compare, for example, "subgroup generated by a set of elements."

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2),$$
$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2),$$

where α is in $\mathfrak{R}'$.

Now we define an order relation in S: Let a, b, c be elements of S, where $a \neq b$, such that there exist λ, μ in $\mathfrak{R}'$ which satisfy

$$c = \lambda a + \mu b,^* \qquad \lambda + \mu = 1, \qquad (0\lambda 1), \qquad (0\mu 1).$$

Then we say (acb).

This definition is of course motivated by the familiar "point of division" formula of elementary cartesian geometry, where μ/λ is the ratio in which point b divides segment ac. It is also suggested by Property E of §3. We observe that the condition $(0\mu 1)$ is redundant in view of Property A of §3, but is included for symmetry of expression.

S, with order as defined, is a partially ordered set. B1, B2, B6 are easily verified. To establish B3 suppose (abc), (bcd). By definition we have

$$b = \lambda a + \mu c, \qquad c = \lambda' b + \mu' d, \tag{1}$$

$$\lambda + \mu = \lambda' + \mu' = 1, \qquad (0\lambda 1), \qquad (0\mu 1), \qquad (0\lambda' 1), \qquad (0\mu' 1). \tag{2}$$

Eliminating c in (1) we get

$$(1 - \mu\lambda')b = \lambda a + \mu\mu' d. \tag{3}$$

We show $1 - \mu\lambda' \neq 0$. By Property B of §3 $(0\,\mu\mu'\,1)$. Hence by Property D we have

$$(0\,\lambda\,\lambda + \mu\mu'), \tag{4}$$

so that

$$1 - \mu\lambda' = 1 - \mu(1 - \mu') = \lambda + \mu\mu' \neq 0 \tag{5}$$

follows. Thus we may solve (3) for b getting

$$b = \alpha a + \beta d \tag{6}$$

where α, β are determined by

$$\alpha = \lambda/(\lambda + \mu\mu'), \qquad \beta = \mu\mu'/(\lambda + \mu\mu'). \tag{7}$$

Clearly $\alpha + \beta = 1$ and (4) implies $(0\alpha 1)$. The supposition $a = d$ implies by (6) $b = d$, contrary to (bcd). Hence $a \neq d$ and we may assert (abd). Similarly (acd) holds. Thus B3 is verified in S. An analogous proof holds for B4. Thus S is a partially ordered set.

We now show that S is a partially ordered *geometry*. G1 is easily verified. To show G2 valid suppose a, b, c do not colline and (bcx), (cya). We seek z satisfying (azb), (xyz). We have

$$c = \lambda' x + \mu' b, \qquad y = \lambda a + \mu c, \tag{8}$$

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* This is really shorthand for $c_i = \lambda a_i + \mu b_i$—the vector notation permits us to drop subscripts.

where the coefficients satisfy (2). Eliminating c in (8) we have

$$y = \mu\lambda' x + (\lambda a + \mu\mu' b). \tag{9}$$

Let z be determined by

$$(1 - \mu\lambda')z = \lambda a + \mu\mu' b. \tag{10}$$

Since (5) holds as in the verification of B3, z is well defined and

$$z = \alpha a + \beta b$$

where α, β are determined by (7). The conclusion (azb) easily follows, $a = b$ being precluded by the non-collinearity of a, b, c. Eliminating $\lambda a + \mu\mu' b$ between (9), (10) we get

$$y = \mu\lambda' x + (1 - \mu\lambda')z.$$

The supposition $x = z$ implies $x = y$. Hence (bcx), (cya) imply by B3 (bca), contrary to the non-collinearity of a, b, c. Hence $x \neq z$ and (xyz) easily follows. This completes the verification of G2.

Before considering G3 we derive the

LEMMA. Let

$$x_i = \lambda_i a + \mu_i b, \qquad \lambda_i + \mu_i = 1, \qquad (i = 1, 2, 3)$$

where a, b are distinct elements of S. Then $(x_1x_2x_3)$ if and only if $(\lambda_1\lambda_2\lambda_3)$.

Proof. Suppose $(\lambda_1\lambda_2\lambda_3)$. By Property E of §3, there exist α, β in $\mathfrak{R}'$ such that

$$\lambda_2 = \alpha\lambda_1 + \beta\lambda_3, \qquad \alpha + \beta = 1, \qquad (0\alpha 1), \qquad (0\beta 1).$$

Hence

$$\begin{aligned} x_2 &= \lambda_2 a + \mu_2 b = \lambda_2(a - b) + b = (\alpha\lambda_1 + \beta\lambda_3)(a - b) + b \\ &= \alpha(\lambda_1(a - b) + b) + \beta(\lambda_3(a - b) + b) = \alpha x_1 + \beta x_3. \end{aligned}$$

It is easily seen that $x_1 \neq x_3$, so that $(x_1x_2x_3)$ holds. The converse is proved similarly.

To verify G3 we must identify *line* in S. This we do in the following

THEOREM. *Line ab, where $a \neq b$, is the set of all x in the form*

$$x = \lambda a + \mu b, \qquad \lambda + \mu = 1, \tag{11}$$

where λ, μ are variables in $\mathfrak{R}'$.

Proof. Suppose x is in line ab. By definition, x is linearly related to a, b. Hence there exists a sequence

$$x_1 = a, x_2 = b, \cdots, x_n = x$$

such that for $2 < i \leqq n$, each x_i bears an order relation to two preceding terms of the sequence. We show by induction on i, that x_i is expressible in the form (11).

This is obvious for $i=1, 2$. Assuming the proposition for $i<k$, where $2<k\leqq n$, we deduce it for $i=k$. We know that one of the relations

$$(x_k x_l x_m), \qquad (x_l x_k x_m) \tag{12}$$

holds for some l, $m<k$. Solving in (12) for x_k in terms of x_l, x_m we get a relation of the form

$$x_k = \lambda' x_l + \mu' x_m, \qquad \lambda' + \mu' = 1. \tag{13}$$

The hypothesis of the induction implies

$$x_l = \lambda_l a + \mu_l b, \qquad \lambda_l + \mu_l = 1, \tag{14}$$

$$x_m = \lambda_m a + \mu_m b, \qquad \lambda_m + \mu_m = 1. \tag{15}$$

Eliminating x_l, x_m between (13), (14), (15) we get an expression for x_k of the form (11). Thus the proposition is true by induction, and $x=x_n$ is expressible in the form (11).

Conversely we show that any x in the form (11) is in line ab. Suppose x expressed in form (11). Since $\mathfrak{R}'$ is linearly ordered (§6), λ is linearly related to 1, 0. Hence there exists a sequence of elements of $\mathfrak{R}'$

$$\lambda_1 = 1, \lambda_2 = 0, \cdots, \lambda_n = \lambda$$

in which each term after the second bears an order relation to two preceding terms. Let x_i be determined by

$$x_i = \lambda_i a + \mu_i b, \qquad \lambda_i + \mu_i = 1 \qquad (i = 1, \cdots, n).$$

Observe that $x_1=a$, $x_2=b$, $x_n=x$. In view of the Lemma, each term of the sequence $x_1, x_2, \cdots, x_n$ after the second bears an order relation to two preceding terms. Thus by definition, x is linearly related to a, b and belongs to line ab.

This result is quite interesting. For, considering λ as a parameter, the form (11) is a parametric equation for the "cartesian line" containing points a, b in the analytic geometry of the (unordered) field $\mathfrak{R}'$. Thus the lines determined in S by our theory of order are precisely the familiar "cartesian lines" defined by linear equations in the two dimensional analytic geometry of $\mathfrak{R}'$. Since it is well known that G3 holds for such "lines," G3 is valid in S and S is a *partially ordered geometry*.

To see whether S is a simply ordered geometry we consider the ordering of the lines of S. Relation (11) establishes a (1-1) correspondence between the set of all λ in $\mathfrak{R}'$ and the set of all x in line ab. In view of the Lemma this correspondence preserves the order relation in both directions. Thus we may say that line ab is *isomorphic* to $\mathfrak{R}'$ with respect to order. Hence all the lines of S are isomorphic as partially ordered sets and are non-simply ordered. Thus we may complete our discussion with the

THEOREM. *S is a partially but not simply ordered geometry.*

DISCUSSIONS AND NOTES

EDITED BY E. F. BECKENBACH, University of California, Los Angeles 24, Calif.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

PAN-MAGIC SQUARES OF THE FOURTH ORDER

E. G. ALLEN, West Newton, Mass.

It is well known* that there are only three essentially different pan-magic (or Nasik, or diabolic) squares of the fourth order:

1	8	10	15
12	13	3	6
7	2	16	9
14	11	5	4

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

All others can be derived from these by reflections, rotations and (cyclic) translations.

It does not seem to have been noticed before that the four neighbors of any particular number are the same in all these three instances, and consequently in all the 384 pan-magic squares; *e.g.*, the number 1 is always surrounded by 8, 12, 14, 15, and the number 16 is always surrounded by 2, 3, 5, 9.

An easy method for finding these associated numbers is as follows. Form a square with the numbers from 1 to 16 in natural order, thus:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

For any given number n, locate the complementary number $17-n$. (This is always symmetrically opposite the given number.) Then take the four numbers that lie in the same row and in the same column as the complementary number but not in the same row or column as the given number. These four are the associated numbers. Thus for the given number 12 we consider the complementary number 5. The numbers in the same row as 5 but not in the same column as 12

* See e.g., M. Kraïtchik, La Mathématique des Jeux, Brussels, 1930, p. 167 (Fig. 181–183). Note that Kraïtchik made some errors in copying these squares into the English version, Mathematical Recreations, New York, 1942, p. 191.

are 6 and 7. The numbers in the same column as 5 but not in the same row as 12 are 1 and 13. Hence the numbers associated with 12 are 1, 6, 7, 13.

A given number in any selected cell, and its four proper neighbors in any one of their 4! possible permutations, determine the entire square. For, the remaining cells can be filled by observing that diagonally alternate numbers must be complementary, and that the four numbers at the corners must add to 34, like the rows, columns, and diagonals. Since the given number may be placed in any one of the 16 cells, while the four associated numbers may be permuted in 24 ways, we obtain in this manner all the $16 \cdot 24 = 384$ pan-magic squares.

CONCERNING A CERTAIN CONGRUENCE

BEN DUSHNIK, University of Michigan

1. Recently* Erdös and Niven pointed out that if the positive integers $x=1, y, z$ satisfy the congruence

$$x^3 + y^3 + z^3 \equiv 0 \pmod{xyz}, \tag{1}$$

then the integers $1, y, (y^3+1)/z$ will also satisfy (1). The derivation of this was not indicated. In this note we state and prove a theorem from which the above fact will follow as a simple corollary. Throughout we may suppose, without loss of generality, that a solution (x, y, z) of the congruence consists of positive integers which are mutually prime.

2. THEOREM 1. *If* (x, y, z) *is a solution of* (1), *and if* u *is any factor of* x^3+y^3 *such that*

$$u^3 \equiv z^3 \pmod{xy}, \tag{2}$$

then (x, y, u) *will satisfy* (1).

Proof. It is readily seen that (1) is equivalent to the system of congruences

$$x^3 + y^3 \equiv 0 \pmod{z},$$

$$x^3 + z^3 \equiv 0 \pmod{y}, \tag{3}$$

$$y^3 + z^3 \equiv 0 \pmod{x}. \tag{4}$$

From (2) combined with (3) and (4), respectively, we get

$$x^3 + u^3 \equiv 0 \pmod{y}, \tag{5}$$

$$y^3 + u^3 \equiv 0 \pmod{x}. \tag{6}$$

By hypothesis, we have

$$x^3 + y^3 \equiv 0 \pmod{u}. \tag{7}$$

Since u is clearly prime to both x and y, we get, from (5), (6), and (7),

$$x^3 + y^3 + u^3 \equiv 0 \pmod{xyu}.$$

* This MONTHLY, vol. 53, 1946, pp. 223–224.

3. To use the above theorem as a "practical" means of deriving new solutions of (1) from a given solution is not feasible, since the theorem asserts nothing about the existence of a number $u \neq z$ with the required property. There are, however, at least two cases where such a number u always exists. One is the case mentioned by Erdös and Niven; indeed, if we put

$$u = \frac{y^3 + 1}{z},$$

then u is a factor of y^3+1, and it is easy to compute that

$$u^3 \equiv z^3 \pmod{y}.$$

Another case is given by the following theorem:

THEOREM 2. *Let* $(1, y, z)$ *satisfy* (1), *and put* $u = y^2+z^2-yz$; *then* (u, y, z) *will also satisfy* (1).

Proof. By the theorem in 2, it is sufficient to show that

$$u^3 \equiv 1 \pmod{yz}, \tag{8}$$

since u is a factor of y^3+z^3. Now, $1+y^3+z^3 \equiv 0 \pmod{yz}$ implies that

$$(y^3 + z^3)^2 \equiv 1 \pmod{yz}. \tag{9}$$

Moreover,

$$(y^3 + z^3)^2 \equiv y^6 + z^6 \equiv (y^2 - yz + z^2)^3 \pmod{yz}. \tag{10}$$

Combining (9) and (10), we obtain (8).

We might add that if $1 \leqq y < z$, then $z < u$, so that (u, y, z) is a new solution.

4. Starting with the solution (1, 1, 2) of (1), we can get, by the remark of Erdös and Niven, the infinite set of different solutions

$$(1, 2, 3), (1, 3, 14), (1, 14, 915), \cdots .$$

From these last we can now get still another infinite set

$$(2, 3, 7), (3, 14, 163), (14, 915, 824611), \cdots .$$

We can thus state that (1) has infinitely many essentially different solutions (x, y, z) such that none of the numbers x, y, z is 1.

Finally, there are solutions of (1) which are not so obtainable, such as:

$$(3, 11, 679), (5, 77, 11138), (19, 26, 905),$$

and many others (obtained in a rather haphazard fashion).

ON THE FEUERBACH POINTS

R. GOORMAGHTIGH, Bruges, Belgium

1. The construction given by S. M. Karmelkar, this MONTHLY, vol. 53, 1946, p. 206, for the in-Feuerbach point Z of a triangle is another form of Mannheim's well known theorem (Bulletin des Sciences mathématiques, 1902, p. 95).

Let, as in the recalled Note, D be the mid-point of BC, P the foot of the altitude from A, I the in-center, L its projection on BC, X the point where the perpendicular bisector of BC meets PI, and Y the point where DI cuts AP.

According to Mannheim's theorem, if T is the image of L in I, the join of T to the mid-point A' of AI passes through Z. But as $AY = IL$, TA' passes through Y; further $TI = IL$, and XD is perpendicular to BC, so this proves that the line XY in the considered construction coincides with the Mannheim line TA'.

2. The fact that the point Y obtained by taking AY equipollent to IL belongs to the Mannheim line corresponding to BC may be extended to the case of the orthopole of any circumdiameter d of ABC, according to the following generalization of Mannheim's theorem which I have given in the Journal de Mathématiques élémentaires (Vuibert), 1913–1914, p. 69:

If Q' is the counter-point of any point Q on d and if the pedal circle $Q_a' Q_b' Q_c'$ of Q' meets again $Q_a' Q'$ at S, then the straight line joining S to the mid-point A'' of AQ' passes through the orthopole of d.

It follows that $A''S$ cuts the altitude AP at a point Y' such that $AY' = SQ' = QQ_a$, Q_a being the projection of Q on BC.

When d is the circumdiameter passing through I, the orthopole of d is the in-Feuerbach point.

3. If Q and Q' are such that QQ' is parallel to BC, these points lie on a cubic Γ, anallagmatic in triangular inversion and having as normal equation

$$cy(x^2 - z^2) + bz(x^2 - y^2) = 0.$$

In that case $Q'S = Q_a' Q'$ and we find the following generalization of Karmelkar's construction:

Let Q be a point on the cubic Γ, Q' its counter-point, Q_a and Q_a' the projections of Q and Q' on BC, Y' the point on the altitude AP such that $AY' = QQ_a$, D' the intersection of $Y'Q'$ with BC, X' the point where the perpendicular at D' to BC meets PQ'; then the projection Z' of Q_a on $X'Y'$ is the orthopole of the circumdiameter passing through Q.

4. This proves that the considered construction applies also to the outer-Feuerbach points.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

A Trigonometric Approximation

A paper on the calculation of the trigonometric ratios printed privately (Elm Press, Calcutta) by Jatis Chandra Ray, a practising neurologist of Calcutta, has been brought to this editor's attention. Although expressed in somewhat different notation, his formulas for calculating the trigonometric ratios for fractions of a right angle are equivalent to the approximation

$$\tan\frac{\pi}{2}x \sim \frac{x}{1-x}\left(\frac{5-3x}{2+3x}\right)^{1/2} \tag{1}$$

and the corresponding formulas immediately derivable for $\sin x$ and $\cos x$. The approximations are exact for angles of 0°, 30°, 45°, 60°, 90°—that is, for $x=0$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, 1—and are not changed by replacing an angle by its complement. The slope in (1) for $x=0$ is $\frac{1}{2}\sqrt{10}$ instead of $\frac{1}{2}\pi$, and for $x=\frac{1}{2}$ is 22/7 instead of π. For angles between 0° and 45° the maximum error is about 0.00048 (near sin 7.5°).

Ray gives an improved approximation by multiplying the exponent $\frac{1}{2}$ in (1) by the factor $1+(3-2x)(2-3x)(3x-1)/420$, but states erroneously that this is "mathematically accurate." The error may be much smaller, but no such finite expression for $\tan(\pi x/2)$ could be exact. With this exponent, the slope at the origin is reduced to 1.569 instead of $\pi/2$.

It may be of interest to note that the approximation

$$\sin\frac{\pi}{2}x \sim x\,\frac{11-3x^2}{7+x^2} \tag{2}$$

gives a better fit for angles less than 30° than Ray's first approximation, and does not involve the extraction of a square root.

CLUB REPORTS 1945–46

Pi Mu Epsilon, Hunter College

Under the directorship of Dr. Annita Tuller, papers were presented on important aspects of geometry.

Geometric points of view, including: *Transformation group point of view, Metric, Affine, Projective geometry, Circular transformations, Linear fractional transformations of a complex variable, Introduction to the study of geometry from the metric point of view.*

Geometric fields of study, including: *Topology*, *Differential geometry of plane curves*, *Space curves*, and *Surfaces*, *Algebraic geometry*.

The following undergraduates contributed to the discussions: Chang Jung, Rae Adelson, Mary Rosania, Edith Weiss, Theresa Danielson, Florence Jaffe, Lorraine Kruglov, Rita Gross, Mary Greene, Carol Podell, Anne Stern, Leila Rubashkin, Josephine Rodstrom, and Ruth Weihrauch.

At initiation meetings in Hunter's North Lounge, 22 members were initiated during the year. The topic at the first of these was

The isograph, by Mr. Robert Dietzold of Bell Laboratories. The second initiation on March 31 was a part of the first meeting of the Metropolitan Intercollegiate Convention, sponsored by the chapters of Hunter College (*Beta*), Brooklyn College (*Gamma*), and New York University (*Delta*), and attended by approximately 200 faculty and students from colleges throughout the city. Papers presented were

Topics in Mathematics, by Professor Edward Kasner of Columbia

Applications of the vector operator del, by Professor Carel van der Merwe of New York University.

Lorraine Kruglov received a copy of Klein's *Elementary Mathematics from an Advanced Standpoint* for presenting the best paper of the year.

A theater party was organized between semesters.

Officers for 1945–46 were: President, Mary Greene; Corresponding Secretary, Sally Rothstein; Recording Secretary, Carol Podell; Treasurer, Leila Rubashkin. Newly elected officers for 1946–47 are: President, Cecile Salwen; Corresponding Secretary, Annette Drucker; Recording Secretary, Ruth Friedman; Treasurer, Mildred Vogt. Professor Lucille Anderson will be Director.

Mathematics Club, Hunter College

An active program included a talk by Professor Edward Kasner, and three other faculty talks

The role of curves in the history of mathematics, by Professor Carl B. Boyer of Brooklyn College

Finite geometry, Professor Jewell Hughes Bushey

The relation of mathematics to the sciences, by Professor Bushey. Reports were given by students on the topics

Transcendentals and imaginaries

Chinese mathematics

Paradoxes

Fun with eight numbers

Polar coordinates

Hyperbolic functions

Vector analysis

Social activities consisted of a Christmas party and a Spring party.

Officers were: President, Sally Rothstein; Vice-President, Dorothy Beck; Treasurer, Cecile Cohen; Secretary, Wilhelmina Fluhr; Publicity Chairman, Ina Pinchuk. The Faculty Adviser was Miss Isabel McLaughlin.

Pi Mu Epsilon, Louisiana State University

The schedule of events included an organization meeting, two initiation meetings, a picnic, and the following program meetings:

Radar, by Dr. L. W. Morris
Projectiles in war, by Dr. Paul White
Prominent mathematicians, by Newton Hawley
Lengths and areas, by Dr. H. L. Smith
Teaching G.I.'s in England, Dr. N. E. Rutt
Empirical probability, by Dr. Peter Carmichael
The problem of the infinity in the eighteenth century, by Dr. Eric Voegelin
Inversion, by Zoe Ellen Sanders.

Winners of the two $15 *Pi Mu Epsilon* Awards were: Newton Hawley, the senior having the best mathematical record from the standpoint of quality of work done and quantity of work taken; and Mary Ellen Rickey, the freshman making the highest score in a competitive examination. The examination committee consisted of Elizabeth Cathey, chairman, Miss Yvonne Jones, Charles McCleskey, Katherine Formusa.

Several new books were added to the *Pi Mu Epsilon* collection of histories, biographies, and popular titles given to the Mathematics Library.

Officers and committee chairmen for 1945–46 were: President and Membership Chairman, Lollie Belle Bienvenu; Vice-President and Program Chairman, Newton Hawley; Secretary, Katherine Formusa; Treasurer and Entertainment Chairman, Lorraine Schertz. Officers for 1946–47 are: President, Ben E. Mitchell; Vice-President, Charles McCleskey; Secretary, Beverly Jean Russell; Treasurer, Kathryn Jumonville. Dr. Houston Karnes served both years as Corresponding Secretary.

Kappa Mu Epsilon, College of St. Francis

The Illinois Delta Chapter of *Kappa Mu Epsilon* was installed at the College of St. Francis, Joliet, on May 22, 1945 by Professor Urbancek of the Illinois Gamma Chapter. Meetings were held on the second Thursday of every month. Topics chosen for discussion were

Zeno's paradoxes
Modern geometry
The discriminant of the quadratic, cubic, and quartic
Applications of the differential calculus
The disciplinary, utilitarian, and cultural aims of the teaching of mathematics

Officers for 1945–46 were President, Lillian Koss; Vice-President, Helena Weigand; Secretary, Margaret Markiewicz; Treasurer, Jeanne Erlenborn; Corresponding Secretary and Faculty Sponsor, Sister M. Claudia Zeller. Six new members were initiated and the following new officers were installed on May 9, 1946: President, Betty Fiene, Vice-President, Marian Masters; Secretary, Betty Lanoue; Treasurer, Patricia Young.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Sequential Analysis of Statistical Data: Applications. Prepared by the Statistical Research Group, Columbia University. Columbia University Press, 1945. $6.25.

Sequential analysis is an important new statistical technique developed initially by A. Wald in 1943 for use in war research. Its basis may be illustrated for non-statisticians by considering the first application described in this book.

In the process known as acceptance sampling, a random sample from a batch of manufactured articles is inspected in order to determine whether the batch is acceptable. Let us suppose that each article in the sample, after inspection or testing, may be classified as "good" or "bad." In an early type of sampling plan, with which sequential sampling will be contrasted, N articles are inspected. The batch is accepted if b or less of these articles are bad and is rejected otherwise. How effective is this sampling plan? This may be seen from a simple formula. If p is the unknown proportion of bad articles in the batch and if the batch is many times larger than N, the probability that the batch is accepted will be given, to a close approximation, by the binomial summation:

$$\sum_{x=0}^{b} \frac{N!}{x!(N-x)!} p^x(1-p)^{N-x}.$$

As p increases from 0 to 1, the probability of acceptance, as given by this formula, decreases steadily from 1 to 0. Nevertheless, and this is true of any sampling plan, some batches with very low p will be rejected and some with relatively high p will be accepted. By choice of N and b, however, a partial control can be obtained on these two types of error. In practice, it is customary to fix two points on the curve. That is, N and b are calculated so that the probability of rejecting a batch with proportion p_1 does not exceed α, while the probability of accepting a batch with proportion p_2 does not exceed β. The quantities p_1, p_2, α, and β are of course selected by the inspector. Since p_1 usually corresponds to a proportion which it is highly desirable to accept and p_2 to one which it is highly desirable to reject, α and β are usually made small.

For many manufacturing processes involved in war production, inspection was both costly and destructive. It was therefore exigent to reduce the inspection to a minimum. In this respect the plan described above is somewhat wasteful, as had been realized previously. For example, with a poor batch, more than b bad articles might appear long before the total N had been inspected. The

sequential method was developed in response to this need for minimum inspection.

With the sequential method, two fundamental changes appear. First, instead of waiting for a decision until all N articles have been inspected, the results to date are examined as each successive article is inspected. From the examination one of three decisions is made: (i) to accept the batch, (ii) to reject the batch, (iii) to continue further inspection. The sampling terminates when a decision for acceptance or rejection is reached. The second important change, which is a corollary of the first, is that the total size of sample N is no longer fixed, but is a random variable depending on the quality of the batch. In general, if p is very low or very high, a decision will be reached quickly. With a borderline p larger samples are needed.

The mathematical problem to be solved in constructing a sequential method is that of developing the rules for decisions (i), (ii) and (iii) above. These rules must satisfy the two "control" conditions mentioned previously, namely that the probability of rejecting a batch with proportion p_1 must not exceed α, and the probability of accepting a batch with proportion p_2 must not exceed β. Further, for the present purpose, the rules must be such that the average amount of inspection needed is a minimum. The rules found by Wald to satisfy these conditions are surprisingly simple. If at any state n articles have been inspected and d are bad, the batch is accepted if

$$\frac{p_1^d(1-p_1)^{n-d}}{p_2^d(1-p_2)^{n-d}} \geqq \frac{1-\alpha}{\beta}$$

and rejected if this quantity is less than or equal to $\alpha/(1-\beta)$. If the quantity falls between these limits, sampling is continued. On taking logarithms the rules become still simpler for practical application, since it is clear that they may be reduced to the following: Accept if $d \leqq -h_1+sn$; Reject if $d \geqq h_2+sn$, where h_1, h_2, and s are simple functions of p_1, p_2, α and β. Consequently the inspector may either apply to a table showing the decisive values of d for given n, or plot the points (d, n) on a graph containing the two decisive parallel lines.

Sequential analysis may be applied to a wide variety of problems in which one of two courses of action is chosen on the evidence provided by a sample of data. Other applications described in this volume are to the choice between two batches or methods according to the proportions of bad articles which they produce, to the acceptance or rejection of batches where quality is measured on a continuous scale, and to the question whether the variability within a batch, as measured by the standard deviation, meets acceptable standards. In each case the principal object is to reduce the average amount of data necessary for a decision with controlled risks of error. While little definite information is given about the actual size of this reduction, values of 40 to 60 percent appear to be attained in certain cases as compared with sampling where N is fixed in advance.

The present volume comprises seven separate pamphlets, largely self-contained. Apart from the first, which is introductory, and the last, which gives certain formulae common to all applications, each pamphlet deals with a specific application. The publication is evidently directed at potential users of the methods who are expert neither in mathematics nor statistics. The rules for using the method are stripped of all mathematical complexities and supplementary tables are given wherever helpful. No attempt is made to present details of the theory, though the common-sense justification for the procedures is included. The mathematician who wishes to learn about sequential analysis will probably find it more profitable to begin with Wald's paper "Sequential tests of statistical hypotheses," Annals of Mathematical Statistics, Vol. 16, part 2, 1945. Thereafter, he may be interested in perusing this volume in order to see how the authors have tackled the many problems involved in putting the technique into the hands of the non-mathematician.

W. G. COCHRAN

Lectures on Probability and Statistics. By E. L. Dodd. Austin, Texas, The University of Texas Press, 1945, 44 pages. No charge.

This includes first a short biographical sketch of Professor Dodd, with a bibliography. Then there are three of his papers, the first two being lectures which were found in manuscript form among his effects, and the third being a copy of a paper sent to the Seventh American Scientific Congress, Mexico City, in 1935. The two lectures are elementary in nature and describe with great clarity various kinds of means and their properties. The last paper, entitled "A survey of statistical means or averages," is the longer and more important part of this brochure. Needless to say, the survey is authoritative, coming from one who had spent much of his mathematical life studying and writing about means. "In general analysis a mean has always signified a number not less than the least number (greatest lower bound) of a set and not greater than the greatest (least upper bound). As regards means used for statistical purposes, however, the notion of intermediacy was originally not fundamental—in all probability. Means arose, more likely, in connection with certain problems requiring a substitution of a single number for a set of numbers. O. Chisini and others have stressed the notion of a substitution, setting forth the equation

$$F(M, M, \cdots, M) = F(x_1, x_2, \cdots, x_n)$$

to define the mean M of $x_1, x_2, \cdots x_n$." For example, if we want to define the arithmetical mean we let F be the sum of the numbers in the parenthesis following it. That is, we can say that the arithmetical mean is one number substituted for each of n numbers, and is such that the sum of these n numbers is not changed by the substitution. Likewise, if we want to define the geometric mean we let F indicate the product of the numbers following it, and then the product is not changed by the substitution. "When M is expressed explicitly as

$$M = f(x_1, x_2, \cdots, x_n),$$

it turns out that the only restriction placed upon f is

$$f(x_1, x_2, \cdots, x_n) = x.$$

Thus, a *statistical* mean need not be intermediate." Many useful means result from this definition. Many means involve summation and the summation may be generalized into an integration. The survey paper also contains a section which expresses a general definition of a statistical mean in the language of point sets.

B. H. CAMP

Elementary Applied Aerodynamics. By P. E. Hemke. New York, Prentice-Hall, Inc., 1946. 7+231. pages. $3.25.

The author succeeds very well in carrying out the objectives announced in his preface: "*Elementary Applied Aerodynamics* has been written to meet the requirements of a first course in technical applied aerodynamics. Such a course is usually given to students who have had preliminary training in mathematics, physics, and mechanics. An adequate preparation in these subjects eliminates the need for an introductory course that may have been simplified to such an extent that it loses its effectiveness and fails to challenge the student. . . . The material is presented in comparatively simple fashion, but this treatment is not merely descriptive when a fundamentally sound analytical treatment is desirable. Illustrative examples are freely used and fully worked out. . . . Numerous problems are included at the end of each chapter."

The table of contents is as follows: 1. Units and Dimensions; 2. Physical Properties of Air; 3. Flow of an Ideal Fluid; 4. Lift and Moment of an Airfoil in an Ideal, Two-Dimensional Flow; 5. Three-Dimensional Flow Around an Airfoil, Induced Effects; 6. Viscosity Effects; 7. Airfoils and Their Properties; 8. Compressibility Effect on Airfoil Characteristics; 9. Propellers; 10. Performance of a Conventional Airplane; 11. Helicopters. Chapter 1 and 2 give the usual discussion of dimensional analysis, pressure and density variation with altitude, viscosity of air, and compressibility of air. They are noteworthy for containing numerical values and units for the standard constants and conversion factors needed to solve problems later in the text; in most texts the values are scattered throughout the text and difficult to locate or else are not given and must be derived before solving the problem.

Chapters 3 and 4 develop the two-dimensional ideal flow for airfoils with particular emphasis on the stream function. The diagrams showing chordwise distributions are helpful. The introduction of the idea of drag is somewhat confused while the discussion of moment and aerodynamic center seems longer and more involved than necessary. Mathematically, the substitution to integrate the integral at the bottom of page 44 needs justification since it involves division by zero at the lower limit.

The discussion of three-dimensional flow, induced effects and viscosity effects in chapters 5 and 6 are above average. The laminar and turbulent boundary layers are examined in detail. Chapter 7 describes some of the characteristics of standard NACA airfoils and gives a good treatment of the flap effects on airfoil characteristics. However, no mention is made of the effects of slots, spoilers, dive brakes, boundary layer control devices, and sweep back on airfoil characteristics. Chapter 8 gives an excellent introduction for the beginner to the subject of compressibility effects on airfoils. Naturally, only the essential outline of the problem of compressibility could be given, for this problem is and will continue to be for a long time the subject of intense study and research by the wind tunnels and aerodynamists of the world.

Chapter 9 outlines the basic procedures in determining propeller characteristics and points out the essential differences between the performance of fixed-pitch and constant-speed propellers. Chapter 10 considers the simplified conventional performance calculations of a conventional airplane, covering take-off time and distance, rate of climb, minimum and maximum flight speeds, range and endurance, landing speed and landing run. Chapter 11 sketches the performance problems of helicopters, discussing hovering and forward flight, and indicating the effects of blade flapping.

Although the book is not difficult to read, there are a considerable number of typographical errors, particularly subscripts being printed level with the symbol. Some of these would prove confusing to the beginner. Also, one misses the smooth diction and well connected explanations of the best texts and is reminded of a lecturer's notes. Nevertheless, the book will undoubtedly take its place as one of the best at the present time for the beginner in aerodynamics.

B. E. GATEWOOD

NEW BOOKS RECEIVED

Calculus. Second Edition. By F. H. Miller. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1946. 16+416 pages. $3.50.

Courbes Algébriques sur Corps Germés de Caractéristique Quelconque. (Acta Salmanticensia, Serie de Ciencias, Seccion de Matematicas, No. 1). By G. Anocchea, Madrid, Univ. de Salamanca, 1946. 39 pages. 10 ptas.

The Development of Trigonometry from Regiomontanus to Pitiscus. (Dissertation, University of Michigan, 1944). By Sister Mary Claudia Zeller. Joliet, Ill., Sisters of St. Francis of Mary Immaculate, 1946. 6+119 pages.

Die Mathematische Denkweise. By A Speiser. Basel Birkhäuser, 1945. 132 pages. s. fr. 14.50.

Les Espaces Normaux. By M. Hachtroudi. Université de Téhéran, 1945. 83 pages.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 736. *Proposed by Paul Erdös, Stanford University*

Let $a_1<a_2<\cdots<a_k\leqq n$, where $k>[(n+1)/2]$, be k positive integers. Then $a_i+a_j=a_r$ is solvable. (*Cf.* No. 3739 [1937, 120]).

E 737. *Proposed by V. L. Klee, Jr., University of Virginia*

Establish the divergence of the series $\sum_{n=1}^{\infty} n^{-a}\cos(b\log n)$ for all values of $a\leqq 1$, regardless of the value of b.

E 738. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that the four spheres passing through the Monge point and the nine point circles of the faces of a tetrahedron are equal to each other.

E 739. *Proposed by L. M. Kelly, University of Missouri*

An ellipse inscribed in the triangle ABC is tangent to AB at D. Show that the midpoints of CD and AB are collinear with the center of the ellipse.

E 740. *Proposed by Esther Szekeres, Shanghai, China*

Let there be given five points in the plane. Prove that we can select four of them which determine a convex quadrilateral.

SOLUTION

Overlapping Progressions

E 696 [1945, 578]. *Proposed by C. H. Wolfe, Lakeside High School, Ohio*

Find four positive integers a, b, c, d such that a, b, c are in geometric progression, b, c, d in arithmetic progression, and $c+d=44$.

Solution by Maud Willey, Denton, Texas. Solutions must satisfy the equations (1) $ac=b^2$, (2) $b+d=2c$, (3) $c+d=44$. Eliminating b and d we obtain $ac=9c^2-264c+1936$. Hence c divides $1936=2^4 11^2$. But from (3), $c<44$, and from (2) and (3), $c>14$. The possible values of c are thus 16 and 22, giving the solutions 1, 4, 16, 28 and 22, 22, 22, 22.

If the restriction that the integers be positive be lifted, there are thirty solutions in all, corresponding to the thirty factors of 1936. In particular, $c=44$

leads to the only other solution in non-negative integers, namely 176, 88, 44, 0.

Also solved by F. A. Alfieri, D. W. Alling, Murray Barbour, P. T. Bateman, R. C. Buck, W. E. Buker, H. N. Carleton, P. L. Chessin, R. E. Crane, J. H. Cross, Monte Dernham, William Douglas, S. E. Field, Mary Goldsmith, Sidney Glusman, Herrick Greenleaf, N. G. Gunderson, Francis Hall, R. H. Hoskins, H. K. Humphrey, Aida Kalish, M. Kirk, H. L. Lee, H. R. Leifer, Eric Michalup, W. Nicholson, C. C. Oursler, S. T. Parker, J. V. Pennington, P. A. Piza, P. W. A. Raine, W. C. Rufus, F. W. Saunders, E. D. Schell, Nathan Schwid, M. J. Sheehy, E. P. Starke, P. D. Thomas, J. A. Tierney, F. Underwood, R. H. Urbano, Hazel Wilson, and the proposer. Monte Dernham suggested the more difficult associated problem: *Replace* 44 *by n. For what values of n are there exactly m solutions?*

Turning an Automobile

E 701 [1946, 36]. *Proposed by N. S. Mendelsohn, Queen's University*

Suppose the distance between the front and rear wheels of a car is b, and the distance between wheels from side to side is a. The right front wheel is turned through a counterclockwise angle θ from its neutral position. Through what angle must the left front wheel be turned so that in the car's subsequent motion all four wheels will roll without slipping? Describe the motion.

I. *Solution by Herbert Reisman, Illinois Institute of Technology.* If the rear axle and the axle of the right front wheel are extended, they will meet at a point P, the instantaneous center of rotation of the car. To prevent slipping, the axle of the left front wheel must also pass through P. Denoting the counterclockwise angle of the left front wheel, measured from its neutral position, by ϕ, it is easily shown that the angle between the rear axle and the left front wheel axle is also equal to ϕ. Denoting the distance from the left rear wheel to the point P by x, the following relationships are obtained:

$$\tan \phi = b/x, \qquad \tan \theta = b/(a + x).$$

Eliminating x and solving for ϕ, we find

$$\phi = \arctan [b/(b \cot \theta - a)].$$

The car will describe a circle about point P.

II. *Solution by Frank Hawthorne, Columbia University.* Since there is to be no slipping, the direction of motion of the center of each wheel must be in the plane of the wheel. If $\theta \neq 0$, the perpendiculars to the planes of the wheels at their respective centers must be concurrent in a point P which is the center of rotation for the car as a whole. P lies on the line of the rear axle, for the rear wheels are normally fixed (except for rotation) with respect to the automobile. In the following we shall consider values of θ between 0° and 90°, although such a range is more than is normally provided.

Case 1. $0 < \theta < \arctan b/a$. P is to the left of the left rear wheel. The corre-

sponding angular displacement of the left front wheel is $\phi = \arctan [b/(b \cot \theta - a)]$. The rotation of the car about P is such that the wheels all rotate in the same sense.

Case 2. $\theta = \arctan b/a$. The car rotates about its left rear wheel, and $\phi = 90°$.

Case 3. $\arctan b/a < \theta < \arctan 2b/a$. P lies on the rear axle between the left wheel and the center. The two front wheels and the right rear wheel rotate in the same sense. The left rear wheel rotates in the opposite sense. For this case $\phi = \arctan [(a - b \cot \theta)/b] + 90°$.

Case 4. $\theta = \arctan 2b/a$. P is at the midpoint of the rear axle. No motion will occur in a rear wheel drive car. For a front wheel drive the car would rotate about this point, the wheels turning oppositely.

Case 5. $\arctan 2b/a < \theta < 90°$. P is on the rear axle to the right of the center. The two front wheels and the right rear wheel rotate in the same sense ("reversed" for "ahead" driving) while the left rear wheel rotates in the opposite sense. Here we have $\phi = \arctan [(a - b \cot \theta)/b] + 90°$.

Case 6. $\theta = 90°$. The car rotates about the right rear wheel and $\phi = \arctan a/b + 90°$.

Also solved by E. F. Allen, W. G. Brady, J. H. Cross, Vladimir Karapetoff, W. A. Rees, P. D. Thomas, and the proposer.

The proposer stated that around the year 1920 numerous papers connected with the turning of automobiles appeared, and, at that time, a number of mechanical devices were invented which would automatically cause the front wheels to align themselves properly when turning. The problem was not appreciated in the very earliest days of the automobile, and was not considered in the days before the automobile, as wagons were constructed with their front wheels perpendicular to the axle, which was pivoted at its midpoint.

Karapetoff remarked that, since the two rear wheels follow circles of different radii, peripheral slippage can be prevented only if the rear axle is broken in two by a differential, as is the case with all trucks and automobiles, or if the two rear wheels are mounted on individual center pins, and have no common revolving axle, as in the case of horse-driven vehicles.

Consecutive Cubes with Difference a Square

E 702 [1946, 36]. *Proposed by J. Brenner, Palmer Laboratory, Princeton, N. J.*

The smallest consecutive cubes whose difference is a square are 8^3 and 7^3. What is the next pair?

Solution by E. P. Starke, Rutgers University. By hypothesis we have

$$(n + 1)^3 - n^3 = 3n^2 + 3n + 1 = r^2, \tag{1}$$

which may be written

$$(2r)^2 - 3(2n + 1)^2 = 1. \tag{2}$$

Since $(2^2 - 3 \cdot 1^2)^k = 1$, r_k and n_k will satisfy (2) if

$$2r_k - \sqrt{3}\,(2n_k + 1) = (2 - \sqrt{3})^k,$$
$$2r_k + \sqrt{3}\,(2n_k + 1) = (2 + \sqrt{3})^k,$$

whence

$$4r_k = (2 + \sqrt{3})^k + (2 - \sqrt{3})^k,$$
$$2\sqrt{3}\,(2n_k + 1) = (2 + \sqrt{3})^k - (2 - \sqrt{3})^k, \tag{3}$$

where k is an odd integer if r_k is to be integral. From (3) it is not difficult to obtain

$$n_{k+4} = 14n_{k+2} - n_k + 6,$$

and thence, with $n_1 = 0$, $n_3 = 7$, we compute $n_5 = 104$, $n_7 = 1455$, $n_9 = 20272$, *etc.* It can also be shown without difficulty that this procedure gives all solutions of (1).

It is worth noting that (1) is a relation for the triangle with integral sides n, $n+1$, r in which the angle opposite r is 120°.

Also solved by D. W. Alling, Norman Anning, Murray Barbour, P. T. Bateman, R. C. Buck, W. E. Buker, Monte Dernham, William Douglas, N. G. Gunderson, R. Hamming, H. L. Lee, W. Nicholson, S. T. Parker, Walter Penney, J. V. Pennington, P. A. Piza, W. C. Rufus, Theodore Running, E. D. Schell, Elijah Swift, F. Underwood, R. H. Urbano, B. F. Yanney, and the proposer.

Bateman, Schell, Alan Wayne, and Yanney pointed out that this problem had been solved by C. Richaud, and gave as reference Vol. II, p. 580, of Dickson's *History of the Theory of Numbers.*

Running noted that for the first few solutions r takes on successively the values 13, 181, 2521, 35113, 489061, 6811741. In each of these, r has the form $b^2 + (b-1)^2$. Does r assume this form for every solution in positive whole numbers?

Anning stated that the values of $n+1$ satisfy the recurrence relation

$$u_1 = 1, \quad u_2 = 8, \quad u_3 = 105,$$
$$u_n = 15u_{n-1} - 15u_{n-2} + u_{n-3}, \qquad n = 4, 5, 6, \cdots.$$

Most solvers applied the standard theory of the Pell equation to equation (2) above.

A Square of the Form abbc

E 703 [1946, 36]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In which scales of notation can a four-digit number *abbc* be the square of a two-digit number *mn*, if $c = a + 1$ and $n = m + 1$? (*Cf.* E 652 [1945, 42 and 459]).

Solution by E. P. Starke, Rutgers University. The method of the former problem can be used here. Let r be the radix, so that by hypothesis we have

$$ar^3 + br^2 + br + c = (mr + n)^2, \tag{1}$$

which, with $c = a + 1$ and $n = m + 1$ can be easily reduced to

$$a(r - 2) + b - m^2 = (2m + b - 3a)/(r + 1).$$

Since the right member is an integer, say I, this gives

$$a(r - 2) + b - m^2 = I, \tag{2}$$

$$2m + b - 3a = I(r + 1), \tag{3}$$

and, upon elimination of b,

$$(a + I)(r + 1) = m^2 + 2m + I. \tag{4}$$

Evidently $|I| \leqq 2$. Suppose $I < 0$. From (3) $a > m$, which contradicts (2) because of $m < n < r$. Suppose $I = 2$. Upon solving (4) for r, substituting in (2) and putting $b < r$, we obtain

$$m^2 - 2(a + 1)m + (3a^2 + 7a + 4) < 0,$$

which is impossible. Hence $I = 0$ or 1. From (4) and the hypothesis we note that

$$1 \leqq a < m \leqq r - 2. \tag{5}$$

Let $I = 1$. Then (4) becomes $(a + 1)(r + 1) = (m + 1)^2$, and if $(m + 1)^2 = u^2v^2k^2$, we can put

$$a = u^2k - 1, \qquad r = v^2k - 1, \qquad m = uvk - 1,$$

and, from (3),

$$b = 3u^2k - 2uvk + v^2k - 1,$$

where $a > 0$ and $b < r$ require $uk > 1$, $2v > 3u$. Thus r can be any number (except 3 and 8) such that $r + 1$ is divisible by a square greater than unity.

Let $I = 0$, then from (3) and (4) we have

$$b = 3a - 2m, \qquad a(r + 1) = m^2 + 2m. \tag{6}$$

If now $b = 0$, we have $3a = 2m$, $r - 2 = 3m/2$, whence m is a multiple of 6 and we can put

$$m = 6k, \qquad a = 4k, \qquad r = 9k + 2.$$

If $b = 4$, we have $3a = 2m + 4$, $r + 1 = 3m/2$, whence $m + 2$ is a multiple of 6 and we can put

$$m = 6k + 4, \qquad a = 4k + 4, \qquad r = 9k + 5.$$

Other values of b are possible but for each there is at most a finite number of values of r, and no general formula appears. However, from (6) it is easy to obtain solutions in terms of a single parameter by setting $a = 2b$, $a = 3b$, $2a = 3b$, *etc.* The following are a few of them: $(a, b, m, r) = (9k + 3, 3k + 1, 12k + 4,$

$16k+7$), ($16k+8$, $8k+4$, $20k+10$, $25k+14$), ($9k+3$, $6k+2$, $21k+7$, $49k+20$), ($49k+28$, $7k+4$, $70k+40$, $100k+59$).

Also solved by W. E. Buker, H. L. Lee and the proposer.

The Sum of a Series

E 704 [1946, 36]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Sum the series $1+1/3-1/5-1/7+1/9+\cdots$.

I. *Solution by P. T. Bateman, Philadelphia, Pa.* By setting $x=\pi/4$ in the well known Fourier series

$$\sin x + (\sin 3x)/3 + (\sin 5x)/5 + \cdots = \pi/4, \quad (0 < x < \pi)$$

we find that

$$1 + 1/3 - 1/5 - 1/7 + 1/9 + \cdots = \pi\sqrt{2}/4.$$

II. *Solution by E. P. Starke, Rutgers University.* The geometric series

$$(1 + x^2) - (x^4 + x^6) + (x^8 + x^{10}) - \cdots = (1 + x^2)/(1 + x^4),$$

is convergent, and has the indicated sum, for $|x| < 1$. Therefore, by integration, the series

$$x + x^3/3 - x^5/5 - x^7/7 + x^9/9 + \cdots \tag{1}$$

converges for $|x| < 1$ to

$$\int_0^x [(1 + x^2)/(1 + x^4)]dx = \tfrac{1}{2}\sqrt{2} \arctan [\sqrt{2}x/(1 - x^2)].$$

Since series (1) also converges for $x=1$, we have, by Abel's continuity theorem,

$$\begin{aligned} 1 + 1/3 - 1/5 = 1/7 + 1/9 + \cdots &= \lim_{x\to 1} \tfrac{1}{2}\sqrt{2} \arctan [\sqrt{2}x/(1 - x^2)] \\ &= \pi\sqrt{2}/4. \end{aligned}$$

III. *Solution by C. D. Olds, San Jose State College.* From the well known expansion (*cf.* Titchmarsh, *Theory of Functions*, p. 113)

$$\csc z = \frac{1}{z} + 2z\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2\pi^2 - z^2},$$

we immediately obtain

$$\frac{\pi}{n} \csc \frac{m\pi}{n} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \cdots.$$

Putting $n=4$ and $m=1$, we get

$$\pi\sqrt{2}/4 = 1 + 1/3 - 1/5 - 1/7 + 1/9 + \cdots.$$

Also solved by D. W. Alling, W. G. Brady, R. C. Buck, W. E. Byrne, W. J. Combellack, H. E. Fettis, Harley Flanders, R. Hamming, H. A. Luther,

Solomon Mitchell, W. O. Pennell, Elijah Swift, P. D. Thomas, F. Underwood, and the proposer.

Olds and the proposer furnished, as an early reference, Newton's second letter to Oldenburg, 1676. The proposer also located the series as an exercise in Chrystal's *Algebra*, part 2, p. 335. W. G. Brady pointed out that the series also appears as problem 32, p. 299, in Hobson's *Plane Trigonometry*, and as problem 108c, p. 268, in Knopp's *Theory and Application of Infinite Series* (1928). Hobson gives the correct answer while Knopp gives $\sqrt{2\pi}/4$. Olds further mentioned that the series is closely related to Gregory's series for $\pi/4$, and belongs to a class of such series involving the reciprocals of the natural numbers. These were extensively studied by J. W. L. Glaisher who obtained many such series (*cf.* Messenger of Math., vol. 32, pp. 12–30). Pennell presented a number of allied series obtained from Fourier expansions. Bateman called attention to a general method given in article 107 of Landau's *Handbuch der Lehre von der Verteilung der Primzahlen*, and to some results in Landau's *Vorlesungen über Zahlentheorie*, Band I, Vierter Teil, Kapitel 8, S. 174–178.

Construction of a Quadrangle

E 707 [1946, 97]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Construct a quadrangle having given the lengths of the sides and of the line joining the midpoints of the diagonals.

Solution by H. E. Fettis, Dayton, Ohio. The problem as here interpreted could more accurately be stated: To construct a complete quadrangle given the lengths of four of the sides and of the line joining the midpoints of the remaining two.

Let AB be the segment joining the midpoints of two sides. The construction may then be effected as follows:

Choose any two of the four given sides as opposite, and construct the parallelogram $AMBM'$ on AB as diagonal, having its non-parallel sides equal respectively to half of each of the two sides of the quadrangle chosen as opposite. In a similar manner, construct the parallelogram $ANBN'$, using half the remaining sides of the quadrangle as non-parallel sides of the parallelogram. Through M and N draw lines respectively parallel to AN and AM. Through M' and N' draw lines respectively parallel to BN and BM, which will intersect the two previously drawn parallels to form the required quadrangle.

The above construction may be effected in either of *two* ways, since the second parallelogram may assume two distinct positions for a fixed position of the first, but changing the position of the first parallelogram evidently will not yield a solution distinct from the first two. Thus, of the four possible figures that could result from the above constructions, only two are distinct.

In addition, there are two other possible choices of pairs of opposite sides. The total number of solutions therefore cannot exceed six.

It is clear, however, that six actual solutions are possible only if triangles can be constructed with AB and half of each of the two shortest sides, and with

AB and half of the shortest and longest sides. If either of the above triangles, but not both, is a flat triangle, the quadrangle has two parallel sides.

Also solved by G. W. Courter, J. H. White, and the proposer. An algebraic approach was indicated by H. N. Carleton.

The solutions by Courter and White were similar to the one given by Fettis. The proposer utilized the Miquel point of a complete quadrilateral as follows. Let the opposite sides BC and DA, AB and CD of a quadrangle $ABCD$ meet, respectively, in E and F, and let P be the Miquel point of the complete quadrilateral thus formed. If I and J are the midpoints of AC and BD we can obtain the relations

$$PA = AB \cdot AD/2IJ,\quad PB = BC \cdot AB/2IJ,\quad PC = BC \cdot CD/2IJ,\quad PD = AD \cdot DC/2IJ$$

The first two of these permit us to draw triangle ABP. The last two then give us C and D.

This problem occurs as Ex. 300 in Petersen's *Methods and Theories for the Solution of Problems of Geometrical Constructions.*

A Number Prime for Several Bases

E 708 [1946, 97]. *Proposed by M. P. de Regt, Todd Pacific Shipyards, Tacoma, Washington*

Is there a succession of digits which represents a prime in all systems of notation of radix r, $2 \leqq r \leqq 10$? If so, find the smallest one.

Note by E. P. Starke, Rutgers University. It will be shown that no such succession exists involving less than thirteen digits. Because of the prohibitive amount of labor involved in testing large numbers as possible primes, the solution will be carried no further.

Let the required numbers be represented by N_r. Since N_r is to have significance when $r=2$, its digits must be exclusively 1's and 0's, and we shall have

$$N_r = r^{a_1} + r^{a_2} + r^{a_3} + \cdots + r^{a_k},$$

where the $a_i (i = 1, 2, \cdots, k)$ are distinct non-negative integers and k is the number of digits 1. Evidently $a_k = 0$, since N_r is to be prime. Now no divisor $d \neq 1$ of k can be a divisor of $r-1$. For, if so, $r = dx+1$, whence each term r^{a_i} is of the form $dy+1$, and N_r, being the sum of k of these, is a multiple of d. Thus k must be at least eleven. However, N_r cannot contain as few as eleven digits (necessarily then all 1's) because if $N_r = (r^{11}-1)/(r-1)$ we have $N_4 = (2^{22}-1)/3$ which, by Fermat's theorem, is divisible by 23.

By special cases we show that N_r must have more than twelve digits. Suppose N_r contains eleven digits 1 and one digit 0. We then have

$$N_r = (r^{12} - 1)/(r - 1) - r^a, \qquad a = 1, 2, 3, \cdots, 10.$$

Now if $a=1$, 17 is a factor of N_5; $a=2$, 11 of N_4; $a=3$, 17 of N_3; $a=4$, 19 of N_3; $a=5$, 17 of N_2; $a=6$, 29 of N_2; $a=7$, 11 of N_4; $a=8$, 11 of N_2; $a=9$, 11 of N_7; $a=10$, 37 of N_2.

Construction of a Tetrahedron

E 709 [1946, 97]. *Proposed by N. A. Court, University of Oklahoma*

In a given sphere to inscribe a tetrahedron so that three concurrent edges shall pass through three given points and the plane of the three remaining edges shall be parallel to the plane determined by the three given points.

Solution by the Proposer. Let $DABC$ be the required tetrahedron whose edges DA, DB, DC pass through the given points P, Q, R, respectively. If the planes ABC and PQR are parallel, the two tetrahedrons $DABC$, $DPQR$ are homothetic, the vertex D being the homothetic center. Hence the sphere $DPQR$ is homothetic to the given circumsphere (O) of $DABC$, and the two spheres are tangent to each other at the homothetic center D. Thus the required vertex D is the point of contact of the given sphere (O) with a sphere belonging to the coaxal pencil of spheres passing through the circle determined by the three given points P, Q, R. Hence D may be found (*cf.* N. A. Court's *Modern Pure Solid Geometry*, art. 599). The problem may have two solutions.

Editorial Note. Recently W. A. Rees sent to this department the problem: In a given circle to inscribe a triangle so that two sides shall pass through two given points and the third side shall be parallel to the line determined by the two given points.

This problem is the two dimensional analogue of the above, and its solution is analogous to the solution of the above.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4215. *Proposed by Hüseyin Demir, Columbia University*

Prove that the Hermite polynomials defined as follows

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2},$$

have the property

$$n! \sum_{p=0}^{n} \frac{H_p^2(x)}{p!} = H_{n+1}^2(x) - H_n(x) H_{n+2}(x).$$

4216. *Proposed by Herbert Robbins, Annapolis, Md.*

Write $\{x_n\} \xrightarrow{c} \alpha$ if $\lim_{n\to\infty}(x_1+\cdots+x_n)/n=\alpha$. A function $f(x)$ is said to be *C. c.*, (Cesàro continuous) at $x=\alpha$ if $\{x_n\} \xrightarrow{c} \alpha$ implies $\{f(x_n)\} \xrightarrow{c} f(\alpha)$. Show that if $f(x)$ is of the form $Ax+B$ then it is *C. c.* at every value of x, and that if $f(x)$ is *C. c.* at even a single value $x=\alpha$, then $f(x)$ is of the form $Ax+B$.

4217. *Proposed by P. A. Piza, San Juan, P. R.*

Let a and b be positive integers whose sum is the square c^2 and whose difference is the cube d^3. Given that each of c^2, d^3, and a is a four digit integer and that the sum of the four digits contained in each is equal to b, find a and b.

4218. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $T\equiv ABCD$, if a point L with the normal coordinates (x, y, z, t) is such that its associates $(-x, y, z, t)$, $(x, -y, z, t)$, $(x, y, -z, t)$, $(x, y, z, -t)$ are on the circumsphere, it coincides with the point whose distances to the planes of the faces BCD, CDA, DAB, ABC are proportional to the radii of the circumcircles of these faces (second Lemoine point for T), and conversely.

4219. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In an orthocentric tetrahedron $ABCD$ with the altitudes AA', BB', CC', DD', let H' be the inverse of the orthocenter H with respect to the circumsphere, which the lines $H'A$, $H'B$, $H'C$, $H'D$ meet again in A_1, B_1, C_1, D_1. Show that the tetrahedrons $A_1B_1C_1D_1$ and $A'B'C'D'$ are similar and that the volume of the first is 27 times that of the second.

SOLUTIONS

Bound for a Finite Sum

4155 [1945, 220]. *Proposed by G. W. Wishard, Norwood, Ohio*

Find a formula for the sum of the series

$$1 + 2^2 + 3^3 + \cdots + n^n.$$

If such a formula is impossible find superior and inferior limits for the sum.

Solution by F. Underwood, University College, Nottingham. Let S_n denote the sum of the given series whose νth term is denoted by u_ν, then a first (crude) upper limit for S_n is given as follows,

$$S_{n-1} < (n-1)u_{n-1} = (n-1)^n u_n; \qquad S_n = S_{n-1} + u_n < 2u_n.$$

The first few cases show the crudity of this limit; thus $u_4=256$, $S_4=288$; $u_5=3125$, $S_5=3413$; and so a much closer inequality can be expected. Now

$$\frac{u_{n-1}}{u_n} = \frac{(n-1)^{n-1}}{n^n} = \frac{1}{n-1}\left(1-\frac{1}{n}\right)^n = \frac{v_n}{n-1}, \qquad v_n = \left(1-\frac{1}{n}\right)^n,$$

$$v_n - v_{n-1} = \left(1-\frac{1}{n}\right)^n - \left(1-\frac{1}{n-1}\right)^{n-1}$$

$$= \left(1-\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^{n-1} - \left(1-\frac{1}{n-1}\right)^{n-1},$$

$$= \left(1-\frac{1}{n}\right)\left(1+\frac{1}{n-1}\right)^{-(n-1)} - \left(1-\frac{1}{n-1}\right)^{n-1}$$

$$= \left(1+\frac{1}{n-1}\right)^{-(n-1)}\left\{\left(1-\frac{1}{n}\right) - \left(1-\frac{1}{(n-1)^2}\right)^{n-1}\right\}.$$

For $n>1$

$$\left(1-\frac{1}{(n-1)^2}\right)^{n-1} < \left(1-\frac{1}{(n-1)^2}\right) < 1-\frac{1}{n}.$$

Hence $v_n - v_{n-1} > 0$ for $n>1$. Also $v_2 = 1/4$ and $\lim_{n\to\infty} v_n = 1/e$. Hence

$$\frac{1}{4} = v_2 < v_3 < v_4 < \cdots < v_n < \frac{1}{e},$$

$$S_{n-1} < 2u_{n-1} < \frac{2u_n}{e(n-1)},$$

$$S_n = S_{n-1} + u_n < u_n\left[1+\frac{2}{e(n-1)}\right], \quad \text{for} \quad n > 1.$$

Again

$$u_{n-1} > \frac{u_n}{4(n-1)}, \quad \text{for} \quad n > 2,$$

$$S_n = S_{n-1} + u_n > u_{n-1} + u_n > u_n\left[1+\frac{1}{4(n-1)}\right].$$

Hence, for $n>2$,

$$n^n\left[1+\frac{1}{4(n-1)}\right] < S_n < n^n\left[1+\frac{2}{e(n-1)}\right].$$

Editorial Note. The proposer gave the inequalities $n^n < S_n < 2n^n$. He also gave a table for the first ten values of S_n: it will suffice here to give the following extracts:

$n =$	6,	7,	8,	9
$S_n =$	50069,	873612,	17650828,	405071317.

The table suggested to him $n^n + n^{(n-1)}/2$ *might* be taken as a superior limit. A third, incomplete, solution uses the inequalities

$$\frac{n!e^n}{\sqrt{2\pi n}}\,\frac{12n-1}{12n} < n^n < \frac{n!e^n}{\sqrt{2\pi n}},$$

which are modifications of inequalities in Serret's *Algebra*, vol. 2, p. 249, §391.

Perfect Squares

4157 [1945, 220]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find the base such that a number of eight digits of the form *ababcdcd* can be the square of a number of four digits *mnmn*, where the numbers of two digits *ab* and *cd*, or *ab* and *mn*, or *cd* and *mn* are consecutive.

Solution by E. P. Starke, Rutgers University. Put B for the base and A, C M for the two-digit numbers *ab*, *cd*, *mn*, respectively. Then the hypothesis gives

$$A(B^6 + B^4) + C(B^2 + 1) = M^2(B^2 + 1)^2,$$

or

$$A(B^2 - 1) + (A + C)/(B^2 + 1) = M^2.$$

The integer $(A+C)/(B^2+1)$ must have the value unity, so that

$$A(B^2 - 1) = M^2 - 1, \tag{1}$$

$$A + C = B^2 + 1. \tag{2}$$

(a) Suppose A and M are consecutive. Then $M=A+1$. ($M=A-1$ is inconsistent with $M<B^2$ and (1).) Hence (1) becomes $M=B^2-2$, and at once $A=B^2-3$, $C=04$. Thus there are squares of the proposed form for every base $B>4$. *e.g.* when $B=10$, $9898^2=97970404$.

(b) Suppose $A=C\pm1$, reducing (2) to $2A\pm1=B^2+1$. We may eliminate A between this equation and (1) with the result

$$2M^2 = B^4 + B^2 \quad \text{or} \quad 2M^2 = B^4 - B^2 + 2.$$

The first of these is impossible since by (2) B is even, and hence M^2 must contain 2 as a factor to an odd power. For the second, solutions are possible: the simplest is $B=4$, $M=11$. Thus, in the system of base 4, $2323^2=20202121$.

(c) Suppose $M=C\pm1$. Using equations (1), (2) and eliminating A and C, we may put the results in the form

$$(5B^2 - 3)^2 + 16 = 5(2M + B^2 - 1)^2 \quad \text{or} \quad (5B^2 + 1)^2 - 16 = 5(2M + B^2 - 1)^2.$$

For the first there seems to be no simple integral solution, but the second yields $B=7$, $M=31$. Thus in the system of base 7 we have $4343^2=26264242$.

Solved also by the proposer.

Editorial Note. The proposer gave three examples, the first two of which are above: base 4, $(2323)^2$; base 7, $(4343)^2$ base 7, $(6565^2)=64640404$.

No derivation was given.

A Partial Differential Equation

4158 [1945, 280]. *Proposed by P. D. Thomas, Lumberton, Miss.*

Integrate the partial differential equation

$$(1+q^2)r - 2pqs + (1+p^2)t + (rt-s^2)/(1+p^2+q^2)^{1/2} + (1+p^2+q^2)^{3/2} = 0;$$

and give a geometrical interpretation of the general integral.

Solution by C. E. Springer, University of Oklahoma. This non-uniform equation is of the form

$$Rr + Ss + Tt + U(rt - s^2) = F. \tag{1}$$

It can be solved by the well known method of Menge (see, for instance, Miller, *Partial Differential Equations*, N. Y., p. 222).

Substitution of

$$r = \frac{dp - sdy}{dx}, \qquad t = \frac{dq - sdx}{dy}$$

into equation (1) gives

$$Rdydp + Tdxdq + Udpdq - Fdxdy = s[R(dy)^2 - Sdxdy + T(dx)^2 + U(dxdp - dydq)]. \tag{2}$$

We regard equation (2) as an identity in s, and seek solutions of

$$\begin{aligned} Rdydp + Tdxdq + Udpdq - Fdxdy &= 0, \\ R(dy)^2 - Sdxdy + T(dx)^2 + U(dxdp + dydq) &= 0. \end{aligned} \tag{3}$$

It can be shown (see reference above) that a value of λ for which

$$R(dy)^2 - Sdxdy + T(dx)^2 + U(dxdp + dydq) + \lambda(Rdydp + Tdxdq + Udpdq - Fdxdy) \tag{4}$$

is factorable, is given by

$$(RT + FU)\lambda^2 + SU\lambda + U^2 = 0. \tag{5}$$

For the given differential equation, the roots of (5) are coincident and are expressed by

$$\lambda = \frac{1}{pq\sqrt{1+p^2+q^2}}. \tag{6}$$

The factors of the expression (4) are found to be

$$\begin{aligned} &Udy + \lambda Tdx + \lambda Udp, \\ &\lambda Rdy + Udx + \lambda Udq. \end{aligned} \tag{7}$$

On using the value of λ in (6) and the coefficients of the given differential equation, there results

$$\begin{aligned}\sqrt{1+p^2+q^2}\,[dx+pdz]+dp&=0,\\ \sqrt{1+p^2+q^2}\,[dy+qdz]+dq&=0.\end{aligned}\tag{8}$$

Use of $dz=pdx+qdy$ in equations (8) leads to the integrals

$$\begin{aligned}p&=(c_1-x)\sqrt{1+p^2+q^2}\\ q&=(c_2-y)\sqrt{1+p^2+q^2}.\end{aligned}\tag{9}$$

The values of p and q from equations (9), can be substituted into $dz=pdx+qdy$ to yield

$$dz[1-(x-c_1)^2-(y-c_2)^2]^{1/2}=(c_1-x)dx+(c_2-y)dy,$$

which can be integrated to give

$$(x-c_1)^2+(y-c_2)^2+(z-c_3)^2=1.$$

This equation represents all unit spheres in three-dimensional space.

Solved also by F. Underwood and the proposer.

Editorial Note. The solution by the proposer is similar to the above. Underwood stated that this problem is Ex. 2, p. 160 in the *Supplementary Volume* to Boole's *Treatise on Differential Equations*, which was revised by Todhunter. He gave a brief sketch of the solution, using the notation of *Differential Equations* by Piaggio, pp. 183, 184.

Determinants of Determinants

4159 [1945, 280]. *Proposed by F. J. Duarte, Caracas, Venezuela*

Given 18 numbers a_i, b_i, c_i, $i=1, 2, 3, 4, 5, 6$ such that

$$\begin{vmatrix}(bc)_{13} & (ca)_{13} & (ab)_{13}\\ (bc)_{25} & (ca)_{25} & (ab)_{25}\\ (bc)_{46} & (ca)_{46} & (ab)_{46}\end{vmatrix}=\begin{vmatrix}(bc)_{24} & (ca)_{24} & (ab)_{24}\\ (bc)_{16} & (ca)_{16} & (ab)_{16}\\ (bc)_{35} & (ca)_{35} & (ab)_{35}\end{vmatrix}$$

where

$$(mn)_{ij}=\begin{vmatrix}m_i & m_j\\ n_i & n_j\end{vmatrix},$$

show that

$$\begin{vmatrix}(bc)_{12} & (ca)_{12} & (ab)_{12}\\ (bc)_{34} & (ca)_{34} & (ab)_{34}\\ (bc)_{56} & (ca)_{56} & (ab)_{56}\end{vmatrix}=0.$$

Solution by John Williamson, Queens College. Let (ijk) denote the determinant $|a_ib_jc_k|$ and $|(ij), (kl), (mn)|$ the determinant $|(bc)_{ij}(ca)_{kl}(an)_{mn}|$.

then

$$|(13), (25), (46)|\,(145) = \begin{vmatrix} 0 & (413) & (513) \\ (125) & (425) & 0 \\ (146) & 0 & (546) \end{vmatrix}$$

$$= -(413)(125)(546) - (513)(425)(146). \tag{i}$$

In the same way we find that

$$|(24), (16), (35)|\,(145) = -(524)(416)(135) - (124)(516)(435) \tag{ii}$$

and

$$|(12), (34), (56)|\,(145) = (412)(534)(156) + (512)(134)(456). \tag{iii}$$

Since by hypothesis both (i) and (ii) are zero,

$$(134)(125)(456) = -(135)(245)(146) = -(124)(156)(345)$$

and this implies that (iii) is zero. Therefore the required result is proved if $(145) \neq 0$. In a similar manner the result is true if $(236) \neq 0$.

Let $(145) = (236) = 0$. We may therefore assume that one of the vectors, say (a_1, b_1, c_1) is a linear combination of the other two (a_4, b_4, c_4) and (a_5, b_5, c_5). Accordingly we may write

$$(a_1, b_1, c_1) = \rho(a_4, b_4, c_4) + \sigma(a_5, b_5, c_5)$$

and

$$(a_2, b_2, c_2) = \lambda(a_3, b_3, c_3) + \mu(a_6, b_6, c_6).$$

Then

$$|(13), (25), (46)| = |\rho(43) + \sigma(53), \lambda(35) + \mu(65), (46)|$$

$$= \rho\lambda|(43), (35), (46)| + \rho\mu|(43), (65), (46)| + \sigma\mu|(53), (65), (46)|, \tag{iv}$$

$$|(24), (16), (35)| = \rho\lambda|(34), (46), (35)| + \lambda\sigma|(34), (56), (35)| + \mu\sigma|(64), (56), (35)| \tag{v}$$

and

$$|(12), (34), (56)| = \rho\mu|(46), (34), (56)| + \lambda\sigma|(53), (34), (56)|. \tag{vi}$$

If $(iv) = (v) = 0$, (vi) is also zero.

Geometrically the result is well known. If a_i, b_i, c_i are the homogeneous coordinates of six points P_i, the hypothesis states that P_1P_3, P_2P_5, P_4P_6 are concurrent and that P_2P_4, P_1P_6, P_3P_5 are also concurrent. These two sets of three lines determine two projective pencils and therefore P_1P_2, P_3P_4, P_5P_6 are concurrent.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Princeton University announces the appointment of Professor Emil Artin of Indiana University to a professorship and the promotion of Associate Professors Salomon Bochner, Claude Chevalley, and A. W. Tucker to professorships.

Syracuse University announces the appointment of Assistant Professor S. S. Cairns of Queens College to the chairmanship of the department of mathematics, the promotion of Associate Professor Charles Loewner to a professorship, and the appointment to assistant professorships of Dr. Roy Dubisch, Dr. D. E. Kibbey and Dr. P. C. Rosenbloom.

The University of California at Los Angeles announces the appointment to a professorship of Professor I. S. Sokolnikoff of the University of Wisconsin, the promotion to a professorship of Associate Professor E. F. Beckenbach, and the promotion to an associate professorship of Assistant Professor P. G. Hoel.

Dr. A. A. Aucoin has been appointed to an associate professorship at the University of Houston.

Dr. W. D. Berg has been appointed to a visiting assistant professorship at Kenyon College.

Assistant Professor D. H. Blackwell of Howard University has been promoted to an associate professorship.

Dr. Archie Blake has been appointed senior statistician in the Office of the Army Surgeon General, Washington, D. C.

Assistant Professor W. E. Bleick of the Postgraduate School, United States Naval Academy, has been promoted to an associate professorship.

Associate Professor Richard Brauer of the University of Toronto has been promoted to a professorship.

Dr. H. K. Brown of Lehigh University has been promoted to an assistant professorship of mechanics.

Dr. A. W. Burks of Swarthmore College has been appointed to an assistant professorship in philosophy at the University of Michigan.

Dr. J. D. Campbell of Rensselaer Polytechnic Institute has been promoted to an assistant professorship.

Dr. D. E. Christie of Bowdoin College has been promoted to an assistant professorship.

Dr. Paul Civin of the University of Buffalo has been appointed to an assistant professorship at the University of Oregon.

Dr. Robert Cortell of the College of the City of New York has been promoted to an assistant professorship.

Dr. V. F. Cowling of Ohio State University has been appointed to an assistant professorship at Lehigh University.

Dr. J. T. Culbertson has been appointed to an assistant professorship in philosophy at Michigan State College.

W. J. A. Culmer has been appointed to an assistant professorship at Hamline University, St. Paul, Minnesota.

Dr. A. B. Cunningham has been appointed to an assistant professorship at West Virginia University.

Assistant Professor John DeCicco of the Illinois Institute of Technology has been appointed to a visiting associate professorship at Columbia University.

Associate Professor C. H. Denbow of Ohio University, Athens, Ohio, has been appointed to an associate professorship at the Postgraduate School, United States Naval Academy.

Associate Professor W. J. Dixon of the University of Oklahoma has been appointed to an associate professorship at the University of Oregon.

Dr. D. M. Dribin has been appointed research analyst with the Army Security Agency, Washington, D. C.

Dr. J. E. Eaton of Queens College, Flushing, New York, has been appointed mathematician at the Naval Research Laboratory.

W. R. Eikelberger has been appointed to an assistant professorship at the University of Denver.

Dr. Herbert Federer of Brown University has been promoted to an assistant professorship.

Assistant Professor D. G. Fulton of the University of New Hampshire has been appointed to an associate professorship at Tufts College.

Dr. B. H. Gere has been appointed to an assistant professorship at the Postgraduate School, United States Naval Academy.

Dr. David Gilbarg, formerly at the Naval Ordnance Laboratory, has been appointed to an assistant professorship at Indiana University.

Assistant Professor B. E. Gillam of Drake University has been promoted to an associate professorship.

Assistant Professor J. W. Givens of Northwestern University has been appointed to an associate professorship at Illinois Institute of Technology.

Dr. L. J. Green of Case School of Applied Science has been promoted to an assistant professorship.

Assistant Professor P. R. Halmos of Syracuse University has been appointed to an assistant professorship at the University of Chicago.

Assistant Professor R. W. Hamming of the University of Louisville has been appointed a member of the technical staff at Bell Telephone Laboratories, Murray Hill, New Jersey.

Dr. E. A. Hedberg has been appointed to an associate professorship at the University of South Carolina.

Assistant Professor A. E. Heins of Purdue University has been appointed to an associate professorship at the Carnegie Institute of Technology.

Dr. R. G. Helsel of Ohio State University has been promoted to an assistant professorship.

Dr. J. G. Herriot of Stanford University has been promoted to an assistant professorship.

Dr. Edwin Hewitt of the Institute for Advanced Study has been appointed to an assistant professorship at Bryn Mawr College.

Dr. F. E. Hohn has been appointed to an assistant professorship at the University of Maine.

Dr. P. L. Hsu has been appointed to an associate professorship at the University of North Carolina.

W. N. Huff of the University of Rochester has been appointed to an assistant professorship at the University of Oklahoma.

Assistant Professor P. M. Hummel of the University of Alabama has been promoted to an associate professorship.

Dr. H. D. Huskey of the University of Pennsylvania has been appointed to an assistant professorship at the University of Oklahoma.

Associate Professor B. O. Koopman of Columbia University has been promoted to a professorship.

Associate Professor R. L. Krueger of Wittenberg College has been promoted to a professorship.

Dr. Mary E. Ladue of Barnard College, Columbia University, has been promoted to an assistant professorship.

Dr. J. P. LaSalle has been appointed to an assistant professorship at the University of Notre Dame.

T. H. Lee of Louisiana State University has been appointed to an adjunct professorship at the University of South Carolina.

Associate Professor D. C. Lewis of the University of New Hampshire has been appointed to a professorship at the University of Maryland.

Assistant Professor C. C. Lin of Brown University has been promoted to an associate professorship.

Professor C. E. Miller of the University of New Brunswick has been appointed to a professorship at the University of Saskatchewan.

R. A. Miller has been appointed to an assistant professorship at the University of Mississippi.

W. B. Morgan has been appointed to an assistant professorship at Bluffton College, Bluffton, Ohio.

Dr. H. T. Muhly of the United States Naval Academy has been promoted to an assistant professorship.

Associate Professor W. R. Murray of Franklin and Marshall College, Lancaster, Pennsylvania, has been promoted to a professorship.

Dr. A. M. Peiser has been appointed to an assistant professorship at Rutgers University.

Dr. C. R. Phelps has been appointed to an assistant professorship at Rutgers University.

Dr. R. S. Phillips of the Massachusetts Institute of Technology has been appointed to an assistant professorship at New York University.

Dr. Harry Pollard of Yale University has been appointed to an assistant professorship at Cornell University.

Associate Professor R. G. Putnam of New York University has been promoted to a professorship.

Associate Professor G. E. Raynor of Lehigh University has been promoted to a professorship.

E. K. Ritter has been appointed to an assistant professorship at the Postgraduate School, United States Naval Academy.

Dr. D. H. Rock has been appointed to an assistant professorship at Iowa State College.

Dr. Robert Schatten of the Institute for Advanced Study has been appointed to an associate professorship at the University of Kansas.

Associate Professor C. H. W. Sedgewick of the University of Connecticut has been promoted to a professorship.

Dr. J. K. Senior of the University of Chicago has been promoted to an associate professorship.

Dr. C. B. Smith has been appointed to an associate professorship at the University of Florida.

Professor C. E. Springer of the University of Oklahoma has been appointed chairman of the department of mathematics.

Assistant Professor I. L. Stright of Baldwin-Wallace College has been appointed associate professor of mathematics and chairman of the department at Northern Michigan College of Education.

Dr. A. C. Sugar of Brown University has been appointed to an associate professorship at Oklahoma Agricultural and Mechanical College.

Dr. Feodor Theilheimer of Trinity College, Hartford, Connecticut, has been promoted to an assistant professorship.

Assistant Professor C. B. Tucker of Kansas State Teachers College of Emporia has been promoted to an associate professorship.

Assistant Professor R. J. Walker of Cornell University has been promoted to an associate professorship.

Professor H. S. Wall of the Illinois Institute of Technology has been appointed to a professorship at the University of Texas.

Dr. W. R. Wasow of New York University has been appointed to an assistant professorship at Swarthmore College.

Associate Professor Max Zorn of the University of California at Los Angeles has been appointed to a professorship at the University of Indiana.

The following appointments to instructorships are announced:
Brown University: Winfield Keck
Columbia University: Dr. T. W. Anderson, J. L. Bogdanoff
Illinois Institute of Technology: Dr. Margaret S. Matchett
Johns Hopkins University: Dr. Murray Mannos, Emanuel Mehr
Mount Holyoke College: Grace E. Bates
Rutgers University: Dr. H. J. Zimmerberg
Syracuse University: Dr. P. W. Gilbert, Dr. Ruth E. Goodman, Kathryn Morgan
United States Naval Academy: K. F. McLaughlin
University of California at Los Angeles: Dr. Leonard Greenstone
University of Michigan: Dr. W. F. Eberlein, Dr. Leonard Tornheim
University of Virginia: V. L. Klee
University of Wisconsin: Dr. R. E. Fullerton

Professor Federigo Enriques of the University of Rome died June 14, 1946.

Assistant Professor A. S. McMaster of the University of Colorado died May 24, 1946.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE PUTNAM MATHEMATICAL COMPETITION FOR 1946

GARRETT BIRKHOFF, Harvard University

The following results of the sixth annual William Lowell Putnam Mathematical Competition held June 1, 1946, have been determined in accordance with the rules of the Competition agreed to by the representatives of the Mathematical Association and the trustees of the William Lowell Putnam Intercollegiate Memorial Fund. The contestants were known to Association officials and to the reader only by number up to the time of this announcement. A total of sixty-seven undergraduate students representing seventeen institutions took part in the Competition.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Canada. The members of the team were Donald Fraser, Harry Seigel and James Stanley; to each of these, a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of the Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were Felix Browder, Eugene Calabi and A. A. Kheiralla; to each of these, a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Brooklyn College, Brooklyn, New York. The members of the team were Julian Keilson, Gerald Washnitzer and Arthur Zeichner; to each of these, a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of Carnegie Institute of Technology, Pittsburgh, Pennsylvania. The members of the team were Louis Bright, George Hinman and Julia Randall; to each of these, a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, were: Felix Browder, Massachusetts Institute of Technology; Eugene Calabi, Massachusetts Institute of Technology; Donald Fraser, University of Toronto; J. Arthur Greenwood, Harvard University; Maxwell Rosenlicht, Columbia University. Each of these will receive a prize of fifty dollars.

The five persons ranking next highest in the examination, named in alphabetical order, were: G. W. Hinman, Carnegie Institute of Technology; H. O. Seigel, University of Toronto; W. T. Sharpe, University of Toronto; J. P. Stanley, University of Toronto; Arthur Zeichner, Brooklyn College. Each of these will receive a prize of twenty dollars.

The following teams won honorable mention: Department of Mathematics, Columbia University, the members of the team being Alexander Heller, Alan Hoffman and Maxwell Rosenlicht; Department of Mathematics, Cornell University, the members of the team being J. E. Hanson, J. B. Hudders and J. E. Storer.

Five individuals are given honorable mention. The names, listed in alphabetical order, are: Julian Keilson, Brooklyn College; Leroy Meyers, Queens College; Norman Nadler, McGill University, Montreal; John Nash, Jr., Carnegie Institute of Technology; L. P. Shapley, Harvard University.

The following is a list, arranged alphabetically, of all colleges and universities which entered teams in the Competition, some also entering individual contestants: Brooklyn College, Brown University, University of California at Los Angeles, Carnegie Institute of Technology, Columbia College, Cornell University, Duke University, Harvard University, McGill University, Massachusetts Institute of Technology, Northwestern University, Stanford, Swarthmore College, University of Toronto. The following additional colleges and universities entered individual contestants only: University of Michigan; Queens College; Radcliffe College.

The Examination was given in two parts on the morning and afternoon of June 1, 1946. Contestants were allowed three hours upon each part. The questions follow.

PART I

1. Suppose that the function $f(x)=ax^2+bx+c$, where a, b, c are real constants, satisfies the condition $|f(x)| \leqq 1$ for $|x| \leqq 1$. Prove that $|f'(x)| \leqq 4$ for $|x| \leqq 1$.

2. If $a(x)$, $b(x)$, $c(x)$, and $d(x)$ are polynomials in x, show that

$$\int_1^x a(x)c(x)\,dx \cdot \int_1^x b(x)d(x)\,dx - \int_1^x a(x)d(x)\,dx \cdot \int_1^x b(x)c(x)\,dx$$

is divisible by $(x-1)^4$.

3. A projectile in flight is observed simultaneously from four radar stations which are situated at the corners of a square of side b. The distances of the projectile from the four stations, taken in order around the square, are found to be R_1, R_2, R_3, R_4. Show that

$$R_1^2 + R_3^2 = R_2^2 + R_4^2.$$

Show also that the height h of the projectile above the ground is given by

$$h^2 = -\frac{1}{2}b^2 + \frac{1}{4}(R_1^2 + R_2^2 + R_3^2 + R_4^2)$$
$$-\frac{1}{8b^2}(R_1^4 + R_2^4 + R_3^4 + R_4^4 - 2R_1^2R_3^2 - 2R_2^2R_4^2).$$

4. Let $g(x)$ be a function that has a continuous first derivative $g'(x)$ for all values of x. Suppose that the following conditions hold: (i) $g(0)=0$. (ii) $|g'(x)| \leqq |g(x)|$ for every x. Prove that $g(x)$ vanishes identically.

5. Find the smallest volume bounded by the coordinate planes and by a tangent plane to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

6. A particle of unit mass moves on a straight line under the action of a force which is a function $f(v)$ of the velocity v of the particle, but the form of this function is not known. A motion is observed, and the distance x covered in time t is found to be connected with t by the formula $x=at+bt^2+ct^3$, where a, b, c have numerical values determined by observation of the motion. Find the function $f(v)$ for the range of v covered by the experiment.

PART II

1. Let K denote the circumference of a circular disc of radius one, and let k denote a circular arc that joins two points a, b on K and lies otherwise in the given circular disc. Suppose that k divides the circular disc into two parts of equal area. Prove that the length of k exceeds 2.

2. Let A, B be variable points on a parabola P, such that the tangents at A and B are perpendicular to each other. Show that the locus of the centroid of the triangle formed by A, B and the vertex of P is a parabola P_1. Apply the same process to P_1, obtaining a parabola P_2, and repeat the process, obtaining altogether the sequence of parabolas P, P_1, P_2, $\cdots$, P_n. If the equation of P is $y^2=mx$, find the equation of P_n.

3. In a solid sphere of radius R the density ρ is a function of r, the distance from the center of the sphere. If the magnitude of the gravitational force of attraction due to the sphere at any point inside the sphere is kr^2, where k is a constant, find ρ as a function of r. Find also the magnitude of the force of attraction at a point outside the sphere at a distance r from the center. (Assume that the magnitude of the force of attraction at a point P due to a thin spherical shell is zero if P is inside the shell, and is m/r^2 if P is outside the shell, m being the mass of the shell, and r the distance of P from the center.)

4. For each positive integer n, put

$$p_n = (1 + 1/n)^n, \qquad P_n = (1 + 1/n)^{n+1}, \qquad h_n = \frac{2p_nP_n}{p_n + P_n}.$$

Prove that $h_1<h_2<\cdots<h_n\cdots$.

5. Show that the integer next above $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1}.

6. A particle moves on a circle with center O, starting from rest at a point P and coming to rest again at a point Q, without coming to rest at any intermediate point. Prove that the acceleration vector of the particle does not vanish at any point between P and Q, and that, at some point R between P and Q, the acceleration vector points in along the radius RO.

THE SUPERVISION OF TEACHING ASSISTANTS

B. W. JONES, Cornell University

Since, in recent years, the department of mathematics at Cornell University has found it necessary to employ an increasing number of teaching assistants (graduate students who teach part time) members of the staff decided to create some plan for the systematic supervision of this teaching with a view to contributing as much as possible towards its improvement. Since inquiry revealed that other institutions have considered the same problem, letters were written to fifteen colleges and universities requesting information; replies were received from all but two. This is a brief digest of these replies.

Of the four institutions having no supervision, two had substantially no teaching assistants; the other two felt a need for supervision but had been prevented by lack of time and staff from formulating any plan of action. Six departments report that they have, or plan to have shortly, some visiting of the classes of assistants by regular members of the staff. In one of the six there is visiting only when there are complaints from students or other sources. At the other extreme, in another institution, formal visits are made on an assistant's

classes; the visitor then confers with the assistant, gives him a grade on his teaching and reports his findings to the other members of the permanent staff. In three of the institutions a regular member of the staff is in charge of each course involving sections taught by assistants, and this staff member, from time to time, calls meetings of those teaching the course and helps the assistants with their teaching problems. A uniform final examination seems to be common for most elementary courses taught in several sections, but it does not seem to be the general practice to give uniform tests during the term. In one university a member of the staff is given the general responsibility of supervising all teaching assistants and he, together with other members of the staff, visits classes taught by assistants, and assists them with their problems.

There seems to be a strong belief that there should be some supervision of beginning teachers, not only on the part of the members of the departments consulted, but among our own teaching assistants who have very definite ideas on the subject. This author would welcome suggestions from mathematicians who have had experience with the supervision of teaching assistants.

APPEAL FOR AID TO DEVASTATED LIBRARIES

About a year ago the Council of the Society appointed a committee charged with recommending a policy for bringing to libraries devastated by the war such aid as it would be possible for mathematicians of this country to give. The committee consists at present of Professors Garrett Birkhoff, H. E. Bray, Arnold Dresden, William Feller, Gabor Szegö, and A. W. Tucker. Recently the Board of Governors of the Association voted to cooperate with this committee in such policies as they might suggest.

A great deal of time and thought has been spent by the committee in obtaining information concerning the status of mathematical libraries abroad, and in making contact with agencies, private and semi-governmental, which have been created for similar purposes. It has become clear that the need is very great, particularly for books and periodicals published during the years 1939 to 1946. A call must be made on individual mathematicians for assistance.

Accordingly, an appeal is now being made for mathematicians to contribute books, journals, and reprints to a collection to be sent to devastated libraries. Please send only material in good condition. Send by express or parcel post, prepaid and addressed as follows:

American Mathematical Society
Butler Library, Columbia University
New York 27, New York
FOR FOREIGN LIBRARIES

It is important not to omit any part of this address.

Contributions in cash, made out to the American Mathematical Society, should be sent to the Society, 531 West 116th Street, New York 27, New York, with the notation "For Foreign Libraries."

THE MATHEMATICAL ASSOCIATION OF AMERICA

APRIL MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held at North Carolina State College, Raleigh, N.C., on Friday and Saturday April 19–20, 1946.

There were one hundred and five present, including the following forty-six members of the Association: Louise Adams, S. R. Baker, Helen Barton, A. T. Brauer, E. T. Browne, R. C. Bullock, E. A. Cameron, C. L. Carroll, Jr., T. C. Carson, J. W. Cell, Helen E. Clarkson, J. B. Coleman, Nelle C. Douglas, F. G. Dressel, L. A. Dye, W. W. Elliott, Tomlinson Fort, L. L. Garner, J. J. Gergen, C. L. Hair, Minnie W. C. Harris, Archibald Henderson, T. F. Hickerson, M. A. Hill, Jr., J. W. Lasley, Jr., Jack Levine, Anne L. Lewis, F. A. Lewis, R. S. Pate, K. B. Patterson, G. E. Reves, J. O. Reynolds, L. V Robinson, C. L. Seebeck, Jr., C. Eucebia Shuler, W. H. Spragens, Ruth W. Stokes, C. F. Strobel, Cora Strong, Mary M. Templeton, J. M. Thomas, G. C. Watson, G. T. Whyburn, W. L. Williams, Y. K. Wong, H. J. Zimmerberg.

Chairman J. W. Lasley, Jr. presided at the Friday afternoon and Saturday morning meetings, and Professor H. A. Fisher, North Carolina State College, presided on Friday evening at the informal dinner given in honor of the visiting speaker, Professor G. T. Whyburn. In the absence of the Secretary, Colonel H. A. Robinson, Professor Ruth W. Stokes acted as Secretary.

At the business session the following officers were elected for the coming year: Chairman, Ruth W. Stokes, Winthrop College; Vice-chairman, J. W. Cell, North Carolina State College; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. W. L. Williams, University of South Carolina, was elected Governor of Region V for two years. The Section voted to hold its 1947 meeting at Winthrop College. Resolutions were passed relative to the passing of Professor W. P. Ott who had held the position of Section Chairman three terms.

The program consisted of the following papers:

1. *On the exact number of primes below a given limit*, by Professor A. T. Brauer, University of North Carolina.

The speaker gave a simple proof of Meisell's formula for the exact number of primes below a given limit, and applied his results to some interesting examples.

2. *Projective flecnode ruled surfaces*, by Professor J. D. Novak, University of South Carolina.

The author gave the results of his study of two classes of ruled surfaces whose two flecnode surfaces are projectively equivalent. Cech's theorem and other generalizations were obtained by use of derivative curves in five dimensions.

3. *Some geometrical constructions of conics*, by Professor L. A. Dye, The Citadel.

Without resorting to the method of rotation of axes, theorems were stated which enable one to construct a conic in general position and to determine the equations and lengths of the axes. These were susceptible of proof by elementary methods not ordinarily introduced in a first analytic geometry course.

4. Retiring Chairman's Address: *The rule of the complex number in elementary mathematics*, by Professor J. W. Lasley, Jr., University of North Carolina.

A transition was outlined from the complex numbers of elementary algebra to the vector representation in the plane, with emphasis upon the concepts of addition and differentiation. Certain relations were formulated with analytic description, followed by geometric interpretation. These relations were applied to the proof of the sum and difference formulas of trigonometry, the rotation formulas of analytics, the plane curvilinear motion of the calculus, and the planetary motion of mechanics.

5. *Value distribution of a function of two complex variables in variables in domain with distinguished boundary surface*, by Professor Josephine Mitchell, Winthrop College, introduced by Professor Stokes.

Using the notion of distinguished boundary surfaces as introduced by S. Bergman, the author investigated meromorphic functions of two complex variables defined in a connected domain with distinguished boundary surface. Inequalities were obtained for various classes of complex functions and for the values which such functions could assume.

6. *Mathematical aspects of some problems in air navigation*, by Professor E. A. Cameron, University of North Carolina.

Certain navigation problems involving relative motion were formulated in terms of moving coördinate reference frames. The author demonstrated how such formulation helps to clarify the nature and significance of problems in air navigation.

7. *The rehabilitation of geometry*, by Professor W. W. Rankin, Duke University.

Professor Rankin stated that geometry is rapidly slipping out of the curriculum. He asserted we should do more to retain it or its equivalent.

8. *Mathematical program for college majors who plan to teach*, Professor Ruth W. Stokes, Winthrop College.

The speaker reviewed recent reports of national committees concerning mathematics in our post-war curriculum. She outlined certain preparatory courses for high school mathematics teachers, and included suggestions on the use of departmental clubs and multisensory aids.

9. *The area of a polygon when the sides and interior angles are known*, by Professor T. F. Hickerson, University of North Carolina.

10. *Surface topology and mappings*, by Professor G. T. Whyburn, University of Virginia.

Using Stokes's theorem as an example, the author showed how topological considerations arise in other branches of mathematics and how they contribute in lending definiteness to results in these fields. A discussion of orientability and Euler characteristic of surfaces was given, leading to the solution of the problem of classifying surfaces into topological types. It was then shown how this, together with the characterization of a surface as a topological space, yielded a complete and satisfactory solution to a most fundamental problem in surface topology. Professor Whyburn concluded with a brief discussion of interior mapping of surfaces and called attention to the as yet unsolved problem of classifying the possible interior images of surfaces of various types.

11. *The accuracy of linear interpolation*, by Professor P. M. Hummel, University of Alabama, read by Doctor Seebeck.

The purpose of this paper was to develop rather sharp upper and lower limits for the error resulting when a function with certain restrictions is evaluated between known values by linear interpolation.

12. *A coefficient of correlation for three variables*, by Doctor J. B. Coleman, Mathematician, S.C. State Highway Department.

In space of three dimensions the sum of the squares of the perpendicular distances of a set of points from a line of best fit was found. Associated with this sum was a formula containing the r's of Pearson's product moment coefficients by pairs. This expression measured the closeness of fit to a line, hence the degree of correlation for the three variables.

13. *A method in the theory of least squares*, by Doctor Y. K. Wong, University of North Carolina.

14. *Equivalent annuities*, by Professor C. L. Seebeck, Jr., University of Alabama.

The speaker proposed a treatment of annuities in the general case by means of equivalent simple annuities. The theory was extended to apply to problems of a type not usually treated in current textbooks. The interpolation theorem for finding the final payment of a simple annuity was generalized.

15. The equation $\ddot{x}+x=0$, by Professor W. H. Spragens, Georgia State Womans College.

Professor Spragens discussed the use of visual aids in teaching and listed sources of mathematical films now available. He showed an English film on the equation $\ddot{x}+x=0$.

16. *Analytic functions and interior transformations*, by Professor G. T. Whyburn.

It was shown first that an analytic function of a complex variable generates an interior transformation from the complex plane or sphere to the complex plane or sphere. This yielded a simple topological proof of the maximum modulus theorems, as well as results concerned with level lines and zeros of an analytic function. A discussion was then given of some recent results of Nevanlinna,

Ahlfors, and others concerning defect relations and totally branched values of meromorphic functions. It was shown also how in certain cases these same results may be obtained in a purely topological way using a special case of a characteristic equation for interior mappings from one surface to another.

17. *Some applications of differential operator methods*, by Professor L. V. Robinson, University of South Carolina.

A brief survey was made of the use of a differential operator in generalizing Taylor's formula and in solving certain types of differential equations.

18. *The solution of a differential equation as the limit of the solution of a difference equation*, by Professor Tomlinson Fort, University of Georgia.

Although the problem is old, it is believed that Professor Fort's novel treatment yielded results not previously known. The difference equations treated were equations in discrete argument (recurrent relations). They were set up in terms of a real parameter and the limit was approached by letting this parameter become infinite. The problem was then one in ordinary analysis. The work was carried out by forming a series for the solution of the difference equation whose terms approached terms of the corresponding series solution of the differential equation. Applications were made to the Sturm-Liouville theory.

19. *Definite differential and integral systems*, by Dr. H. J. Zimmerberg, North Carolina State College.

The author summarized the results obtained by Bliss and Reid for differential systems in matrix form under various definitions of definiteness. Then he discussed results for integral systems of Friedholm type under corresponding assumptions.

20. *Differential equations with quadrilateral envelope, cuspidal, and nodal loci*, by Professor Archibald Henderson, University of North Carolina.

This paper has appeared in the *National Mathematics Magazine*, Vol. XX, No. 2, pp. 1–18.

H. A. ROBINSON, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The thirtieth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on April 4, 1946. Professor J. B. Brandeberry, Chairman of the Section, presided.

Sixty-eight persons attended, including the following forty-three members of the Association: Grace M. Bareis, H. M. Beatty, Louis Brand, J. B. Brandeberry, O. E. Brown, V. B. Caris, Florentina M. Clinton, Wayne Dancer, H. P. Fawcett, M. P. Fobes, B.C. Glover, H. H. Hartzler, R. G. Helsel, R. C. Hildner, Margaret E. Jones, L. C. Knight, A. C. Ladner, H. E. Menke, E. J. Mickle, H. J. Miser, Max Morris, F. D. Murnaghan, C. V. Newsom, Emma J. Olson, Tibor Radó, J. F. Randolph, F. Mozelle Rankin, S. E. Rasor, Hortense Rickard, R. F. Rinehart, S. A. Rowland, K. C. Schraut, Samuel Selby, E. S. Smith,

H. E. Stelson, J. L. Synge, H. S. Toney, D. R. Whitney, R. B. Wildermuth, C. O. Williamson, C. R. Wylie, Jr., E. P. Vance, C. H. Yeaton.

The following officers were elected for the coming year: Chairman, S. A. Rowland, Ohio Wesleyan University; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of Executive Committee, E. J. Mickle, Ohio State University; Program Committee, F. B. Wiley, Denison University, C. H. Yeaton, Oberlin College, Wayne Dancer, University of Toledo. It is expected that the next meeting will be held at Ohio State University on April 3, 1947.

The following program was presented:

1. *Imaginary branches of real curves*, by Professor J. B. Brandeberry, University of Toledo.

By introducing an iY axis through the origin and perpendicular to the XY plane, the imaginary branches of curves can be graphed in the same manner as the real branches. Intersections, maxima and minima, points of inflection, *etc.*, when imaginary, can be illustrated and explained by means of these graphs. The student gains a better understanding of "excluded values" and the real branches of curves after studying their behavior in the imaginary plane.

2. *Finding the term in annuity equations*, by Professor H. E. Stelson, Kent State University.

The speaker discussed the meaning of the term of an annuity when a fractional value is obtained by solving an annuity equation. The interpolated value can be assigned a meaning in both the present value and final value annuity formulas. Solution by logarithms gives a result which bears a relationship to the interpolated result.

3. *A computer for celestial navigation*, by Professor O. E. Brown, Case School of Applied Science.

Professor Brown described a computer consisting of a transparent sheet pivoted to an opaque sheet. The opaque sheet carries an azimuthal equidistant projection of the celestial sphere displaying a row of dots for each of forty-four navigational stars, each row containing one dot for each integer degree of local hour angle. The transparent sheet carries curves of constant azimuth and curves of constant altitude. Rotation of the transparent sheet corresponds to latitude. The assumed local hour angle and the name of the star select one dot on the opaque sheet. When the rotor is set to the assumed latitude, the appearance of the selected dot among the curves of the rotor permit the altitude and azimuth of the star to be estimated with amazing accuracy. Symmetry considerations and the use of both sides of the opaque sheet make it possible to employ only one quadrant of the otherwise circular figures.

4. *Motors as dual vectors*, by Professor Louis Brand, University of Cincinnati.

A vector $\mathbf{a}$ localized in a line $\mathbf{r}\times\mathbf{a}=\mathbf{a}_0$ may be specified by its Plücker coordinates $(\mathbf{a}, \mathbf{a}_0)$; these satisfy the relation $\mathbf{a}\cdot\mathbf{a}_0=0$. The *moment vector* depends upon the choice of 0. Referred to P the moment vector is given by the *shift formula* $\mathbf{a}_P=\mathbf{a}_0+\overrightarrow{PO}\times\mathbf{a}$.

The ∞^4 lines of space may be represented by Plücker coördinates for which $\mathbf{a}\cdot\mathbf{a}=1$, $\mathbf{a}\cdot\mathbf{a}_0=0$. These may be combined into a *dual* vector $\mathbf{A}=\mathbf{a}+\epsilon\mathbf{a}_0$, where the algebraic unit ϵ has the property $\epsilon^2=0$. If we expand $(\mathbf{a}+\epsilon\mathbf{a}_0)\colon(\mathbf{a}+\epsilon\mathbf{a}_0)$ by applying the distributive law, $\mathbf{A}\cdot\mathbf{A}=1$. Thus the dual vector which represents a line is a *unit* vector.

The ∞^5 line vectors of space may be represented as real multiples of unit line vectors: $\mathbf{B}=\lambda(\mathbf{a}+\epsilon\mathbf{a}_0)$. The Plücker coördinates $\mathbf{b}=\lambda\mathbf{a}$, $\mathbf{b}_0=\lambda\mathbf{a}_0$ still satisfy $\mathbf{b}\cdot\mathbf{b}=0$ and $\mathbf{b}_P$ is computed from the shift formula.

The ∞^6 *motors* $\mathbf{M}$ of space are now defined as *dual* multiples of unit line vectors: $\mathbf{M}=(\lambda+\epsilon\lambda')\mathbf{A}$. Thus $\mathbf{M}=\lambda\mathbf{a}$, $\mathbf{M}_0=\lambda\mathbf{a}_0+\lambda'\mathbf{a}$; $\mathbf{M}_P$ is still given by the shift formula but $\mathbf{M}\cdot\mathbf{M}=\lambda\lambda'$. Motors may be classified as *screws* (λ, $\lambda'\neq 0$), *line vectors* ($\lambda\neq 0$, $\lambda'=0$), *couples* ($\lambda=0$, $\lambda'\neq 0$), or *zero* ($\lambda=0$, $\lambda'=0$).

The products $\mathbf{M}\cdot\mathbf{N}$ and $\mathbf{M}\times\mathbf{N}$ are now defined by applying the distributive law to the dual vectors. All rules of calculation for this motor algebra are in precise analogy with the rules of ordinary vector algebra. By means of the concept of *dual* angle, $\mathbf{M}\cdot\mathbf{N}$ and $\mathbf{M}\times\mathbf{N}$ may be expressed exactly as the corresponding vector products. Finally this algebra permits of reciprocal sets of motors which have all the properties of reciprocal sets of vectors.

5. *The teaching of college mathematics*, by Professor F. D. Murnaghan, Johns Hopkins University.

This paper appears in the present issue of the MONTHLY.

6. *A report of the committee on the teaching of elementary mathematics*, by Professor H. P. Fawcett, Ohio State University.

Requirements for the certification of elementary teachers in Ohio include, among other things, 24 semester hours of work in special subjects, 15 in the field of English, 8 in the field of science. There are no specific requirements in mathematics, although under the subject of arithmetic the current elementary teacher training curriculum prescribed in 1938 states that "the teacher in training shall possess mastery of the four fundamentals, common and decimal fractions, and percentage, including the special case of interest." Teacher training institutions are then asked to provide "guarantees that the graduates of the elementary teacher training program possess certain primitive knowledge and skills deemed essential to successful teaching."

To deal with this problem the teacher training institutions of Ohio are following a variety of practices, ranging all the way from a few weeks spent on the teaching of arithmetic to the use of a carefully prepared diagnostic test followed by six hours of required courses in professionalized subject matter.

Following a comprehensive study of the situation, this committee has proposed a tentative course of study in mathematics for prospective teachers in the elementary schools of Ohio. The plan has been submitted to the teacher training institutions for criticism and study. The committee asks for further development of this proposal, and appropriate action.

RUFUS CRANE, *Secretary*

THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the U. S. Naval Academy, Annapolis, Maryland, on Saturday, December 8, 1945. Professor J. B. Scarborough, Chairman of the Section, presided at morning and afternoon sessions.

The attendance was one hundred, including the following fifty members of the Association: H. C. Ayres, R. A. Baumgartner, W. E. Bleick, H. H. Campaigne, Randolph Church, G. R. Clements, Abraham Cohen, J. F. Daly, Alexander Dillingham, J. A. Duerksen, P. J. Federico, E. J. Finan, B. H. Gere, Michael Goldberg, R. A. Good, J. R. Hammond, D. C. Harkin, L. M. Kells, D. M. Krabill, Solomon Kullback, R. C. Lamb, A. E. Landry, J. F. Locke, G. A. Lyle, Carol V. McCamman, E. S. Mayer, D. D. Miller, Helen K. Milleson, J. F. Milos, T. W. Moore, Eugenie M. Morenus, H. M. Nahikian, W. H. Norris, Jr., C. R. Phelps, Grace S. Quinn, C. H. Rawlins, Jr., C. E. Rhodes, J. N. Rice, R. E. Root, C. A. Rupp, J. B. Scarborough, E. D. Schell, Veryl G. Schult, Hillel Spitz, E. G. Swafford, O. M. Thomas, V. J. Varineau, R. W. Wagner, Hazel S. Wilson, R. H. Wilson, Jr.

The first four of the following papers were read at the morning session. The last paper was read at the afternoon session by Lieut. Comdr. J. F. Daly at the invitation of the Section.

1. *Magic rectangles modulo p*, by Professor E. J. Finan, Catholic University. This paper has since appeared in this MONTHLY.

2. *Approximation to an analytic function*, by Ensign E. N. Nilson, U.S.N.R., introduced by the Secretary.

A problem of the best approximation to an analytic function in the sense of least pth powers was discussed for the case of least squares with the function analytic in the unit circle of the complex plane. The approximating functions were restricted to be analytic within, and of bounded integral mean on, a larger circle. The measure of approximation was taken to be the integral mean, over a circle within the unit circle, of the modulus of the difference between the given function and the approximating function.

3. *On a set of almost orthogonal trigonometric functions*, by Lieut. R. P. Bailey, U.S.N.R., introduced by the Secretary.

The speaker discussed the properties of a set of trigonometric functions $\chi_m(x) = \cos \lambda_m x / \cos \lambda_m \pi$, $(m = 0, 1, 2, \cdots)$ arising in a boundary value problem connected with the heat equation. The characteristic numbers $\{\lambda_m\}$ appear as the roots of the equation $\tan \lambda\pi = \frac{1}{2}M\lambda$. It was shown that the non-harmonic "Fourier" expansion

$$\sum \chi_m(x) \left[\int_{-\pi}^{\pi} f(t)\chi_m(t)dt - Mf(\pi) \right] \cdot \left[\int_{-\pi}^{\pi} \chi_m^2(t)dt - M \right]^{-1}$$

for an arbitrary even periodic function $f(x)$ belonging to L^2 on $(-\pi, \pi)$ always converges to $f(\pm\pi)$ at the endpoints of the interval, and in the interior has exactly the same convergence (or summability) properties as the ordinary Fourier cosine expansion for the function.

4. *Mathematics in the Washington Veterans High School Center*, by Carol V. Coleman, Washington Veterans High School Center.

This paper dealt with some of the problems apt to confront any mathematics teacher whose class is at the high school or freshman college level and composed principally of veterans.

5. *Some applications of probability theory to the inspection of materials*, by Lieut. Comdr. J. F. Daly, U.S.N.R.

The speaker discussed the application of certain basic concepts of probability theory to the control of quality of manufactured articles, with particular emphasis on the effectiveness of rational methods of lot acceptance inspection. The notion of the operating characteristic curve of a sampling plan was explained, and was used to clarify the meaning of such quality control terms as average outgoing quality limit, acceptable quality level, consumer's risk, producer's risk, and tolerance percent defective. Illustrations were taken from the wartime experience of both industry and government to show the improvement in quality which can be achieved through the systematic use of even the most elementary statistical methods.

E. J. FINAN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirtieth Annual Meeting, Swarthmore, Pa., December 26–27, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., October, 1946
ILLINOIS, Peoria, May 9–10, 1947
INDIANA,
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., December 7, 1946
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Claremont, March 8, 1947
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Madison, May, 1947

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No. 4. *Projective Geometry,* by PROFESSOR J. W. YOUNG. (First Impression, 1930; Second Impression, 1938.)

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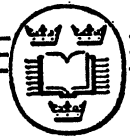
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MODELS IN THE THEORY OF SEVERAL COMPLEX VARIABLES

STEFAN BERGMAN, Harvard University*

1. Introduction. The theory of functions of a single complex variable has exerted a valuable unifying and systematizing influence on the field of analysis, even in domains which are primarily concerned with real function theory. This is especially true in such fields as analytic number theory, the theory of linear differential equations, power series, algebraic functions, *etc.*, *etc.*; more important still it has been a powerful instrument of research in both pure and applied nathematics, in mathematical physics, and in engineering.

Many leading mathematicians—Weierstrass, Poincaré, Picard, and Osgood being among the most eminent—appear to have felt that a theory of analytic functions of several complex variables would be similarly valuable in various other branches of mathematics, say, non-linear differential equations, and that such a theory might well be the key to many important but as yet unsolved problems. Nevertheless, while the theory of analytic functions of one complex variable has been making a continuous and rapid progress and has become one of the most important and useful tools in the standard equipment of the mathematician, the theory of functions of several complex variables has by no means enjoyed such spectacular development; its progress is slow, halting and unintegrated; and relatively few mathematicians have any considerable knowledge of this latter field.

There are many reasons why there should be such a marked disparity in the states of advancement of these two theories, despite their close affinities. Perhaps the most obvious is the greater formal complexity of the functions of many variables; in the author's opinion one of the most significant factors is to be found in our lack of direct intuition of spaces of more than three dimensions.

In the first half of the nineteenth century the theory of analytic functions of a complex variable was formal, manipulative and very difficult. The modern reader is frequently baffled by the complications of analysis and the demands made on the ingenuity of the mathematician of that time to prove theorems which do not seem difficult to the contemporary mathematician. Much of the credit for this simplification and increased power must be ascribed to the introduction and elaboration of geometric methods in function theory. The mathematician's innate geometric intuition serves him by enabling him to interpret many involved and frequently subtle analytic properties of functions as relations between easily grasped geometric configurations, while the use of geometric language renders his statements both clearer and briefer.

This invaluable aid is almost totally wanting when we attack problems of functions of many variables. This note discusses a possible means of circumventing some of the difficulties of picturing spaces of more than three dimensions.

* The author wishes to express his thanks to Mr. M. H. Slud for friendly criticism and many helpful suggestions.

It will be shown how models* may be used for the representation of four-dimensional manifolds encountered in the theory of functions of two complex variables, and how appropriate training can create a kind of intuition for domains in space of four and more dimensions.

2. The role of geometrical interpretation. Interpretation of complex numbers as points in the x, y-plane has proved highly fruitful in the theory of functions of a complex variable. It is only an easy step beyond this to interpret the domain of definition of a function as a region in the plane and to express theorems on analytic functions in geometric terms. For instance, one of the most important and fundamental results in the theory of functions may be put into the following simple geometric form: Every simply connected domain with at least two boundary points may be mapped bi-uniquely and conformally on a circle. Naturally these (geometric) theorems are correctly and completely expressive of the full (analytic) content only if the geometric notions have been axiomatized, *i.e.* described in a completely abstract manner. Despite this the geometric formulation possesses various advantages connected with the fact that we do have intuition of two-dimensional space.

As mentioned before, while operating with ideas we must be able to keep them constantly in mind; as we often have a better understanding of geometrical relationships than of the same relationships in abstract terms† it goes without saying that whenever one must work with something which allows of geometrical representation, a geometrically minded mathematician will prefer to use the geometrical model. Many qualitative properties then become apparent such as, for example, whether the domain is connected or consists of several pieces, whether a curve is continuous or discontinuous, whether the domain is convex or not, or what symmetry properties it possesses, *etc.* Most properties which are associated with first and second derivatives may be seen in this way, such as whether a curve has a continuously turning tangent or corners and cusps, and whether there are maxima and minima. If several configurations $A, B, C \cdots$ are given then we can readily see the nature and properties of the configurations symbolized by $A+B+C+\cdots$, $A \subset B \subset C$, $A \cap B$, *etc.*

Moreover if the elements have been ordered in some manner, we can immediately see their mutual relations. To a certain extent we can also apprehend certain metric relations, such as estimating which of two curves is longer or which of two domains has the larger area.

---

* Models of this kind are described in the author's papers: 1. Zur Veranschaulichung der Kreiskörper und Bereiche mit ausgezeichneter Randfläche, Jahresbericht der deutschen Mathem. Vereinigung vol. 42, 1933, pp. 238–252; 2. The visualization of domains of the theory of functions of two complex variables, Journal of Mathematics and Physics (M.I.T.) vol. 20, 1941, pp. 107–117.

† An obvious analogy is the difference between hearing a remark in one's native language or hearing the same statement in a language in which one is not altogether at ease. The second case demands considerable more mental effort.

3. Models in abstract mathematical theory. In Section 2 we have indicated the advantages for a geometrically minded mathematician, of using appropriate models in the theory of functions of one variable (two-dimensional case). Before discussing models in four-dimensional space it would be useful to clarify the relationship between the abstract geometrical notion and a model.

In order to describe geometrical configurations verbally or analytically, we often need a very considerable number of words or symbols. The most convenient method of describing a geometrical manifold is frequently by making use of equalities and inequalities.

For example, the set of points on the straight line which bisects the angle of the first quadrant may be represented by the symbol $E[x-y=0]$;* and the interior of the unit circle with center at the origin may be denoted by $E[x^2+y^2<1]$. Unfortunately, it is rarely possible to characterize a manifold so simply. It would be difficult to give an analytic expression for the interior of a polyhedron, while it would be impossible to characterize many of the manifolds used by the analyst with a finite number of symbols. Even in those cases where an analytic expression can be found, it is usually very long, and it is all but impossible to deduce the geometric properties of the figure from its analytic representation.† These difficulties are often circumvented by simply using an arbitrary symbol of some sort, most usually a letter, to denote the configurations which occur most commonly in mathematical discussions. Of course the symbol used has none of the properties of the entity it represents, any more than the letters of a word in any way resemble the object to which the word refers.

A further step in this direction is the use of geometric models. (Two-dimensional geometric models are usually called diagrams.) Naturally we may regard a model as a sort of symbol for the abstract entity known as a geometric manifold, but it is much more than that. The model (diagram) is a material object which has certain of the properties of the geometric concept it represents, and that is why it is useful. The representation is never perfect, however. The model seldom has *all* the properties of the original, and at the same time it has properties of its own which its original does not possess. Therefore care must be taken that the properties of the model shall not be mistaken for the properties of the original, a very common source of error. Because almost all geometric concepts were originally suggested by the properties of real figures, it is customary to say that the abstract concept is an "idealization" of the model. After the

* By $E[\cdots]$ we denote a set of points whose coordinates satisfy the equalities or inequalities indicated in brackets.

† We wish to emphasize that the ability to deduce properties depends to a large extent upon the individual, and can be increased by appropriate training. The question of whether in the future some "machines" can be constructed which will facilitate our operating with long formulae, and automatically give us information about the properties of the manifold remains at the present time open.

abstract concept has become familiar, the model is regarded as a "realization" of the concept.

The most important disadvantage of diagrams is that they are not able to describe the manifold completely, *e.g.* to indicate whether a circle is to be taken with its boundary or without it. Because of this during the transition from the diagram to the abstract notion we often commit mistakes by attributing to the manifold properties which are possessed only by the model.

A means of checking our conclusions would be to repeat our reasoning but now operating exclusively with abstract symbols, *e.g.* of the type indicated earlier in this section. This situation necessitates a certain kind of training and particular flexibility of mind, enabling one to pass from an intuitive model to an abstract formulation of notions and back.

These difficulties arise naturally in operating with models of four-dimensional space; in addition to this there occur further difficulties, which we shall indicate in what follows.

4. The advantage of models in teaching and research. Despite the difficulties mentioned before it seems that in many cases the use of models for a geometrically minded individual is of great value in research. Two important reasons may perhaps be indicated:

First, our interest in a thing is determined to a considerable extent by the amount of mental energy which is required for working with it. We involuntarily prefer to think in terms of familiar things which can be conceived without special effort. In other words, the extent of our interest in any subject depends to a certain degree on our previous associations—and here again we see the advantage of being able to use intuitive geometry to help in reasoning about abstract relations.

Second, if a relationship is known, it is not too important how we formulate it, provided only that our formulation is sufficiently precise. Research, on the other hand, is the search for new relations and in that case it becomes highly significant in which way our problems are formulated. If we can succeed in stating our problem in terms of a class of objects with which we have gained great familiarity, then we may be able to exploit our previous experience with these things. The form of the problem in the new representation may suggest comparisons and hypotheses and further questions, and our previous experience will enable us to see new relationships all the more readily. Thus, the moment that complex numbers are regarded as points in a plane, the analogy with integrals of real functions leads to the formulation of curvilinear integrals in the complex plane—the most fruitful approach in the theory of functions. Furthermore, even after some fact has been discovered and it has been reduced to abstract form which completely and rigorously describes it, even then, the intuitive processes by means of which we have worked are of importance in further research because of the ready associations and the many problems which in this way arise almost spontaneously.

5. Models of four-dimensional manifolds. It would, perhaps, be of interest to illustrate the foregoing by a description of a type of model used for visualization of four-dimensional space.*

These models are moving three-dimensional pictures, that is to say time is chosen as one of the coordinates. A four-dimensional domain is then represented as a moving picture. The continuous process is thus recorded by a finite number of "frames," and this last is in turn often replaced by a number of stills.

In order to have a better idea as to how intuitive these models are, let us put ourselves in the position of beings ("Flatlanders") who are able to "see" two-dimensional domains but who do not possess intuition for three-dimensional space.

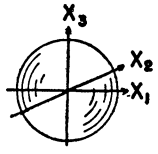


FIG. 1. A sphere in an isometric representation.

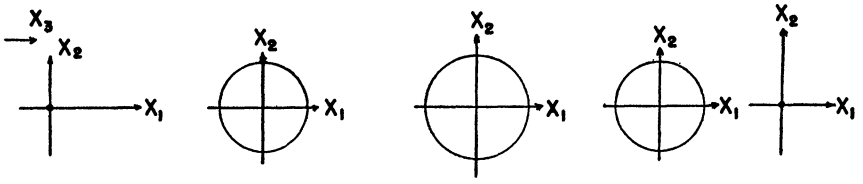


FIG. 2a–2e. Successive plane sections of a sphere. (Series of sections perpendicular to the x_1-axis.)

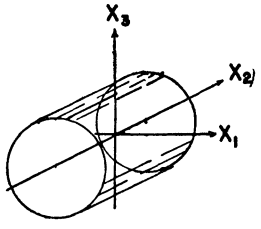


FIG. 3. A cylinder in an isometric representation.

Let us consider two examples of corresponding models of three-dimensional space, namely that of a cylinder $E[-1<x_2<1,\ x_1^2+x_3^2<1]$ and of a sphere $E[x_1^2+x_2^2+x_3^2<1]$.

Assuming that x_3 has been chosen as time we obtain films whose "stills" are represented in Figures 2a–2e and 4a–4e.

Obviously Figures 1 and 3 are for us much "better" representations of the above manifolds. However this impression is mostly due to the fact that we *do*

* Naturally various other types of models are possible and it depends on the purpose what type will be most suitable.

have intuition for three-dimensional space and are sufficiently trained to "imagine" from a two-dimensional projection (see Figures 1 and 3) the corresponding three-dimensional manifold. In the four-dimensional case, the corresponding projection of a four-dimensional manifold on the three-dimensional space (without the long training necessary to "reading" such projections) will give us almost no information about the properties of the four-dimensional manifold.

In Figures 5a–5e and 6a–6e stills of the models described above of the "hypersphere" $E[x_1^2+x_2^2+x_3^2+x_4^2<1]$ and of the "bicylinder" $E[x_1^2+x_2^2<1, x_3^2+x_4^2<1]$ are given.

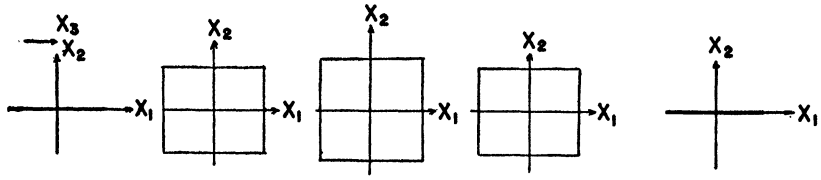


FIG. 4a–4e. Successive longitudinal sections of a cylinder. (Series of sections perpendicular to the x_1-axis.)

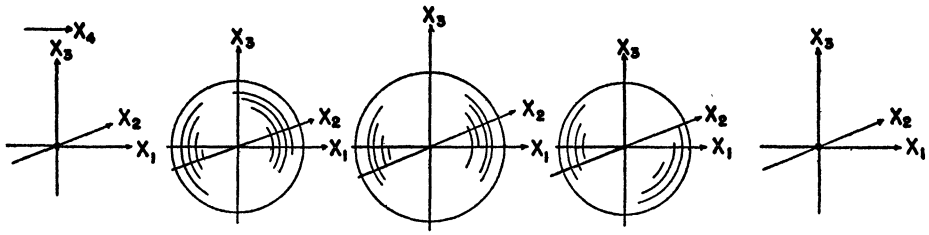


FIG. 5a–5e. Successive sections of a (four-dimensional) hypersphere by a series of (three-dimensional) hyperplanes perpendicular to the x_4-axis.

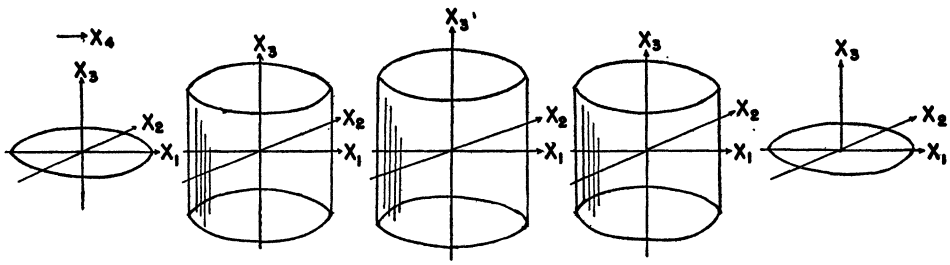


FIG. 6a–6e. Successive sections of a (four-dimensional) bicylinder by a series of (three-dimensional) hyperplanes perpendicular to the x_4-axis.

6. The training necessary to "read" models. As mentioned before the above (or other) models can be successfully used only after sufficiently long training in "reading" the properties of the manifold represented by the model.

Therefore the problem arises of working out a program for such training, in particular in connection with our models. Obviously this is a very complicated

question of a partially psychological nature, and cannot be answered without extensive experimental study.

It is the opinion of the author (who has only limited experience in this direction) that training should proceed in such a way that we systematically construct at first the stills for a series of various manifolds which are of interest in the theory of functions of two complex variables.*

The next stage of such training should consist of working out rules for changing a model if the manifold which it represents undergoes a linear transformation, at first say a translation or rotation, and later more complicated transformations.

Some preliminary investigations of this kind can be found in the papers mentioned in the footnote (page 496). On the other hand only actual use of a great variety of models can determine their value as well as give further indications concerning the lines along which training should proceed.

It seems therefore that this investigation should proceed on a considerably larger scale, and use more powerful means, *e.g.*, actual moving pictures.

The author hopes that the present article will contribute toward the realization of such a program.

AN INEQUALITY OF JENSEN

E. F. BECKENBACH, University of California, Los Angeles

1. Introduction. Let (a) denote a set $a_1, a_2, \cdots, a_n$ of n positive values, $n \geqq 2$. It has been shown by Pringsheim [7], who attributed his proof to Lüroth, and by Jensen [5, p. 192], that for $r > 1$ we have

$$\left(\sum_{j=1}^{n} a_j^r\right)^{1/r} < \sum_{j=1}^{n} a_j. \tag{1}$$

We write

$$\left(\sum_{j=1}^{n} a_j^r\right)^{1/r} \equiv \mathfrak{S}_r(a), \qquad (r \neq 0). \tag{2}$$

If in (1) we replace a_j by a_j^s, for $s > 0$, and rs by t, and take the sth root of both members of the inequality, we obtain

$$\mathfrak{S}_t(a) < \mathfrak{S}_s(a), \qquad (t > s > 0). \tag{3}$$

The inequality (3) is sometimes called [3; 4, p. 28] *Jensen's inequality*; it can be obtained directly as follows [4, p. 4]:

* We omit here the discussion of models of n-dimensional space for $n > 4$. We wish however to indicate that diagrams similar to $2a-2e$ and $4a-4e$ with several series of stills representing the intersections of the manifold with $x_4 = \text{const.}$, those with $x_5 = \text{const.}$, *etc.* can be constructed and used as models of manifolds of a space with more dimensions.

$$\frac{\left(\sum_{j=1}^{n} a_j^t\right)^{1/t}}{\left(\sum_{j=1}^{n} a_j^s\right)^{1/s}} = \left[\sum_{j=1}^{n} \frac{a_j^t}{\left(\sum_{k=1}^{n} a_k^s\right)^{t/s}}\right]^{1/t} = \left[\sum_{j=1}^{n} \left(\frac{a_j^s}{\sum_{k=1}^{n} a_k^s}\right)^{t/s}\right]^{1/t}$$

$$< \left(\sum_{j=1}^{n} \frac{a_j^s}{\sum_{k=1}^{n} a_k^s}\right)^{1/t} = 1.$$

Let

$$m \equiv \min_{j=1,2,\cdots,n} a_j, \qquad M \equiv \max_{j=1,2,\cdots,n} a_j.$$

From (2) we obtain

$$M < \mathfrak{S}_t(a) < n^{1/t} M, \qquad (t > 0),$$

so that [4, p. 29]

$$\lim_{t \to +\infty} \mathfrak{S}_t(a) = M.$$

Since

$$\lim_{t \to 0} \sum_{j=1}^{n} a_j^t = n,$$

it follows [4, p. 29] that

$$\lim_{t \to +0} \mathfrak{S}_t(a) = +\infty.$$

Hence $\mathfrak{S}_t(a)$ *decreases from* $+\infty$ *to* M *as* t *increases from* $+0$ *to* $+\infty$.

Since

$$\mathfrak{S}_{-t}(a) = \frac{1}{\mathfrak{S}_t(a)},$$

it follows from the above results that $\mathfrak{S}_t(a)$ *decreases from* m *to* 0 *as* t *increases from* $-\infty$ *to* -0.

On the other hand, the mean value function

$$\mathfrak{M}_t(a, q) \equiv \left(\sum_{j=1}^{n} q_j a_j^t\right)^{1/t}, \qquad (t \neq 0),$$

$$\mathfrak{M}_0(a, q) \equiv \prod_{j=1}^{n} a_j^{q_j},$$

where $q_1, q_2, \cdots, q_n$ are fixed positive values with

$$\sum_{j=1}^{n} q_j = 1,$$

increases [4, p. 26] *continuously from m to M, or remains constant if all the a's are equal, as t increases from* $-\infty$ *to* $+\infty$.

It is known [4, pp. 27, 72] that the function

$$\log\,[\mathfrak{M}_t(a, q)]^t \equiv \log \sum_{j=1}^{n} q_j a_j^t \tag{4}$$

is a convex function of t for $-\infty < t < +\infty$. Since we have

$$\log \sum_{j=1}^{n} a_j^t \equiv \log\,[\mathfrak{M}_t(a, 1/n)]^t + \log n, \tag{5}$$

it follows [4, p. 28] that the continuous function (5), which coincides with $\log\,[\mathfrak{S}_t(a)]^t$ for $t \neq 0$, *is a convex function for t for* $-\infty < t < +\infty$.

We offer the following alternative to the proofs usually given of the convexity of (4) (or of (5)). Since

$$\log q_j a_j^t \equiv \log q_j + t \log a_j,$$

the logarithm of $q_j a_j^t$ is a linear, and therefore a convex, function of t. But the class of positive functions having convex logarithms, like the class of positive functions $p(x, y)$ having subharmonic logarithms [2, p. 651], is closed under the operation of addition. Hence (4) is convex; and similarly (5) is convex.

As implied above, the increasing function $\mathfrak{M}_t(a, q)$ has two horizontal asymptotes; but it is not necessarily convex for $-\infty < t < 0$, and is not necessarily concave for $0 < t < +\infty$ [1, 6].

Though the decreasing function $\mathfrak{S}_t(a)$, $-\infty < t < 0$, has a horizontal asymptote, it is not necessarily concave; in particular, for

$$a_1 = a_2 = \cdots = a_n = \mathfrak{a}, \tag{6}$$

we find that $d^2\mathfrak{S}_t(a)/dt^2$ has exactly one zero, namely at the negative value

$$t = \log 1/\sqrt{n}. \tag{7}$$

We note that for (a) and t given by (6) and (7) respectively, we have

$$\mathfrak{S}_t(a) = \mathfrak{a}/e^2,$$

so that, in this case, *the ordinate of the inflection point is independent of n.*

One well might ask concerning bounds on the number of possible inflection points of $\mathfrak{M}_t(a, q)$ for $-\infty < t < +\infty$, and of $\mathfrak{S}_t(a)$ for $-\infty < t < 0$.

The decreasing function $\mathfrak{S}_t(a)$, $0 < t < +\infty$, has a vertical asymptote and a horizontal asymptote. It is our purpose to show that the function

$$\log \mathfrak{S}_t(a) \equiv \log \left(\sum_{j=1}^{n} a_j^t \right)^{1/t}$$

is a convex function of t for $0<t<+\infty$, and consequently that $\mathfrak{S}_t(a)$ *is a convex function of t for $0<t<+\infty$*.

2. Lemma. We shall use the following lemma.

LEMMA. *If $f(t)$ and $g(t)$ are positive non-increasing convex functions for $b<t<c$, then the function*

$$\phi(t) = f(t)\cdot g(t)$$

is convex in the interval (b, c).

Proof. Let t_1, t_2 satisfy $b<t_1<t_2<c$. Since $f(t)$ and $g(t)$ are non-increasing, we have

$$[f(t_2) - f(t_1)][g(t_1) - g(t_2)] \leqq 0,$$

or

$$f(t_1)\cdot g(t_2) + f(t_2)\cdot g(t_1) \leqq f(t_1)\cdot g(t_1) + f(t_2)\cdot g(t_2). \tag{8}$$

Since $f(t)$ and $g(t)$ are convex, we have

$$f\left(\frac{t_1 + t_2}{2}\right) \leqq \frac{1}{2}[f(t_1) + f(t_2)], \tag{9}$$

and

$$g\left(\frac{t_1 + t_2}{2}\right) \leqq \frac{1}{2}[g(t_1) + g(t_2)]. \tag{10}$$

Since $f(t)$ and $g(t)$ are positive, from (8), (9), and (10) we obtain, by multiplication,

$$\phi\left(\frac{t_1 + t_2}{2}\right) \leqq \frac{1}{2}[\phi(t_1) + \phi(t_2)],$$

so that $\phi(t)$ is convex in the interval (b, c).

3. Convexity theorems. We shall use the lemma in §2 in establishing the following result.

THEOREM 1. *The function* $\log \mathfrak{S}_t(a)$ *is convex for* $0<t<+\infty$.

Proof. Let t_0 be an arbitrary positive value, and consider the functions

$$f(t) \equiv 1/t,$$

$$g(t) \equiv \log\left[\mathfrak{S}_t\left(\frac{a}{\mathfrak{S}_{t_0}(a)}\right)\right]^t,$$

for $0<t<t_0$. Clearly the hypotheses of the lemma concerning $f(t)$ are satisfied.

As observed in the Introduction, $g(t)$ is a decreasing convex function of t for $0<t<+\infty$. Since for any positive constant k we have

$$\mathfrak{S}_t(ka) = k\mathfrak{S}_t(a),$$

and since $\mathfrak{S}_t(a)$ is a decreasing function of t for $0<t<+\infty$, it follows that

$$\mathfrak{S}_t\left(\frac{a}{\mathfrak{S}_{t_0}(a)}\right) \equiv \frac{\mathfrak{S}_t(a)}{\mathfrak{S}_{t_0}(a)} > 1, \qquad (0 < t < t_0),$$

so that $g(t)$ is positive. Therefore, by the lemma, the function

$$f(t)\cdot g(t) \equiv \log \mathfrak{S}_t(a) - \log \mathfrak{S}_{t_0}(a)$$

is convex for $0<t<t_0$. Hence $\log \mathfrak{S}_t(a)$ is convex for $0<t<t_0$ and therefore for $0<t<+\infty$.

THEOREM 2. *The function $\mathfrak{S}_t(a)$ is convex for $0<t<+\infty$.*

Proof. It is well known [2, p. 651] that if a function $p(t)$ is positive and $\log p(t)$ is convex, then also $p(t)$ is convex.

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Insomnia. Again there was the procession of figures, parentheses, fractions, equations, cube and square roots. The second sleepless night was flying by imperceptibly.—Merejkowski, *The Romance of Leonardo De Vinci.*

Murder. "In order to understand these crimes," he began, "we must consider the stock-in-trade of a mathematician, for all his speculations and computations tend to emphasize the relative insignificance of this planet and the unimportance of human life."—S. S. Van Dine, *The Bishop Murder Case.*

The casual geometer. I wish he had explained why, but he merely threw out the statement with just the same take-it-or-leave-it casualness as Euclid used when he made his celebrated observation about parallel straight lines.—Somerset Maugham, *Cakes and Ale.*

Not only the Dutch. Bartoline was as dull at drawing inferences as any Dutch professor of mathematics.—Walter Scott, *Heart of Mid-Lothian.*—*E. D. Schell.*

THE MODULI OF THE ROOTS OF AN ALGEBRAIC EQUATION

ELIZABETH BEAMAN, Duke University

1. Introduction. The purpose of this paper is to discuss the polynomial $g(n)$ whose zeros are the norms of the zeros of a given polynomial $f(z)$ with complex coefficients, the norm of a number being defined as the product of the number and its complex conjugate or equivalently as the square of the modulus of the number.

For the quadratic

$$f(z) = az^2 + 2bz + c, \tag{1}$$

the explicit form of the polynomial is

$$g(n) = (Na)n^2 - 2(N\sqrt{d} + Nb)n + Nc, \tag{2}$$

where $4d$ is the discriminant $4(b^2-ac)$ and N denotes the norm. The polynomial whose roots are the moduli of the roots of $f(z)$ is

$$m(r) = |a|r^2 - (\sqrt{2}\sqrt{|b^2 - ac| + |b|^2 + |ac|})r + |c|. \tag{3}$$

Finding $g(n)$ for the cubic is reduced to the problem of finding $\sum Nz$ for the reduced cubic. The value $\sum Nz$ for the reduced cubic is one of the roots of a specified cubic.

The general case is not completely treated. The polynomial $g(n)$ for an equation of degree m is shown, however, to be a factor of a polynomial $R(n)$ whose degree is m^2. As a by-product, there is found a method for obtaining the equation of degree mp, whose roots are the products of the m roots of a polynomial $f(z)$ by the p roots of a polynomial $g(z)$.

2. The quadratic. The most direct way to find the polynomials for the quadratic case is perhaps to compute the elementary symmetric polynomials Nz_1+Nz_2, Nz_1Nz_2, and $|z_1|+|z_2|$, $|z_1|\cdot|z_2|$. The geometric theorem that the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the four sides gives

$$Nz_1 + Nz_2 = \tfrac{1}{2}[N(z_1 + z_2) + N(z_1 - z_2)]. \tag{4}$$

Now

$$z_1 + z_2 = -\frac{2b}{a}, \qquad a^2(z_1 - z_2)^2 = 4d,$$

whence

$$N(z_1 + z_2) = \frac{4Nb}{Na}, \qquad N(z_1 - z_2) = \frac{4N\sqrt{d}}{Na},$$

so that

$$Nz_1 + Nz_2 = \frac{2(Nb + N\sqrt{d})}{Na}. \tag{5}$$

From

$$z_1 z_2 = \frac{c}{a}$$

we get at once

$$Nz_1 Nz_2 = \frac{Nc}{Na}.$$

Extracting the non-negative square root of the last relation we have

$$|z_1| \cdot |z_2| = \frac{|c|}{|a|}.$$

Substituting in the identity

$$(|z_1| + |z_2|)^2 = Nz_1 + 2|z_1| \cdot |z_2| + Nz_2$$

and extracting the non-negative square root we get

$$|z_1| + |z_2| = \sqrt{2}\,\frac{\sqrt{|b^2| + |ac| + |b^2 - ac|}}{|a|}.$$

Hence the coefficients in (2), (3) have the values stated.

3. The cubic. $g(n)$ for the cubic

$$f(z) = az^3 + bz^2 + cz + d$$

can be discussed as follows:

We may write

$$g(n) = (Na)n^3 - xn^2 + yn - Nd,$$

where x and y are to be determined and, in particular,

$$xNa = \phi(a, b, c, d).$$

The polynomial whose roots are the reciprocals of Nz_1, Nz_2, and Nz_3 is the polynomial $g(n)$ corresponding to

$$z^3 f\left(\frac{1}{z}\right) = dz^3 + cz^2 + bz + a,$$

that is,

$$(Nd)n^3 - yn^2 + xn - Na,$$

and

$$yNd = \phi(d, c, b, a).$$

From this,

$$\sum Nz_1Nz_2 = \frac{y}{Na} = \frac{\phi(d, c, b, a)}{Nad}.$$

Therefore, the problem is reduced to that of finding

$$\sum Nz_1 = \frac{x}{Na}.$$

Let A, B, and C be vectors representing z_1, z_2, and z_3, respectively. By computing

$$(A + B + C)^2 + (A + B - C)^2 + (A - B + C)^2 + (-A + B + C)^2,$$

we obtain the formula

$$N\sum z_1 + \sum N(-z_1 + z_2 + z_3) = 4\sum Nz_1. \tag{6}$$

By (4) we have

$$Nz_1 + N(z_2 - z_3) = \tfrac{1}{2}[N(z_1 + z_2 - z_3) + N(z_1 - z_2 + z_3)],$$

whence

$$\sum Nz_1 + \sum N(z_2 - z_3) = \sum N(-z_1 + z_2 + z_3).$$

Use of this reduces (6) to

$$\sum N(z_2 - z_3) = 3\sum Nz_1 - N\sum z_1. \tag{7}$$

The left member of this equation is invariant under the translation

$$z_1' = z_1 + h, \qquad z_2' = z_2 + h, \qquad z_3' = z_3 + h.$$

Choose h so that $\sum z_i' = 0$. This gives the reduced cubic, for which (7) becomes

$$\sum N(z_2 - z_3) = 3\sum Nz_1'.$$

Substituting in (7) gives the formula

$$\sum Nz_1 = \sum Nz_1' + \tfrac{1}{3}N\sum z_1 \tag{8}$$

which expresses $\sum Nz_1$ in terms of the sum of the roots of $f(z)$ and the sum of the norms of the roots of the reduced $f(z)$.

Accordingly, the problem of finding the norms for the general cubic has been reduced to that for the reduced cubic.

4. The reduced cubic. The polynomial

$$f(z) = z^3 + 3pz + 2q$$

has the roots*

$$t + u, \qquad \omega t + \omega^2 u, \qquad \omega^2 t + \omega u, \tag{9}$$

* J. M. Thomas, Theory of Equations, New York, 1938, p. 105.

where

$$tu = -p, \tag{10}$$

t^3, u^3 are the roots of

$$T^2 + 2qT - p^3 = 0, \tag{11}$$

and ω is a primitive cube root of unity. Let 108Δ be the discriminant of the cubic:

$$\Delta = -p^3 - q^2.$$

Direct substitution from (9) in (7) gives

$$\sum Nz_1 = 3(Nt + Nu).$$

Now

$$(Nt + Nu)^3 = Nt^3 + Nu^3 + 3NtNu(Nt + Nu). \tag{12}$$

From (10)

$$NtNu = Np.$$

Applying (5) to (11), whose roots are t^3, u^3, gives

$$Nt^3 + Nu^3 = 2(Nq + N\sqrt{-\Delta}).$$

Hence (12) can be written

$$(Nt + Nu)^3 = 2(Nq + N\sqrt{-\Delta}) + 3Np(Nt + Nu),$$

or if we write

$$V = Nt + Nu = \tfrac{1}{3}\sum Nz_1,$$

the equation for determining $\sum Nz_1$ can be taken as

$$V^3 - 3Np \cdot V - 2(Nq + N\sqrt{-\Delta}) = 0. \tag{13}$$

This cubic has real coefficients. Its resolvent quadratic, like (11), is

$$V^2 - 2(Nq + N\sqrt{-\Delta})V + Np^3 = 0, \tag{14}$$

whose discriminant is

$$4[(Nq + N\sqrt{-\Delta})^2 - Np^3]. \tag{15}$$

By computing the elementary symmetric polynomials for

$$t\bar{t} + u\bar{u}, \qquad \omega t\bar{t} + \omega^2 u\bar{u}, \qquad \omega^2 t\bar{t} + \omega u\bar{u},$$

where $\bar{t}$ denotes the complex conjugate of t, it is readily verified that these numbers are the roots of (13). These numbers can be all real only if the last two are equal. Hence the solution of the cubic (13) never falls into the so-called irreducible case.

It immediately follows that the roots of (14) are $(t\bar{t})^3$, $(u\bar{u})^3$. Since both these numbers are real, from (15) we infer the truth of the following inequality:

$$Np^{3/2} \leqq Nq + N(p^3 + q^2)^{1/2} \tag{16}$$

for all complex p, q.

5. Multiplication of roots. Consider simultaneously a polynomial $f(z)$ of degree m in the indeterminate z and a polynomial $\bar{f}(\bar{z})$ of degree p in the indeterminate $\bar{z}$, where for the moment z and $\bar{z}$, f and $\bar{f}$ are unrelated. Put

$$z = rs, \qquad \bar{z} = r/s, \qquad s \neq 0.$$

The condition that $f(rs)$ and $s^p\bar{f}(rs^{-1})$ have a root s in common is the vanishing of their resultant R, which can be shown to be a polynomial of degree $2mp$ in r. Moreover, since changing r into $-r$ can be compensated by changing s into $-s$, R is a polynomial of degree mp in r^2, or in n, if we define n as r^2. Since

$$z\bar{z} = n,$$

the roots of $R(n)$ are precisely the mp products $z_j\bar{z}_k$, where z_j is a root of $f(z)$ and $\bar{z}_k$ a root of $\bar{f}(\bar{z})$.

If now $s = e^{i\theta}$ and r is real, $\bar{z}$ is the complex conjugate of z and n is Nz. If further $\bar{f}(\bar{z})$ is made the polynomial in $\bar{z}$ whose coefficients are the complex conjugates of those of $f(z)$, then $m = p$ and $R(n)$ is a polynomial of degree m^2 in n with roots which can be grouped into m sets of m each:

$$\begin{aligned}
&z_1\bar{z}_1, \quad z_1\bar{z}_2, \cdots, z_1\bar{z}_m,\\
&z_2\bar{z}_1, \quad z_2\bar{z}_2, \cdots, z_2\bar{z}_m,\\
&\cdots\cdots\cdots\cdots\\
&z_m\bar{z}_1, \quad z_m\bar{z}_2, \cdots, z_m\bar{z}_m.
\end{aligned}$$

The numbers on the main diagonal are the norms of the roots of $f(z)$.

6. The general case. The polynomial $R(n)$ can be written

$$R(n) = g(n)h(n),$$

where $g(n)$ is the polynomial sought and $h(n)$, of degree $m(m-1)$, is a polynomial whose roots occur in $\frac{1}{2}m(m-1)$ pairs, the members of each pair being conjugate imaginary or real and equal. The factorization of $R(n)$ therefore will lead to a solution of the general problem.

The viewpoint of factorization can be successfully adopted in the quadratic and cubic cases. The explicit form of the factor $h(n)$ can readily be obtained in the quadratic case:

$$h(n) = (Na)n^2 + 2(N\sqrt{d} - Nb)n + Nc, \qquad m = 2. \tag{17}$$

It can be verified directly that the factors of this $h(n)$ are conjugate imaginary except when they are equal and that $g(n)$, $h(n)$ have a root in common only when $f(z)$ has equal roots or a zero root.

ELLIPTIC EULEROIDS

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1. Introduction. Let X, Y be the ordinary orthogonal cartesian coordinates of a point P on the ellipse

$$X = a\cos\theta, \qquad Y = b\sin\theta \tag{1}$$

whose foci are the points $F(c, 0)$, $F'(-c, 0)$. It will be recalled that the Euler line of a triangle contains the circumcenter, the orthocenter, and the centroid of the triangle. Let l be the Euler line of the triangle PFF'. It is the purpose of this paper to investigate the envelope of the Euler line l as the point P moves along the ellipse (1).

Parametric equations of the envelope are deduced in §2. Its general properties are considered in §3. The last section contains a brief study of four special cases in which interesting examples of algebraic curves are exhibited.

2. Elliptic euleroids. A simple calculation suffices to show that the Euler line l of the triangle PFF' has the equation

$$y - \frac{b}{3}\sin\theta = \frac{(3a^2 - b^2)\sin^2\theta - 3b^2}{ab\sin 2\theta}\left(x - \frac{a}{3}\cos\theta\right), \tag{2}$$

where x, y are the coordinates of a variable point on the line. We now apply the usual method of finding envelopes and so differentiate equation (2) partially with respect to θ. The resulting equation and equation (2) yield, on placing $\sin\theta = u$ and introducing the eccentricity e of the ellipse, the following parametric equations of the envelope of the line l, namely:

$$\begin{aligned} x &= \pm\frac{a(1-u^2)^{3/2}(1-e^2+e^2u^2)}{(7e^2-4)u^2+3(1-e^2)}, \\ y &= \frac{a}{2\sqrt{1-e^2}}\,\frac{u[(1-e^2)^2-2(1-e^2)(1-4e^2)u^2-e^2(e^2+2)u^4]}{(7e^2-4)u^2+3(1-e^2)}, \end{aligned} \tag{3}$$

where the parameter u is defined on the range $-1 \leqq u \leqq 1$, and $0 \leqq e < 1$. We propose to call the curve defined by equations (3) an *elliptic euleroid.*

3. General properties. It is evident that the character of the particular curve of the family defined by equations (3) depends upon the value of the eccentricity e. Furthermore, it is geometrically obvious that euleroids possess symmetry properties which are identical with those of the ellipse.

We readily verify that every euleroid meets the x-axis in the points $(\pm a/3, 0)$ for which $u = 0$. With the exception of the curve obtained when $e = \frac{1}{2}$, all euleroids have two double points on the x-axis. The parameter values for these points are the two real roots of the equation

$$e^2(e^2+2)u^4 + 2(1-e^2)(1-4e^2)u^2 - (1-e^2)^2 = 0. \tag{4}$$

The only intersections with the y-axis are obtained when $u = \pm 1$, so that these points have the coordinates

$$\left(0, \pm \frac{a(1 - 2e^2)}{2\sqrt{1 - e^2}}\right). \tag{5}$$

From equations (3) we find

$$\frac{dy}{dx} = \pm \frac{1}{2\sqrt{1 - e^2}} \frac{(e^2 + 2)u^2 - 3(1 - e^2)}{u(1 - u^2)^{1/2}}. \tag{6}$$

Thus the curve has horizontal tangents at points whose parameter values are obtained from the equation

$$u = \pm \sqrt{3}\left(\frac{1 - e^2}{2 + e^2}\right)^{1/2}. \tag{7}$$

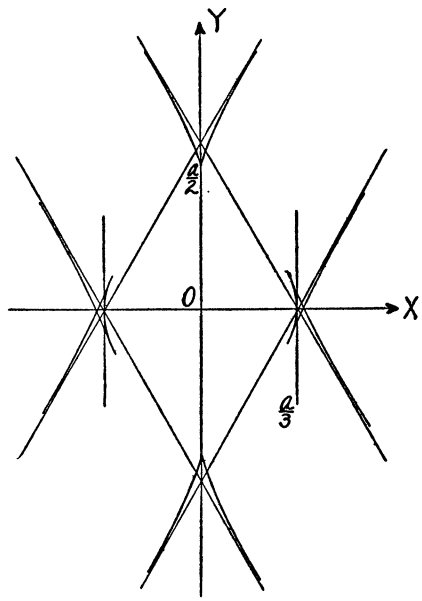


FIG. 1.

Since $|u| \leqq 1$, it is an easy matter to show that no horizontal tangents exist when $e < \frac{1}{2}$. However, when $\frac{1}{2} \leqq e < 1$, points always exist at which the curve has horizontal tangents. The only euleroid with horizontal tangents at the points where $u = \pm 1$ is the curve with $e = \frac{1}{2}$.

All euleroids have the lines $x = \pm a/3$ for vertical tangents at the points where $u = 0$. The y-axis is also a vertical tangent, except when $e = \frac{1}{2}$.

The existence of real asymptotes depends upon parameter values which satisfy the equation

$$u = \pm \sqrt{3}\left(\frac{e^2 - 1}{7e^2 - 4}\right)^{1/2}. \tag{8}$$

This equation enables us to show that all euleroids for which $e<\frac{1}{2}$ possess real asymptotes. No real asymptotes exist when $\frac{1}{2}\leqq e<1$. When $e<\frac{1}{2}$, we find that there are four real asymptotes which are parallel in pairs (see Fig. 1). The equations of the asymptotes are found to be

$$\pm \sqrt{3}\,(1 - 4e^2)^{1/2}(4 - 7e^2)^{1/2}x \pm (4 - 7e^2)^{1/2}y = a(2 - 5e^2)/\sqrt{3}. \tag{9}$$

It is evident from (9) that the slopes of the two pairs of asymptotes are given by the expression $\pm\sqrt{3}(1-4e^2)^{1/2}$, which is imaginary when $e>\frac{1}{2}$.

It is of some interest to follow the change in the position of the asymptotes as e varies from $e=0$ to $e=\frac{1}{2}$. The maximum absolute value of the slopes of the asymptotes is $\sqrt{3}$, and occurs when $e=0$. The asymptotes intersect orthogonally (so that their slopes are ± 1) when $e=\sqrt{6}/6$. Finally, when $e\to\frac{1}{2}$, the limiting value of the slopes is zero.

The expression for the y-intercepts of the asymptotes may be obtained immediately from equation (9). From this result we readily find that the maximum absolute value of the y-intercepts is $a/\sqrt{3}$, occurring when $e=0$. Moreover, when e increases in value the y-intercepts decrease in numerical value but approach $a\sqrt{3}/6$ as a limit, when $e\to\frac{1}{2}$.

It may be remarked, in passing, that y is an integral rational function of the parameter u in the cases $e=0$, $e=\frac{1}{2}$, and $e=2/\sqrt{7}$.

In general, when the parameter is eliminated from equations (3), we obtain an algebraic equation of the tenth degree in which the term involving x^{10} is given by

$$576e^8(e^2 + 2)^8(1 - 4e^2)^4(1 - e^2)^4x^{10}. \tag{10}$$

The coefficient of this term vanishes, so that the degree is less than ten if, and only if, $e=0$ or $e=\frac{1}{2}$. In each of these two cases we later prove that the euleroid is an algebraic curve of order six.

4. Special cases. In this section we consider four special cases which appear to be of some interest.

(a) *The case when* $e=0$. It follows at once that equations (3) reduce to

$$x = \pm \frac{a(1 - u^2)^{3/2}}{3 - 4u^2}, \qquad y = \frac{au(1 - 2u^2)}{2(3 - 4u^2)}. \tag{11}$$

This euleroid meets the x-axis in the points $(\pm a/3,\ 0)$ for which $u=0$, and at these points the curve has vertical tangents. The points $(\pm a\sqrt{2}/4,\ 0)$ are double points which correspond to the parameter values $u=\pm 1/\sqrt{2}$. The slopes of the tangents to the curve at each of the double points are ± 2, as may be verified by means of the derivative.

$$\frac{dy}{dx} = \pm \frac{2u^2 - 3}{2u(1 - u^2)^{1/2}}. \tag{12}$$

Incidentally, it may be remarked that the absolute value of the derivative in (12) is always greater than unity.

The intersections with the y-axis occur at the points $(0, \pm a/2)$ which are cusps on the curve, the y-axis being the common cuspidal tangent. The curve also has four cusps at the points $(\pm 3a\sqrt{3}/16, \pm a/16)$ where the parameter values are $u = \pm\frac{1}{2}$. To demonstrate the existence of these four cusps it will be sufficient to consider the one which lies in the first quadrant. From equations (11) we find

$$dx/du = N(1 - 2u), \qquad dy/du = P(1 - 2u), \tag{13}$$

where we have placed

$$N = -\frac{au(1 + 2u)(1 - u^2)^{1/2}}{(3 - 4u^2)^2}, \qquad P = \frac{a(1 + 2u)(3 - 2u^2)}{(3 - 4u^2)^2}. \tag{14}$$

When u increases from $u = 0$ to $u = \frac{1}{2}$, the signs of the derivatives in (13) show that y increases while x decreases. Both derivatives in (13) vanish when $u = \frac{1}{2}$. Moreover, when $u > \frac{1}{2}$, y decreases while x increases. Since the slope of the tangent to the curve at the point where $u = \frac{1}{2}$ is $-5/\sqrt{3}$, we conclude from these results that the curve has a cusp at the point in the first quadrant where $u = \frac{1}{2}$.

The curve has four real asymptotes whose equations are obtained by placing $e = 0$ in equation (9). The resulting equations are

$$y = \pm \sqrt{3}\, x \pm a/\sqrt{3}. \tag{15}$$

Elimination of the parameter from equations (11) yields the following algebraic equation of the sixth degree:

$$64(x^2 + y^2)(3x^2 - y^2)^2 - 208a^2x^4 - 128a^2x^2y^2 - 48a^2y^4 + 25a^4x^2 + 12a^4y^2 - a^6 = 0. \tag{16}$$

The resulting graph is sketched in Fig. 1.

(b) *The case when* $e = \frac{1}{2}$. Placing $e = \frac{1}{2}$ in equations (3) shows that the equations of the euleroid in this case are

$$x = \pm (a/9)(u^2 + 3)(1 - u^2)^{1/2}, \qquad y = (\sqrt{3}\, au/12)(1 + u^2). \tag{17}$$

With the aid of the derivative

$$\frac{dy}{dx} = \pm \frac{3\sqrt{3}}{4} \frac{(1 - u^2)^{1/2}}{u}, \tag{18}$$

we readily determine the nature of the graph. The curve has vertical tangents at the points $(\pm a/3, 0)$ where it meets the x-axis. Furthermore, we find that the

points $(0, a\sqrt{3}/6)$, $(0, -a\sqrt{3}/6)$ are respectively maximum and minimum points with horizontal tangents to the curve at these points.

The result of eliminating the parameter from equations (17) shows that this euleroid is an algebraic curve of the sixth order represented by the equation

$$3(27x^2 + 16y^2)^3 - 18{,}225a^2x^4 - 13{,}834a^2x^2y^2 + 5{,}120a^2y^4 + 1{,}872a^4x^2 + 256a^4y^2 - 64a^6 = 0. \tag{19}$$

The character of the graph is shown in Fig. 2.

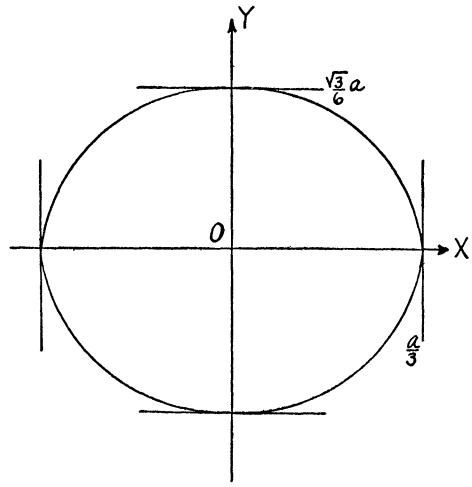


FIG. 2.

(c) *The case when* $e = 1/\sqrt{2}$. The parametric equations of the curve in this case are found to be

$$x = \pm \frac{a(1 + u^2)(1 - u^2)^{3/2}}{3 - u^2}, \qquad y = \frac{\sqrt{2}\,a}{4} \frac{u(1 + 5u^2)(1 - u^2)}{3 - u^2}. \tag{20}$$

From these equations we find

$$\frac{dy}{dx} = \pm \frac{\sqrt{2}}{4} \frac{5u^2 - 3}{u(1 - u^2)^{1/2}}, \tag{21}$$

so that this euleroid not only has vertical tangents at the points $(\pm a/3, 0)$, but also a vertical tangent at the origin. In fact, we readily verify that the curve has two cusps at the origin, the y-axis being the common cusp tangent. Next we find that horizontal tangents exist at points of the curve where the parameter values are $u = \pm\sqrt{15}/5$. The points with coordinates $(\pm 4a\sqrt{10}/75, a/\sqrt{30})$ are maximum points, while the points $(\pm 4a\sqrt{10}/75, -a/\sqrt{30})$ are minimum points.

Elimination of the parameter from equations (20) yields an algebraic equation of the tenth degree, but we shall not write this equation here.

The character of the graph is indicated in Fig. 3.

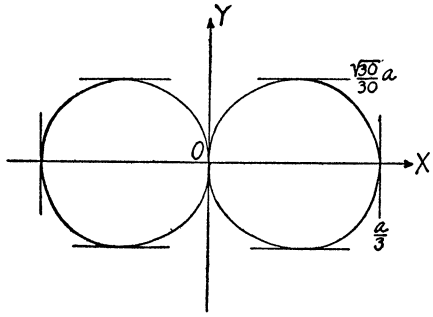


FIG. 3.

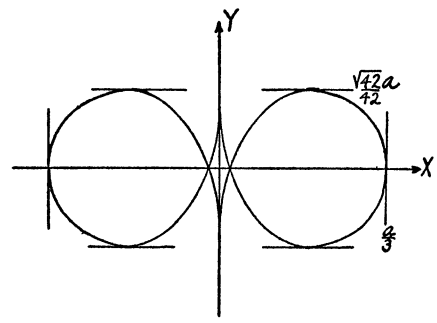


FIG. 4.

(d) *The case when* $e=2/\sqrt{7}$. In this case the euleroid is represented parametrically by the equations

$$x = \pm\,(a/9)(3 + 4u^2)(1 - u^2)^{3/2}, \qquad y = (au\sqrt{21}/42)(1 + 6u^2 - 8u^4). \tag{22}$$

From these equations we find

$$\frac{dy}{dx} = \mp\,\frac{3\sqrt{21}}{14}\,\frac{(1 - 2u^2)}{u(1 - u^2)^{1/2}}. \tag{23}$$

The curve meets the x-axis in the points $(\pm a/3,\ 0)$ where it has vertical tangents. Moreover, the curve has two double points on the x-axis at which the parameter values are $u=\pm\frac{1}{4}(6+2\sqrt{17})^{1/2}$. The coordinates of the double points are found to be

$$\left(\pm\,\frac{(71 - 17\sqrt{17})^{1/2}}{18}\,a,\ 0\right).$$

The slopes of the double point tangents are $\pm 3/\sqrt{42}/14(\sqrt{17}-1)^{1/2}$.

This curve meets the y-axis in the points $(0,\ \pm a\sqrt{21}/42)$, and at each of these points the curve has a cusp with the y-axis as the common cusp tangent.

The curve has horizontal tangents at the points where the parameter values are $u=\pm 1\sqrt{2}$. We find that the points $(\pm 5a\sqrt{2}/36,\ a/\sqrt{42})$ are maximum points, and that the points $(\pm 5a\sqrt{2}/36,\ -a/\sqrt{42})$ are mimimum points on the curve.

If the parameter is eliminated from equations (22), an algebraic equation of the tenth degree is obtained, but we shall omit the writing of this result.

This analysis enables us to sketch the graph indicated in Fig. 4.

A GEOMETRIC APPROACH TO THE COVARIANTS OF A CUBIC*

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1. The covariants. The fundamental covariants of a binary cubic, $a_0x^3+a_1x^2y+a_2xy^2+a_3y^3$, are usually obtained algebraically [1]. If the form is equated to zero the three roots of the resulting cubic equation in x and y can be represented by three points A, B, C, of the complex plane. The cross ratio of these three points and a fourth point D is written as (AB, CD) and defined by

$$(1) \qquad \alpha = (AB, CD) = \frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z_4 - z_2}{z_4 - z_1},$$

where the coordinates of A, B, C, D, are z_1, z_2, z_3, z_4, respectively. If $\alpha=a+bi$ is given and the points A, B, C, are fixed, and distinct, then the position of D is uniquely determined by (1).

If three points A, B, C, and a value of α are given, then six points $A' \cdots C''$ are determined by $(AB, CC')=\alpha$, $(BC, AA')=\alpha$, $(CA, BB')=\alpha$ $(BA, CC'')=\alpha$, $(AC, BB'')=\alpha$, $(CB, AA'')=\alpha$. The triads $A'\,B'\,C'$, $A''\,B''\,C''$ are covariants of the triad ABC for each α, since a linear functional transformation

$$(2) \qquad w = \frac{az + b}{cz + d} \qquad ad - bc \neq 0$$

which transforms a given triad ABC into $A_1B_1C_1$ will transform $A'B'C'$ and $A''B''C''$ into $A_1'B_1'C_1'$ and $A_1''B_1''C_1''$ with $(A_1B_1, C_1C_1')=\alpha$, *etc.*

In this article are developed geometric relations between the triads ABC, $A'B'C'$, and $A''B''C''$ where α is restricted to $|\alpha|=1$. The general relations are developed by a study of the relations which exist for a particular triad ABC and then we use the linear fractional transformations (2) and the properties which they possess [2, Chap. V].

In the final section we obtain the equation of the cubic whose roots are the coordinates of A', B', C', in terms of the coefficients of the cubic whose roots are A, B, C, and α, where α is any complex number.

2. Principal theorems. In this section we present a number of theorems.

THEOREM 1. *Any three distinct points A, B, C, in the complex plane together with a cross ratio α, where α is a complex number such that $|\alpha|=1$, determine two triads of points $A'B'C'$, $A''B''C''$ such that a point of either of the triads is the inverse of the corresponding point to the other triad with respect to the circle determined by A, B, and C.*

* This paper is part of a thesis, written under the direction of Professor F. E. Woodard, presented in partial fulfillment of the requirements for the Master of Arts degree at the University of Oregon.

In the special values $A \equiv -1$, $B \equiv 1$, $C \equiv k$, where k is any real number, and $\alpha = a + bi$, lead to values for A' and for A''. Use of the fact that $|\alpha| = 1$ makes $A' = \overline{A}''$, where $\overline{A}$ is the conjugate imaginary of A. The linear fractional transformation will take the axis of reals into a circle. A' and A'' will be inverse points with respect to the circle ABC. This property is maintained by the linear fractional transformation, so that the theorem is true for any three distinct points A, B, C.

THEOREM 2. *The locus of the points A' of Theorem* 1, *as α varies under the restriction $|\alpha| = 1$, is a circle orthogonal to the circle ABC at the point A.*

In the proof of Theorem 1 let us take $k = 0$ and obtain an expression for A' in terms of a and b. A' is a complex number which we may represent by $A' = x + yi$. If we equate the real parts and the imaginary parts we then obtain two equations from which a and b can be eliminated. The result is the equation of a circle with center on the axis of reals passing through the point A. Therefore the circle upon which A' moves is orthogonal to the circle ABC (the axis of reals) at the point A. Using the linear fractional transformation the result may be generalized to prove the theorem. Similarly the circle on which B' moves and the circle on which C' moves may be determined.

COROLLARY. *Both A' and A'' move on a single circle; as do B' and B'', C' and C''.*

A'' moves on a circle inverse to the circle on which A' moves. The circles coincide since an orthogonal circle is its own inverse.

The cross ratio of the four points A, C, A', C', in the particular case $(-1\ 1,\ 0C') = (1\ 0,\ -1A') = a + bi$ is real. Therefore the four points lie on a circle. In general $ACA'C'$, $BCB'C'$, and $ABA'B'$, lie on circles as do $ACA''C''$, $BCB''C''$, and $ABA''B''$.

THEOREM 3. *If θ is the positive acute angle between the circle ABC and the circle $ACA'C'$ we have $\theta = \arctan |b/1-a|$.*

The equation of the circle through -1, 0, A', is

$$x^2 + y^2 + x + \frac{1-a}{b} y = 0. \tag{3}$$

The circle (3) cuts the circle ABC (the axis of reals) at -1 and 0. The slope of the tangent at these points is $dy/dx = b/1-a$ which in this case gives the tangent of the angle between the two curves. To obtain the positive acute angle we use $|dy/dx|$. The linear fractional transformation preserves the angle between the two curves; thus the theorem is true in general.

THEOREM 4. *There are six circles making an angle θ with circle ABC, four through each of the points A, B, C. These circles are tangent in pairs at the points A, B, C, or intersect by pairs in the points $A' \cdots C''$, provided that $\alpha \neq \pm 1$.*

At a point, say C, there are two lines making an angle θ with circle ABC. Circle $ACA'C'$ is tangent to one of these lines. Of the other two circles through C but not through A, one is tangent to circle $ACA'C'$ at C. The other has the two points C and C' in common with it.

THEOREM 5. *When* $\alpha=-1$, $A'\equiv A''$, $B'\equiv B''$, $C'\equiv C''$, *and the three points lie on the circle* ABC.

From Theorem 3 we have $\theta=0$. The six circles of Theorem 4 are identical with circle ABC. $(BC, AA')=\alpha$, $(CB, AA')=1/\alpha$, $(CB, AA'')=\alpha$; but $\alpha=-1$, so $A'\equiv A''$.

Let us take $\alpha=\omega_1$, where ω_1 is an imaginary cube root of minus one. We define the Hessian points of the triad ABC, H_1 and H_2, by $(AB, CH_1)=\omega_1$,, $(BA, CH_2)=\omega_1$. (Algebraically the triad ABC may be considered as the roots of a cubic equation. Each of the points H_1 and H_2 is known [1, p. 197] to form an equianharmonic range with the triad ABC.) It can be shown for $\alpha=\omega_1$ that $A'\equiv B'\equiv C'\equiv H_1$ and $A''\equiv B''\equiv C''\equiv H_2$; the Hessian points are inverse points with respect to circle ABC.

THEOREM 6. *The Hessian points,* H_1 *and* H_2, *lie on circles through* AB, BC, CA, *making angles of* $60°$ *with the circle* ABC, *and conversely.*

This theorem is an immediate consequence of Theorem 4.

THEOREM 7. *There is an infinite number of triads on the circle* ABC *having the Hessian points* H_1, *and* H_2.

Take a fixed circle O with a pair of inverse points H_1 and H_2. A circle Q through H_1 and H_2 cuts circle O in two points, say A and A_1. Two circles through A and H_1 each making an angle of $60°$ with circle O at A will cut circle O at B and at C. ABC so determined will have Hessian points H_1 and H_2 since the conditions of Theorem 6 are satisfied. There is an infinite number of circles through H_1 and H_2. We see that there is an infinite number of triads on circle O having Hessian points H_1 and H_2.

From Theorem 2 a construction for the Hessian points can be easily indicated. We know that when $\alpha=\omega_1$, ω_2 then $A'\equiv B'\equiv C'\equiv H_1$ and $A''\equiv B''\equiv C''\equiv H_2$. The loci of Theorem 2 have two points in common, the Hessian points.

To obtain the locus of A' as α varies we use: (i) $\alpha=1$, then $A'\equiv A$; (ii) $\alpha=-1$ then $A'\equiv A_1$ which can readily be constructed; (iii) the orthogonal condition. Similarly the loci of B' and C' may be obtained.

We state the following lemma without proof.

LEMMA. *If we have four distinct points* A, B, C, C_1, *such that* $(AB, CC_1)=-1$) *then* C *and* C_1 *are inverse points with respect to the circle which meets the circle* ABC *orthogonally at* A *and* B.

The lemma leads to constructions for the fourth harmonic point to three given distinct points in the complex plane. When A, B, and C are not collinear

draw circle ABC. Construct the tangent at A and at B. They intersect at a point O. If we draw CO it will cut circle ABC in C_1. When ABC are collinear draw circle ABC, a straight line, and construct a circle S orthogonal to circle ABC at A and B. If we construct a circle S' orthogonal to circle S and orthogonal to circle ABC at C, it will cut circle ABC again at C_1. The second construction also applies if A, B, C, lie on a circle. The circle S' is easily obtained if we use its center as O' at the intersection of the line AB and the tangent to the circle ABC at C'.

The construction for the Hessian points described here is more easily indicated than the construction given by Grace and Young [1, p. 211]; however in both constructions the same circles are used.

3. Algebraic considerations. In this section we regard the triad ABC as the roots r, s, t, *of a cubic equation*

$$a_0z^3 + a_1z^2 + a_2z + a_3 = 0. \tag{4}$$

Using the cross ratio α, any complex number, and taking the roots in cyclic order, $(rs, tt') = \alpha$, $(st, rr') = \alpha$, $(tr, ss') = \alpha$ we obtain the roots r', s', t', of a cubic equation

$$A_0z^3 + A_1z^2 + A_2z + A_3 = 0. \tag{5}$$

The roots r', s', t', are cyclic symmetric functions of r, s, and t. By use of the elementary symmetric functions we obtain (5) in the form

$$(Pz^3+Qz^2+Rz+S)(\alpha^2-\alpha) \mp (2\alpha^3-3\alpha^2-3\alpha+2)(a_0z^3+a_1z^2+a_2z+a_3)\sqrt{\Delta}=0. \tag{6}$$

where

$$\begin{aligned}
P &= 2a_1^3 - 9a_0a_1a_2 + 27a_0a_3 \\
Q &= 2a_1^2a_2 + 27a_0a_1a_3 - 18a_0a_2^2 \\
R &= 18a_1^2a_3 - 3a_1a_2^2 - 27a_0a_2a_3 \\
S &= 9a_1a_2a_3 - 27a_0a_3^2 - 2a_2^3 \\
\Delta &= 18a_0a_1a_2a_3 - 4a_0a_2^3 + a_2^2a_1^2 - 4a_1^3a_3 - 27a_3^2a_0^2.
\end{aligned}$$

The ambiguity of sign in (6) will give either the cubic with roots r', s', t', or with roots r'', s'', t''.

By consideration of (6) we note that the associated cubic with roots r', s', t', is a combination of the covariant cubic [3, p. 175] and the original cubic. The covariant cubic is obtained when $\alpha = -1$, $1/2$, 2, the three values gotten if and only if the four elements forming the cross ratio are a harmonic set.

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1. J. H. Grace and A. Young, Algebra of Invariants, Cambridge University press, 1903.
2. E. J. Townsend, Functions of a Complex Variable, New York, Henry Holt and Co., 1915.
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DISCUSSIONS AND NOTES

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The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE EXACT NUMBER OF PRIMES BELOW A GIVEN LIMIT*

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1. *The problem.* In the solution of Problem 4083 (this MONTHLY, vol. 51, 1944, pp. 479–480) the following question is raised: How can we prove that there are 50,847,478 primes less than one billion although these primes are not individually known?

The prime number theorem gives an analytic formula for the approximate number of primes which are less than or equal to a given limit x. But it does not enable us to determine the exact number $\pi(x)$ of such primes.

Let $p_1 < p_2 < \cdots < p_n$ be the primes less than or equal to $x^{1/2}$. Then

$$\pi(x) = [x] + n - 1 - \sum_{\kappa} \left[\frac{x}{p_\kappa}\right] + \sum_{\kappa<\lambda} \left[\frac{x}{p_\kappa p_\lambda}\right] - \cdots$$

where $p_\kappa, p_\lambda, \cdots$ run over the primes $p_1, p_2 \cdots, p_n$. This follows at once from a formula already given by Legendre. But for large values of x this formula is completely impracticable.

2. *A solution.* The problem in question was solved by E. Meissel in 3 papers.† In the first paper, he proved the following theorem which enables us to find $\pi(x)$ if the primes $q \leqq x^{1/2}$ are known.

Let $\phi(x, r)$ be the number of positive integers less than or equal to x which are relatively prime to each of the first r primes. Set $\pi(x^{1/3}) = m$, $\pi(x^{1/2}) = n$, and $n - m = s$. Then

$$(1) \qquad \pi(x) = \phi(x, m) + m(s + 1) + \frac{s(s-1)}{2} - 1 - \sum_{\sigma=1}^{s} \pi\left(\frac{x}{p_{m+\sigma}}\right).$$

Using (1) Meissel actually calculated $\pi(20{,}000)$, $\pi(500{,}000)$, $\pi(10^6)$, $\pi(10^7)$, $\pi(10^8)$, and $\pi(10^9)$.

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* Presented at the meeting of the Southeastern Section of the Mathematical Association of America, Raleigh, April 19, 1946.

† Über die Bestimmung der Primzahlmenge innerhalb gegebener Grenzen, Mathematische Annalen, vol. 2, 1870, pp. 636–642.—Berechnung der Menge von Primzahlen, welche innerhalb der ersten Hundert Millionen natürlicher Zahlen vorkommen, Mathematische Annalen, vol. 3,1871, pp. 523–525.—Berechnung der Menge von Primzahlen, welche innerhalb der ersten Milliarde natürlicher Zahlen vorkommen, Mathematische Annalen vol. 25, 1885, pp. 251–257.

Later other proofs for (1) were given and similar formulas were proved by A. Lugli,* F. Rogel,† and A. Baranowski.‡ But (1) remains the simplest formula for the calculation of $\pi(x)$. It was also used to find errors in the tables of primes.§

Meissel's method is outlined in a number of books, for instance in Mathews, *Theory of Numbers*, 1892, pp. 273–278, Uspenski and Heaslet, *Elementary Number Theory* 1939, pp. 117–125, Wertheim, *Elemente der Zahlentheorie*, 1887, pp. 21–25, and Wertheim, *Anfangsgründe der Zahlenlehre*, 1902, pp. 56–63.

The following very short and simple proof of (1) may be of interest.

We have $\phi(x, k-1)$ integers t which satisfy $t \leqq x$ and $(t, p_1p_2 \cdots p_{k-1}) = 1$; exactly $\phi[x/(p_k), k-1]$ of them are divisible by p_k. Hence

$$\phi(x, k) = \phi(x, k-1) - \phi\left(\frac{x}{p_k}, k-1\right). \tag{2}$$

Adding (2) for $k = m+1, m+2, \cdots, n$, we obtain

$$\phi(x, n) = \phi(x, m) - \sum_{\sigma=1}^{s} \phi\left(\frac{x}{p_{m+\sigma}}, m+\sigma-1\right). \tag{3}$$

From the definition of m, n, and s it follows that

$$x^{1/3} < p_{m+\sigma} \leqq x^{1/2} \leqq \frac{x}{p_{m+\sigma}} < x^{2/3} \qquad (\sigma = 1, 2, \cdots, s). \tag{4}$$

Now, let a and b be numbers such that $a \leqq b \leqq a^2$. The integers less than or equal to b and relatively prime to each prime less than or equal to a are the number 1 and the $\pi(b) - \pi(a)$ primes q for which $a < q \leqq b$. Applying this for $b = x$ and $a = x^{1/2}$, we obtain

$$\phi(x, n) = 1 + \pi(x) - \pi(x^{1/2}) = \pi(x) - n + 1. \tag{5}$$

For $b = x/p_{m+\sigma}$ and $a = p_{m+\sigma} - \epsilon$ where $0 < \epsilon < \min(1, p_{m+\sigma} - x^{1/3})$, we obtain, by (4),

$$\begin{aligned} \phi\left(\frac{x}{p_{m+\sigma}}, m+\sigma-1\right) &= 1 + \pi\left(\frac{x}{p_{m+\sigma}}\right) - \pi(p_{m+\sigma} - 1) \\ &= \pi\left(\frac{x}{p_{m+\sigma}}\right) - (m+\sigma-2). \end{aligned} \tag{6}$$

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* Sul numero dei numeri primi da 1 ad n, Giornale di Matematiche, vol. 26, 1888, pp. 86–95.

† Zur Bestimmung der Anzahl Primzahlen unter gegebenen Grenzen, Mathematische Annalen, vol. 36, 1890, pp. 304–315.—Die Bestimmung der Anzahl der unter einer gegebenen Grenze liegenden Primzahlen, Archiv der Mathematik und Physik (2), vol. 17, 1900, pp. 225–237.

‡ O wzorach sluzacych do oblizenia liczby liczb pierwszych nie przekraczajacych danej granicy. Rozprawy Akademii Umiejetnosci w Krakowie (2), vol. 8, 1895, pp. 192–219. (Extract in German in Anzeiger der Akademie der Wissenschaften in Krakau (1894), pp, 280–281.

§ J.-P. Gram, Rapport sur quelques calculs entrepris par M. Bertelsen et concernant les nombres premiers, Acta Mathematica, vol. 17, 1893, pp. 301–314.

It follows from (5), (3) and (6) that

$$\pi(x) = \phi(x, n) + n - 1 = \phi(x, m) - \sum_{\sigma=1}^{s} \phi\left(\frac{x}{p_{m+\sigma}}, m + \sigma - 1\right) + m + s - 1$$

$$= \phi(x, m) - \sum_{\sigma=1}^{s} \pi\left(\frac{x}{p_{m+\sigma}}\right) + \sum_{\sigma=1}^{s} (m + \sigma - 2) + m + s - 1$$

$$= \phi(x, m) + m(s + 1) + \frac{s(s-1)}{2} - 1 - \sum_{\sigma=1}^{s} \pi\left(\frac{x}{p_{m+\sigma}}\right).$$

This proves (1).

The most difficult step in the use of (1) is the calculation of $\phi(x, m)$. For this calculation we may use (2) and the simple fact that for $[x] = gp_1p_2 \cdots p_k \pm r$ where g and r are non-negative integers, we have

$$\phi(x, k) = g \cdot \phi(p_1p_2 \cdots p_k) \pm \phi(r, k). \tag{7}$$

This holds since each complete system of residues (mod $p_1p_2 \cdots p_k$) contains $\phi(p_1p_2 \cdots p_k)$ integers relatively prime to the first k primes, and since with z also $gp_1p_2 \cdots p_k \pm z$ is relatively prime to $p_1p_2 \cdots p_k$. For the positive sign this was already used by Meissel, for the negative sign by Hossfeld.* Moreover, we may use

$$\phi(p_2p_3 \cdots p_k, k) = \tfrac{1}{2}\phi(p_1p_2 \cdots p_k, k) = \tfrac{1}{2}\phi(p_1p_2 \cdots p_k). \tag{8}$$

3. *An example.* As an example, let us calculate $\pi(2000)$. Since $[2000^{1/3}] = 12$, $[2000^{1/2}] = 44$, we have $m = 5$, $n = 14$, and $s = 9$, hence

$$\pi(2000) = \phi(2000, 5) + 50 + 36 - 1 - \sum_{\sigma=1}^{9} \pi\left(\frac{2000}{p_{5+\sigma}}\right).$$

Now $\pi(2000/13) = 36$, $\pi(2000/17) = 30$, $\pi(2000/19) = 27$, $\pi(2000/23) = 23$, $\pi(2000/29) = 19$, $\pi(2000/31) = 18$, $\pi(2000/37) = 16$, $\pi(2000/41) = 15$, $\pi(2000/43) = 14$, hence $\pi(2000) = \phi(2000, 5) - 113$.

The computation of $\phi(2000, 5)$ will become simpler than that of Wertheim in his book, *Anfangsgründe der Zahlenlehre*, p. 57, by the use of (7) and (8). We have

$$\phi(2000, 5) = \phi(2310) - \phi(310, 5) = 480 - \phi(310, 5),$$

$$\phi(310, 5) = \phi(310, 4) - \phi(28, 4) = \phi(210) + \phi(100, 4) - 6 = 42 + \phi(100, 4),$$

$$\phi(100, 4) = \phi(105, 4) - 2 = \tfrac{1}{2}\phi(210) - 2 = 22,$$

$$\phi(2000, 5) = 480 - 42 - 22 = 416,$$

$$\pi(2000) = 416 - 113 = 303.$$

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* Bemerkungen über eine zahlentheoretische Formel, Zeitschrift für Mathematik und Physik, vol. 35, 1890, pp. 382–384.

ON THE ORTHOPOLE

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1. In a paper published recently in this MONTHLY, vol. 53, 1946, p. 200, I have generalized the following theorem considered by Murnaghan, Mathesis, vol. 41, 1927, p. 27; Godeau, Mathesis, vol. 41, 1927, p. 72; and Ramler, this MONTHLY, vol. 47, 1940, p. 140:

When two triangles inscribed in the same circle are such that the sides of the first triangle have the same orthopole as to the second triangle, the sides of the second one have also the same point as common orthopole with respect to the first triangle; and that common orthopole is the mid-point of the segment joining the orthocenters of the two triangles.

The theorem had already been given in the following more general form, different from the one I have mentioned, by Kantor, Mathesis, vol. 26, 1906, p. 144:

If two triangles $A_1A_2A_3$, $B_1B_2B_3$ are inscribed in a circle, the orthopoles A_1', A_2', A_3' of the sides of $A_1A_2A_3$, with respect to the triangle $B_1B_2B_3$, and the orthopoles B_1', B_2', B_3' of the sides of $B_1B_2B_3$, with respect to the triangle $A_1A_2A_3$, are six points on a circle having as center the mid-point of the segment joining the orthocenters of the triangles.

Absolonne has proved, Mathesis, vol. 27, 1907, p. 23, that, R being the radius of the common circumcircle and θ the angle formed by the homologous sides of $A_1A_2A_3$ and $A_1'A_2'A_3'$, $B_1B_2B_3$ and $B_1'B_2'B_3'$, the radius of the circle containing the six orthopoles is $R\cos\theta$.

2. In a recent article published in the Bulletin de la Société royale des Sciences de Liége, 1946, p. 119. I have mentioned the notion of orthopole of an $(n-1)$-sided polygon inscribed in a circle as to an n-sided polygon inscribed in the same circle, and given several theorems, one of which generalizes the just recalled Kantor theorem.

It is known that the orthopole of a chord B_1B_2 of a circle Γ as to the inscribed triangle $A_1A_2A_3$ is the intersection of the Simson lines of B_1 and B_2; when a chord B_1B_2 and a quadrilateral $A_1A_2A_3A_4$ incribed in Γ are given, the orthopoles of B_1B_2 as to the four triangles obtained by omitting one vertex in $A_1A_2A_3A_4$ are on a straight line, which we call the orthopolar line of B_1B_2 as to $A_1A_2A_3A_4$; when a triangle $B_1B_2B_3$ and the quadrilateral $A_1A_2A_3A_4$ are given, both inscribed in Γ, the orthopolar lines of the chords forming $B_1B_2B_3$ as to $A_1A_2A_3A_4$ meet at a point, which we call the orthopole of $B_1B_2B_3$ as to $A_1A_2A_3A_4$; continuing the process we get to the notion of orthopole of an $(n-1)$-sided polygon as to an n-sided one, both inscribed in Γ.

Further, to *any* n-sided polygon $A_1A_2\cdots A_n$ inscribed in Γ may be associated an n-sided *regular* polygon $L_1L_2\cdots L_n$ as follows: Ω being any point on Γ, any vertex L_i is given by

$$\Omega L_i = \frac{1}{n}(\Omega A_1 + \Omega A_2 + \cdots + \Omega A_n),$$

multiples of 2π being added to the various arcs considered. The polygon $L_1L_2 \cdots L_n$ is the same for any position of Ω on Γ and may be called the *mean* polygon of $A_1A_2 \cdots A_n$.

It may finally be noted that, if O is the center of Γ, we may call *orthocenter* of the cyclic polygon $A_1A_2 \cdots A_n$ the extremity of the resultant of the forces represented by OA_1, $OA_2 \cdots$, OA_n.

Then the generalization of Kantor's theorem which I have given is as follows:

When two n-sided polygons $A_1A_2 \cdots A_n$ and $B_1B_2 \cdots B_n$ are inscribed in a circle Γ, the n orthopoles, with respect to $A_1A_2 \cdots A_n$, of all the polygons obtained by omitting one of the vertices of $B_1B_2 \cdots B_n$ and the n orthopoles, with respect to $B_1B_2 \cdots B_n$, of all the polygons obtained by omitting one of the vertices of $A_1A_2 \cdots A_n$ are 2n points on a circle having as center the mid-point of the segment joining the orthocenters of the polygons. If ϕ is the angle formed by the mean polygons of the given polygons, the radius of the circle equals that of Γ multiplied by $\cos \frac{1}{2}n\phi$ *when n is even and by* $\sin \frac{1}{2}n\phi$ *when n is odd.*

FERMAT'S THEOREM AS A PROBLEM IN PROBABILITY

E. C. Molina, East Orange, New Jersey

Consider an urn containing a white and b black balls, where a and b are both positive integers. Let n drawings be made, the ball being replaced in the urn after each drawing. Then

$$P = \left(\frac{a}{a+b}\right)^n + \left(\frac{b}{a+b}\right)^n$$

is the probability that the balls drawn will be *all white or all black.*

Now let the balls be repainted so that A are white and B black, where

$$A + B = a + b.$$

Then repeating the n drawings, we obtain

$$P' = \left(\frac{A}{A+B}\right)^n = \left(\frac{A}{a+b}\right)^n$$

as the probability that the balls drawn will be *all white.*

Fermat's theorem may then be stated as follows. *It is impossible so to repaint the balls that for $n>2$, we have*

$$P = P';$$

for this would give

$$a^n + b^n = A^n.$$

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, N. Y.

CLUB REPORTS 1945–46

Pi Mu Epsilon, Northwestern University

There were twelve meetings during the academic year 1945–1946, two of which were held during the summer of 1945. The main event of the year was an open house and banquet sponsored in conjunction with the northwestern Mathematical Colloquium to celebrate the moving of the mathematics department into new quarters. Members of the mathematics departments of the University of Chicago and the Illinois Institute of Technology were invited to a tea. The following papers were presented after the tea and after the banquet:

Index theorems in the calculus of variations, by Professor M. R. Hestenes of the University of Chicago.

Experiences of an editor, by Professor L. R. Ford of the Illinois Institute of Techology.

This year *Illinois Beta* sponsored an undergraduate mathematical competition. Four prizes consisting of books were awarded as follows: First prize: Henry F. Hunter, senior; second prizes: Marjorie L. Smith, senior; James B. Serrin, junior; and Edmond U. Cohler, sophomore.

The place of mathematics in a liberal education, by Professor E. P. Northrop of the University of Chicago, was the title of the talk following the February initiation of twenty-three members. Seven others were initiated in May.

Program meetings for the year were held as follows:

Multisensory aids in the teaching of mathematics, by Professor E. H. C. Hildebrandt.

Finite differences, by Miss D. M. DeWitt.

Transformations, by R. R. Reynolds.

Some mathematical aspects of atomic energy, by Professor H. T. Davis.

Infinite products, by Martinus Esser.

Quality control, by Professor M. E. Wescott.

Greek mathematics in relation to modern mathematics, by Professor E. Hellinger.

Transcendental numbers, by H. F. Hunter.

Introduction to lattic theory, Professor J. W. Givens.

Officers for the year 1945–1946 were: President, H. F. Hunter; Vice-President, Margaret McWhirter; Secretary, Nelda Gulbransen; Treasurer, Helmut Abt. Officers for the year 1946–1947 are: President, Helen Gray; Vice-President, Rose Ann Grundman; Secretary, Marjorie Smith; Treasurer, Arlene Lindahl.

Pi Mu Epsilon, St. Louis University

Lectures were given at the three regular meetings of the Missouri Gamma Chapter of *Pi Mu Epsilon* during 1945–46:

The calculus of variations, by the Rev. Walbert Kalinowski, O.S.B.

The elementary transcendental functions, by Mr. Clarence Wallen, S.J.

The shot effect, by Miss Helen Ann Jackson.

The theory of braids was the title of Professor Emil Artin's talk, which was the Annual Mathematics Lecture, following the April initiation in the Medical School Auditorium. More than 250 members and guests of the Chapter attended.

At the banquet which culminated the day's activities, the Rev. Patrick J. Holloran, S.J., President of St. Louis University, delivered the principal address, welcoming the returning members from the armed services and expressing his personal and the University's interest in the growth of the mathematical activities on the campus.

The officers for 1945–46 were: Director, Miss Helen Ann Jackson; Vice-Director, Herbert Gebhart; Secretary-Treasurer, Mary Nawrocki. Professor Francis Regan has been the Corresponding Secretary and Faculty Adviser.

Kappa Mu Epsilon, Upsala College

Nine meetings were held throughout the year, at four of which the following papers were presented by the members.

Mathematical terminology, by Audrey Richter.

The trisection of an angle, by Arnold Goust.

The psychology of arithmetic, by Natalie Manno.

Mathematics and the imagination, by Marian Larsen.

Cooperative discussions and a trip to the Hayden Planetarium of New York City occupied three meetings. At the midyear special meeting a discussion of probability was presented through

The concept of the random walk, with applications to war problems, by Mr. Edward Molina, formerly research engineer with the Bell Telephone Laboratories.

Our annual banquet and initiation was held on May 31, when three new members were admitted to the chapter.

Mathematics as related to the work on an actuary, given by Mr. Walter Klem of the Mutual Life Insurance Company of New York City, was the evening address.

Officers for the year 1945–1946 were: President, Audrey Richter; Vice-President, Mrs. Mary McKim; Treasurer, Marian Larsen; Secretary, Natalie Manno; Historian, Betty Rudeback; Faculty Sponsor, Professor M. A. Nordgaard.

Pi Mu Epsilon, Michigan State College

For the initiation meeting in May 1945, the program consisted of a series of five minute talks on

Interesting mathematical curves, by the initiates Barbara Brisbin, Portia Treend, David Zaukelies, William Coleman, Patricia Wilson, Barbara Gregory, Martha Driver, Rosalie Manz, Carolyn Trimm, and Joan Bowman.

After an inactive summer term, programs were continued as follows:

Computation of life insurance premiums, by Barbara Brisbin, with computations by Dr. J. S. Frame of net premiums on policies suggested by members of the audience.

Riddles in mathematics, presented from Northrup's book of that title by the initiates Charles Costa, Wesley Porterfield, Marjorie Vis, Clarence Healey, Raymond Hurlburt, William Krupp, Robert Jackson, Katherine Weiss, Mary Fuss, Eleanor Bessonen, Beverly Holcomb, Virginia Snyder, Joanne Attwood, and Doris Rinehart.

Zeller's congruence, by Rosalie Manz, and

Kraitchik's nomogram to solve Zeller's congruence, by Barbara Gregory.

String constructions, by Dr. B. M. Stewart at the fall picnic in the forestry cabin, with all hands strung up in attempted imitation.

Doughnuts, by Dr. B. M. Stewart.

Secret and urgent, presented from Fletcher Pratt's book of that title by the initiates Harlan Ogle, Phyllis Rowe, Floyd Lamb, and Joyce Conklin.

Selected topics from statistics, including

Multinomial theorem applied to smelling salts tests, by Mary Baumgarten.

Interesting curves pertaining to the parabola, by Margaret Fox.

Comparison of the old witch to the normal curve, by Marjorie Vis.

Fitting a straight line to data by least squares, by Jane Walker.

A tribute to Georg Cantor, by Dr. F. Herzog, presented after the winter term banquet.

The arbelos or shoemaker's knife, by Eleanor Bessonen.

The trimetric ruler, by Eunice Anderson.

Coin weighing, by Robert Jackson.

Three Louis C. Plant scholarship awards were presented at the February banquet to outstanding Juniors majoring in mathematics and deemed most worthy on the basis of scholarship, interest in mathematics, and helpfulness to the Department of Mathematics. Barbara Gregory received a first prize of fifty dollars, and Rosalie Manz and Carolyn Trimm split the second prize, receiving fifteen dollars each. A spring picnic was held in the East Lansing State Park.

The following officers were elected for 1946–47: President, Barbara Gregory; Vice-President, Virginia Snyder; Secretary-Treasurer, Charles Costa; Membership Committee, Rosalie Manz, Carolyn Trimm, and Phyllis Rowe.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Analytic Geometry, Third Edition. By F. S. Nowlan. New York, McGraw-Hill Book Co., 1946. 14+355 pages. $2.75.

Here is a rarity indeed: an elementary geometry text which seems to be mathematically sound throughout. The book contains all that anyone would be likely to give in a first course on plane analytic geometry, together with enough of the three-dimensional theory to form a basis for a first course in calculus. It seems unnecessary to go into detail; however we shall point out certain unusual features of the book.

In the plane geometry the emphasis is laid upon direction cosines, though the slope concept is carefully discussed. Equality of two ordered pairs of real numbers $x:y$, $a:b$ is defined as follows: $x:y=a:b$ if and only if $x=ka$, $y=kb$ for some non-zero k. By use of this symbolism the inelegant possibility of zero denominators is entirely avoided. However, the author is not unconscious of the need for the notion of slope in other connections; indeed a later chapter discusses ordinary and partial derivatives of polynomials. These concepts are introduced from a geometrical point of view, with as much rigour as a student is likely to encounter in any first course in calculus.

One would hardly expect to find in these days a real contribution to the theory of polar coordinates; yet the author has treated this subject from a fresh and worthwhile point of view. He conceives of three types of polar coordinates, one in which the radius vector is non-negative, one in which it is non-positive, and finally a "mixed" type. As a result he is able to handle all questions both with rigour and with unusual ease, and his methods, moreover, are particularly useful in the calculus.

The distinction between a necessary and a sufficient condition is pointed out at an early stage in the text and is used throughout. This is particularly obvious in two contrasting chapters: Chapter 3, entitled "Loci and Their Equations" and Chapter 4, of which a partial title is "Equations and Their Loci."

A great deal of care seems to have gone into this book. The reviewer noted a somewhat obscure sentence near the bottom of p. 67, an incomplete statement about a limit at the top of p. 81, and a slight slip in the statement of Corollary 1, p. 107, but could find nothing else amiss. Apparently a textbook need not be incorrect to be elementary!

R. H. BRUCK

Analytic Geometry. By F. D. Murnaghan. New York, Prentice-Hall, Inc., 1946. 8+402 pages. $3.25.

This is the most comprehensive American text on ordinary analytic geometry. Within its covers is much mathematics that is not usually to be found in such texts—much, in fact, that seems to be the author's own development. The method of vectors and matrices is used throughout and the applications of analytic geometry to engineering and physical problems are continually kept in view. The standards of rigor are high, the theoretical explanations are detailed.

The order of presentation of the material is strictly rational. The first chapter is concerned with points and vectors on a line, the second with points, vectors and lines in a plane, the third, fourth and fifth with plane vectors and vectors in space. Chapter five is an excellent treatment of determinants and matrices. It is a pleasure to read in an elementary text that one must "distinguish clearly between the *matrix*, which is a block of four numbers, and the *determinant*, which is a single number."

Now come the conics. Chapter VI is devoted to the circle and the sphere, Chapters VII, VIII and IX to the parabola, ellipse and hyperbola, respectively. The treatment of these familiar curves is not always along conventional lines, however. In the chapter on the circle, for instance, we find the concepts of point and line at infinity and the principle of duality, ideas which are usually not included in Cartesian geometry.

Chapter X is concerned with the quadric surfaces. The method of vectors and matrices has certain definite advantages in this treatment, but it may seem strange to one brought up in the old school to see the equation of a hyperboloid written in the form

$$(v'Av) - 1 = 0.$$

The concluding two chapters are devoted to that difficult subject, the reduction of the general equation of the second degree to canonical forms under translations and rotations by means of their invariants. Such reductions are to be found in the mathematical literature, but this seems to be the first textbook on analytic geometry which gives a complete reduction with characterization of the parameters.

At one point the author's desire for careful statement seems to have overshot the mark. Like several other recent writers, he speaks of *an* equation of a definite locus. This implies that there can be equivalent equations which are not identical, and immediately makes it necessary to explain when two equations are identical. Presumably $2x+3y-5=0$ and $4x+6y-10=0$ are not identical, but what about $2x+3y-5=0$ and $2x+3y=5$, or $3y+2x-5=0$, or $x+x+3y-5=0$? Any definition of identity of two equations would have to be artificial, for the only genuine relation of equality among equations is that of equivalence. It seems proper, therefore, to speak of *the* equation of a given straight line, for there is but one such equation in the sense of equivalence.

Upon reading the author's preface, the reviewer finds some ideas which he wishes to second heartily, and some which raise grave doubts. He has only the desire to echo the statement that "The demands of modern physics and engineering for a knowledge of the elements of calculus are so pressing that it is no longer practicable to postpone the teaching of calculus to the sophomore year." If our American high schools paid proper attention to the education of the gifted pupils, they could easily send them to college with a knowledge of analytic geometry. European schools have done this, and so can we.

It is only when the author states that "the methods used here are not too advanced for college Freshmen, and in fact are suitable for presentation in high school" that the reviewer, with tears and regrets, is forced to disagree. The book before us is too complete, too advanced in its concepts and too sophisticated in its notation for the high school pupil or for the student fresh from high school. Only the genius would survive, but he would then be a mathematician. But the book would seem to be ideal for a second course in analytics.

Since perfect rigor is a will-o'-the-wisp that seems to be able to stay just beyond our grasp, would it not be justifiable in our elementary teaching to admit that there are levels of rigor suitable for students of differing degrees of development? High school geometry is not rigorous, yet it has had an effective influence upon our philosophy. As we progress up the educational ladder, one of the sources of greatest pleasure is in looking down from a higher rung upon the more laborious and less sophisticated methods which we at one time so admired.

"There are also celestial bodies, and bodies terrestrial: but the glory of the celestial is one, and the glory of the terrestrial is another."

C. C. MACDUFFEE

NEW BOOKS RECEIVED

College Algebra. By A. A. Albert. New York, McGraw-Hill Book Co., 1946. 12+278 pages. $2.75.

College Algebra. By M. W. Keller, Boston, Houghton Mifflin Co., 1946. 471+12 pages. $3.00.

Curves. By R. C. Yates. West Point, U. S. Military Academy, 1946. 230+6 pages.

Differential and Integral Calculus. Second Edition. By R. R. Middlemiss. New York and London, McGraw-Hill Book Co., 1946. 8+497 pages. $3.25.

Mathematics and Life. By F. B. Knight, J. W. Studebaker, and G. Tate. Chicago, Scott Foresman Co., 1946. Book 1: 480 pages, $1.04. Book 2: 512 pages, $1.08.

Trigonometry. Second Edition. By H. K. Hughes and G. T. Miller. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd. 7+175+117 pages, 1946. $2.50.

SOLUTIONS

Exponents Yielding Only Odd Binomial Coefficients

E 706 [1946, 97]. *Proposed by D. H. Browne, Buffalo, N. Y.*

For what values of n does $(a+b)^n$ yield only odd coefficients?

I. *Solution by E. P. Starke, Rutgers University.* In order that all coefficients be odd, each ratio of consecutive coefficients,

$$\frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}, \cdots, \frac{n-k}{k+1}, \cdots, \frac{1}{n}, \tag{1}$$

must reduce to a fraction in simplest terms with both numerator and denominator odd. Evidently n itself is odd. Let $n=2^s u-1$, where u is odd. Then the ratio of the (2^s+1)th term to the 2^sth term is

$$\frac{n-(2^s-1)}{2^s} = \frac{2^s(u-1)}{2^s} = \frac{u-1}{1},$$

which reduces to a form having an even numerator over an odd denominator. This is contrary to our requirements unless $u=1$ so that $n=2^s-1$ and there is no (2^s+1)th term. Conversely, if $n=2^s-1$, the general term of the sequence (1) is $[2^s-(k+1)]/(k+1)$ with $k<n$. If $k+1=2^t v$, v odd, this ratio becomes $(2^{s-t}-v)/v$, which meets our requirements. Thus n is a power of 2 diminished by unity.

II. *Solution by Paul Bateman, Philadelphia, Pa.* We claim that if the binary expansion of n contains no zeros (*i.e.*, if n is of the form 2^k-1), then all the binomial coefficients are odd, and that if, on the contrary, the first zero counting from the right in the binary expansion of n is in the rth place, then the binomial coefficient $\binom{n}{2^{r-1}}$ is even (and is, in fact, the first such).

To this end we note that $\binom{n}{j+1}$ is obtained from $\binom{n}{j}$ by multiplying by $(n-j)/(j+1)$, $j=0, 1, \cdots, n-1$. If the first zero counting from the right in the binary expansion of j is in the r_jth place, then $j+1$ has a 1 in the r_jth place and zeros to the right thereof. Thus $j+1$ has 2 as a factor r_j-1 times. If the last r_j digits (at least) in the binary expansion of n are 1's, then $n-j$ has a 1 in the r_jth place an all zeros to the right thereof, and consequently has 2 as a factor r_j-1 times. Thus in going from $\binom{n}{j}$ to $\binom{n}{j+1}$ we do not pick up a factor 2. On the other hand, if the binary expansion of n has a zero in the rth place and only 1's to the right thereof, and if j is that number whose binary expansion consists of just $r-1$ digits 1, then $n-j$ will have 2 as a factor at least r times, whereas $j+1=2^{r-1}$. Thus in going from $\binom{n}{j}$ to $\binom{n}{j+1}$ we do pick up (for the first time) a factor 2.

III. *Solution by M. S. Knebelman, Washington State College.* The answer to the question stated is that n must be of the form 2^s-1. Since this is a special case of a more general problem that leads to an interesting conclusion, we restate the problem: What must be the form of n so that for all k, C_k^n is relatively prime to a prime $p<n$?

We shall say that a set of number $\{N_k\}$ is relatively prime to p if each member of the set is relatively prime to p. Let α be the greatest exponent such that $p^\alpha \leqq n+1$. Then since $kC_k^n = C_{k-1}^n(n+1-k)$ and $\{C_k^n\}$ is relatively prime to p, if we let $k=p^\alpha$ it follows that $n+1$ must be a multiple of p^α. Hence $n=ap^\alpha-1$, where $a<p$. This condition is necessary; that it is sufficient can be observed from the fact that $C_k^n = C_{k-1}^n(ap^\alpha-k)/k$. For in this case C_k^n cannot contain p as a factor if $k<p$, and if k contains p as a factor, $(ap^\alpha-k)/k$ does not, since $a<p$. If we require that $\{C_k^n\}$ be also relatively prime to another prime q, then $n=ap^\alpha-1=bq^\beta-1$, $b<q$. Hence $ap^\alpha=bq^\beta$. Suppose $q>p$. Then, from the last equation, we must have $b=cp^\alpha$, so that $a=cq^\beta>p$, which is a contradiction. Hence the set $\{C_k^n\}$ can be relatively prime to at most one prime p. We can combine these results into one theorem:

Let $n+1=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, *where* $p_1<p_2<\cdots<p_r$. *Then if* $p_1^{\alpha_1}\cdots p_{r-1}^{\alpha_{r-1}}<p_r$, $\{C_k^n\}$ *is relatively prime to* p_r *and is not relatively prime to any other prime less than* n. *If* $p_1^{\alpha_1}\cdots p_{r-1}^{\alpha_{r-1}}>p_r$, *the set* $\{C_k^n\}$ *is not relatively prime to any prime less than* n.

Also solved by D. W. Alling, Murray Barbour, R. C. Buck, H. S. Grant, H. D. Grossman, A. E. Karp, Nicholas Kushta, E. M. Michalup, P. A. Piza, Irene Price, W. J. Robinson, E. D. Schell, C. W. Topp, J. H. White, B. F. Yanney, and the proposer. Several solvers made use of Pascal's arithmetic triangle.

The Inverse of a Matrix

E 710 [1946, 97]. *Proposed by D. H. Lehmer, Aberdeen Proving Ground, Md.*

Find the inverse of the symmetric matrix of the nth order: $A=\|a_{ij}\|$, in which $a_{ij}=i/j$ for $i\leqq j$.

I. *Solution by D. M. Smiley and M. F. Smiley, Bethlehem, Pa.* Let A_n denote the matrix described and let $\|r_{ij}(n)\|$ denote its inverse (if, indeed, it has one). Direct computation for $n=2$, 3 leads one to suspect that $r_{ij}(n)=0$ if $i\geqq j+2$, and that $r_{ij}(n)=r_{ij}(n-1)$ if $i, j=1, \cdots, n-1$, and $(i, j)\neq(n-1, n-1)$. To test this, introduce numbers a, b, c, and set

$$C_n = \begin{Vmatrix} (A_{n-1})^{-1}+aB_{n-1} & b(T_{n-1})' \\ bT_{n-1} & c \end{Vmatrix},$$

in which T_{n-1} is the row matrix $(0, \cdots, 0, 1)$ of order $n-1$ and B_{n-1} is the square matrix $(0, \cdots, 0, (T_{n-1})')$ of order $n-1$. In order that $A_nC_n=I$, we must have $c=n^2/(2n-1)$, $b=-n(n-1)/(2n-1)$, and $a=(n-1)^2/(2n-1)$. Conversely, these values yield $A_nC_n=I$. From this result we easily find that

$$r_{ii} = 4i^3/(4i^2 - 1), \qquad (i = 1, \cdots, n-1),$$
$$r_{nn} = n^2/(2n - 1),$$
$$r_{i\,i+1} = -i(i+1)/(2i+1), \qquad (i = 1, \cdots, n-1),$$
$$r_{ij} = 0 \quad \text{for} \quad j > i+1.$$

II. *Solution by John Williamson, Queens College.* Let $A=(a_{ij})$ be a symmetric matrix of order n where $a_{ij}=a_i/a_j$, $i \leqq j$, $a_i \neq 0$ and $a_i \neq a_j$ if $i \neq j$. If

$$B = \begin{Vmatrix} 1 & -b_1 & 0 \cdots 0 & 0 \\ 0 & 1 & -b_2 \cdots 0 & 0 \\ \cdot & \cdot & \cdot \cdots \cdot & \cdot \\ 0 & 0 & 0 \cdots 1 & -b_{n-1} \\ 0 & 0 & 0 \cdots 0 & 1 \end{Vmatrix},$$

where $b_i=a_i/a_{i+1}$, $i=1, 2, \cdots, n-1$, $BAB'=D$, where $D=[d_1, d_2, \cdots, d_n]$ is a diagonal matrix $d_i=1-b_i^2$, $b_n=0$. Further, $(B')^{-1}A^{-1}B^{-1}=D^{-1}$, so that $A^{-1}=B'D^{-1}B$. Since D^{-1} is a diagonal matrix $[k_1, k_2, \cdots, k_n]$, $k_i=1/d_i$, a simple matrix calculation gives $A^{-1}=R=(r_{ij})$, where

$$r_{ii} = k_i + b_{i-1}^2 k_{i-1}, \qquad b_0 = 0,$$
$$r_{i\,i+1} = r_{i+1\,i} = -b_i k_i,$$
$$r_{ij} = 0, \qquad |i-j| \geqq 2.$$

On substituting for k_i and b_i their values we find

$$r_{ii} = a_i^2(a_{i+1}^2 - a_{i-1}^2)/(a_{i+1}^2 - a_i^2)(a_i^2 - a_{i-1}^2), \qquad i = 1, 2, \cdots, n-1,$$
$$r_{nn} = a_n^2/(a_n^2 - a_{n-1}^2),$$
$$r_{i\,i+1} = -a_{i+1}a_i/(a_{i+1}^2 - a_i^2),$$

and if $a_i=i$,

$$r_{ii} = 4i^3/(4i^2 - 1),$$
$$r_{nn} = n^2/(2n - 1),$$
$$r_{i\,i+1} = -i(i+1)/(2i+1),$$

$i=1, 2, \cdots, n-1$.

Also solved by D. W. Alling, Paul Bateman, E. A. Cameron, H. S. Grant, and H. P. Hitchcock.

Cyclic Numbers in Geometric Progression

E 714 [1946, 157]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a system of numeration of base B, find the sets of cyclic numbers of three

different digits such that the numbers of each set are successive terms of a geometric progression.

If $B=3p+1$ show that the three digits of at least two of these sets are consecutive terms of an arithmetic progression. Apply to the case $B=10$.

Solution by E. P. Starke, Rutgers University. The present problem is a generalization of E 658 [1945, 461] such that the earlier problem provides the solution in case $B=10$. As in that solution, let us put

$$x = B^2a + Bb + c, \qquad y = B^2b + Bc + a, \qquad z = B^2c + Ba + b.$$

If y is the geometric mean, we have $y^2=xz$, which is easily reduced to

$$B = (a^2 - bc)/(b^2 - ac), \tag{1}$$

or

$$c = (B^3 - 1)a^2/(Ba - b) - Bb - B^2a. \tag{2}$$

(In the solution of E 658 it was erroneously assumed that a common divisor of a, b is necessarily also a divisor of c. Even without this assumption, however, no further results are found.)

For a specified value of B, solutions may be found by trial: assign a value to a, determine values of $b(\neq a)$ for which the fraction in (2) is an integer; a solution results if $0\leqq a, b, c<B$. For example if $B=11$, (2) becomes

$$c = 1330a^2/(11a - b) - 11b - 121a.$$

Let $a=1$; then $11-b$ divides 1330 for $b=1, 4, 6, 9$. Disregarding $b=a=1$, all other values of b give $c>11$. Let $a=2$; then $22-b$ divides 5320 for $b=2, 3, 8$. Here $b=3$ yields $c=5$. Continuing thus we find all the solutions: $(a, b, c) = (2, 3, 5), (4, 6, X), (6, 3, 1), (9, 4, 1)$, where X is the digit ten. This procedure is tedious for large values of B, but no more efficient method appears.

If $B=6$, 8, 12 or any value for which $B-1$ and B^2+B+1 are both prime, it is easy to see that there are no solutions.

The second part of the problem is shown immediately upon observing that

$$\begin{aligned} &B = 3p + 1, \quad a = p + 1, \quad b = p, \quad c = p - 1; \\ &B = 3p + 1, \quad a = 2p + 2, \quad b = 2p, \quad c = 2p - 2; \end{aligned} \tag{3}$$

reduce (1) to an identity. (Other, less simple, identical solutions are easy to find: *e.g.* $b=2c-1$, $a=4c-4$, $B=14c^2-31c+16$.)

Conversely, suppose a, b, c are consecutive terms of an arithmetic progression, so that $a=2b-c$. Then (1) becomes

$$B = (4b - c)/(b - c) = 1 + 3b/(b - c).$$

Now $b-c<0$, $b-c\geqq 3$ lead to contradiction of the requirement $0\leqq a, b, c<B$. If $b-c=1$, 2, we are led at once to (3). There are never more than two sets.

Also solved by Murray Barbour, E. D. Schell, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, N. J. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4220. *Proposed by P. Erdös, University of Michigan*

Let m be a positive integer with no prime factors greater than the integer n and $m \leqq n^{(k+1)/2}$. Then m can be written as the product of k integers $\leqq n$; the exponent $(k+1)/2$ is the best possible.

4221. *Proposed by P. Erdös, University of Michigan*

Let $\phi(n) = n\Pi_{p/n}(1-1/p)$ be Euler's function. Prove that for every k the equation $\phi(x) = k!$ is solvable.

4222. *Proposed by J. H. Butchart, Grinnell College*

If points are numbered and if $\overline{12}$ denotes the distance from 1 to 2, then

$$\begin{vmatrix} 0 & \overline{12}^2 & 1 \\ \overline{21}^2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2L^2,$$

where L is the length of the segment. Generalize this for the triangle and tetrahedron showing that the corresponding determinants are $-16A^2$ and $288V^2$ respectively, where A is the area of the triangle and V is the volume of the tetrahedron.

4223. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron the harmonic plane of the point L whose normal coordinates are proportional to the radii of the circumcircles of the triangles of the faces (second Lemoine point), coincides with the polar plane with respect to the circumsphere.

Dedicated to N. A. Court.

4224. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $ABCD$; (1) The right cones with vertices at the orthogonal projections of the second Lemoine point L on the axes of the circumcircles of the faces and with these circles as bases have the same base angle V (Brocard angle). (2) The symmedians AL, BL, CL, DL meet the circumsphere in the ver-

tices of the tetrahedron $A'B'C'D'$ having the same Brocard angle V and the same Lemoine point L as $ABCD$.

Dedicated to N. A. Court.

SOLUTIONS

A Differential Operator

4156 [1945, 220]. *Proposed by Charles M. Stein, New York City*

Show that if $f(u)$ is a polynomial in u and $D=d/dx$, then for any positive integer n

$$\left\{1+\frac{[(k-1)n]!}{[kn]!}(y^n-x^n)D^n\right\}\cdots\left\{1+\frac{1}{n!}(y^n-x^n)D^n\right\}f(x^n)=f(y^n).$$

If possible, generalize the above result to analytic functions $f(u)$ as in the special case $n=1$ in 3985 [1942, 345].

Solution of first part by the Proposer. We have

$$\left\{1+\frac{[(s-1)n]!}{[sn]!}(y^n-x^n)D^n\right\}x^{ns}=x^{ns}+(y^n-x^n)x^{(s-1)n}=y^nx^{(s-1)n}.$$

Consequently

$$\left\{1+\frac{(y^n-x^n)}{n!}D^n\right\}\cdots\left\{1+\frac{[(s-1)n]!}{[ns]!}(y^n-x^n)D^n\right\}x^{ns}=y^{ns}.$$

But the factors in the operational product are commutative, since any linear operator, and in particular $(y^n-x^n)D^n$, is commutative with powers of itself. Thus

$$\cdots\left\{1+\frac{[(k-1)n]!}{[kn]!}(y^n-x^n)D^n\right\}\cdots\left\{1+\frac{(y^n-x^n)}{n!}D^n\right\}x^{ns}$$
$$=\cdots\left\{1+\frac{[sn]!}{[(s+1)n]!}(y^n-x^n)D^n\right\}y^{ns}=y^{ns}.$$

This shows that the theorem is true for any finite linear combination of powers of x^n, *i.e.*, any polynomial in x^n.

Conics

4162 [1945, 345.] *Proposed by H. S. M. Coxeter, University of Toronto*

Prove, by the methods of real projective geometry, that if a projectivity $P\overline{\wedge}P'$ on a conic is not an involution, the envelope of PP' is a conic. (For a complex proof, see H. F. Baker, *Principles of Geometry*, Cambridge 1922 or 1930, p. 52.)

Solution by R. J. Walker, Cornell University. It is convenient to break the proof into parts.

LEMMA. *Given a conic* Γ *and two lines* a *and* b *not conjugate with respect to* Γ *and not intersecting on* Γ, *there is a conic* Δ *tangent to* a *and such that any point of* b *has the same polar with respect to both* Γ *and* Δ.

Proof. Let $P=ab$, and let Q be any other point of b not on Γ and not conjugate to P with respect to Γ. Let p and q be the polars of P and Q with respect to Γ and let $A=ap$. Let AQ meet q in X and let A_1 be the harmonic conjugate of A with respect to Q and X. Let A_1P intersect p in Y and let A_2 be the harmonic conjugate of A_1 with respect to P and Y. Let A_1P intersect q in Z and let A_3 be the intersection of AZ and A_2Q. It is easily verified that no three of A, A_1, A_2, A_3 are collinear and that none of A_1, A_2, A_3 lie on a. Hence there is a conic Δ containing A, A_1, A_2, A_3 and tangent to a at A. It is also readily verified that p and q are the polars of P and Q with respect to Δ. Now let R be any point of b. The point $B=pq$ is the pole of b with respect to both Γ and Δ, and so the polars of R with respect to Γ and Δ both pass through B. Γ and Δ determine the same involution of conjugate points on b, since the two pairs P, bp and Q, bq are conjugate with respect to both conics. Hence there is a point R' on b conjugate to R with respect to both conics. The line BR' is thus the polar of R with respect to both Γ and Δ.

THEOREM. *Let* A, A' *be a pair of distinct points in a projectivity (not an involution) on a conic* Γ, *and let* b *be the axis of the projectivity. Then the conditions of the Lemma are satisfied with* $a=AA'$, *and every line* PP' *joining a pair of points of the projectivity is tangent to* Δ.

Proof. Let PA' and $P'A$ meet at Q on b. Let S be the intersection of AA' and PP', and R of AP and $A'P'$. Then SR is the polar of Q with respect to Γ and hence also with respect to Δ. Therefore SR and SQ are harmonic with SA and the other tangent to Δ from S. But from the quadrangle $AA'PP'$, SQ and SR are harmonic with SA and SP. Hence $SP(=PP')$ is tangent to Δ.

Solved also by Howard Eves and James Jenkins.

Cubes with Twelve Digits

4164. [1945, 346]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find a number of twelve digits $a_1a_2a_3a_4b_1b_2b_3b_4c_1c_2c_3c_4$ which is a perfect cube, and the cube root is the sum of the numbers $a_1a_2a_3a_4$, $b_1b_2b_3b_4$, $c_1c_2c_3c_4$.

Solution by E. P. Starke, Rutgers University. Let $A=a_1a_2a_3a_4$, $B=b_1b_2b_3b_4$, $C=c_1c_2c_3c_4$. Then

$$(1) \qquad (A+B+C)^3 = 10^8A + 10^4B + C \equiv A + B + C \pmod{9999}.$$

This congruence is equivalent to the three similar simultaneous congruences having the moduli 9, 11, 101. These are satisfied respectively if and only if

$$A + B + C \equiv 0, 1, -1 \pmod 9,$$

$$A + B + C \equiv 0, 1, -1 \pmod{11},$$

$$A + B + C \equiv 0, 1, -1 \pmod{101}.$$

Thus the 27 solutions of congruence (1) are easily obtained. They are 1, 100, 505, 2222, 2223, 2322, 2727, 2728, 2827, 4445, 4544, 4949, 4950, 5049, 5050, 5455, 5554, 7172, 7271, 7272, 7677, 7776, 7777, 9494, 9899, 9998, 9999.

Since $(A+B+C)^3$ must be a 12-digit number, we have also

$$4641 < A + B + C < 10000.$$

Upon finding the cubes of the above numbers we have the results:

$$4949^3 = 121213882349; \qquad 1212 + 1388 + 2349 = 4949.$$
$$5049^3 = 128711132649; \qquad 1287 + 1113 + 2649 = 5049.$$
$$5455^3 = 162324571375; \qquad 1623 + 2457 + 1375 = 5455.$$
$$5554^3 = 171323771464; \qquad 1713 + 2377 + 1464 = 5554.$$
$$7172^3 = 368910352448; \qquad 3689 + 1035 + 2448 = 7172.$$

1, 2322, 2728, 4445, 4544 also satisfy the requirements except for the necessity of supplying one or more zeros as initial digits of the cube. All other numbers cited above are such that

$$(A + B + C)^3 = 10^8A + 10^4B + C - 9999.$$

Solved also by Hugo Brandt and the proposer without stating methods of derivation, Brandt giving one example with the remark "Here is one needle in the haystack." The proposer gave three.

Circles Intersecting on a Parabola

4166. [1945, 400]. *Proposed by J. H. Butchart, Grinnell College*

Given the straight lines l, l' and a point F. A variable circle through F and the intersection of l, l' cuts l and l' in A and A' respectively. The circles through F and tangent to l, l' at A, A' respectively meet again on a parabola tangent to l and l' and having F as focus.

I. *Solution by H. F. Sandham, Trinity College.* The circumcircles of the triangles formed by a complete quadrilateral intersect in the focus of the unique parabola touching its sides. This is a limiting case when two of the sides coincide. Take two of the sides to be l, l' and let the other two intersect in P and tend to coincidence with l''. Then the circles passing through P and touching l, l' at their points of intersection with l'', *i.e.* A, A' respectively, and the circumcircle of l, l', l'' intersect in the focus F of the unique parabola touching l, l', and l'' at P.

II. *Solution by Howard Eves, College of Puget Sound.* As a special case of the first theorem in Butchart's article *Some Properties of the Limaçon and Cardioid* (this MONTHLY [1945, 384]), we have the theorem:

Given two circles l and l' intersecting in points F and E. Through E we draw an arbitrary secant cutting the circles l and l' again in A and A' respectively.

The tangents to l and l' at A and A' meet in a point P on a cardioid tangent to l and l' and having its cusp at F.

Inversion of this theorem with respect to F as center of inversion produces the required result.

Solved also by Mrs. Leslie Davis and P. D. Thomas.

Editorial Note. The solution by Mrs. Davis is analytic, and Thomas gave two solutions, one analytic and the other synthetic. Another solution follows.

The tangents at two points P_1 and P_2 on a parabola meet in the point C, then it is known that the tangent segments CP_1 and CP_2 subtend equal angles at the focus F; that in the quadrilateral P_1FP_2C the points P_1, P_2 are on opposite sides of CF; and the equal angles at F are equal to the exterior angle at C of the quadrilateral.

Now let the tangent at a third point P_3 meet the straight line of CP_2 in A and CP_1 in B, forming the triangle ABC with the interior angles α, γ at A, C, respectively, and the exterior angle β at B. Then by the theorem above P_3A and AP_2 subtend the angle α at F; P_2C and CP_1 subtend the angle γ at F; and P_3B and BP_1 subtend the angle β at F. Thus at F there are three pairs of equal angles such that $2\alpha+2\gamma=2\beta$; and hence $\angle AFC=\beta$, which says that the circumcircle of ABC passes through F. Obviously the circle through F and tangent to CA at A passes through P_3, and similarly the circle through F and tangent to CB at B goes through P_3, *etc.*

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

In memory of Professor Federigo Enriques a group of his students have established a "Grant Federigo Enriques." The object of the fund is to assist some of his pupils to continue their scientific careers. Those interested in assisting in this enterprise are asked to write to Casa Editrice Zanichelli, Bologna, Italy.

Ohio State University announces the following: Assistant Professor Grace M. Bareis has retired, Assistant Professor Marshall Hall of Yale University has been appointed to an associate professorship, and Associate Professor H. H. Alden of the University of Wyoming has been appointed to an assistant professorship.

Dr. Joshua Barlaz of Ohio State University has been appointed to an assistant professorship at Rutgers University.

B. K. Brown of Syracuse University has been appointed to an associate professorship at the James Millikin University.

Dr. R. S. Burington, Director of the evaluation and analysis groups of the research and development division of the bureau of ordnance, Navy Department, has been appointed chief mathematician, bureau of ordnance.

P. C. Cox has been appointed to an assistant professorship at Roanoke College.

Assistant Professor Howard Eves of the College of Puget Sound has been appointed to an assistant professorship at Oregon State College.

Assistant Professor W. J. Harrington has received a leave of absence from Pennsylvania State College to accept a visiting assistant professorship at Cornell University.

J. F. Hubbard has been appointed to an assistant professorship at Massachusetts State College at Fort Devens.

Associate Professor H. S. Kaltenborn of the University of Idaho has been appointed professor of mathematics and head of the department at Memphis State College.

Associate Professor V. S. Lawrence of Cornell University has been appointed to an associate professorship at Virginia Polytechnic Institute.

Professor Deane Montgomery of Smith College has been appointed to an associate professorship at Yale University.

Dr. A. R. Poole has been appointed to an assistant professorship at Oregon State College.

Assistant Professor R. G. Sanger of the University of Chicago has been appointed professor of mathematics and head of the department at Kansas State College. Professor W. T. Stratton has retired from his position as head of the department.

Dr. R. L. Westhafer has been appointed to an assistant professorship at New Mexico College of Agricultural and Mechanic Arts.

The following appointments to instructorships are announced:
Clark Junior College, Vancouver, Washington: R. H. Stair
Duke University (department of Physics): Rachael La Roe
Lafayette College: Dr. J. C. Smith

Professor Richard Baldus of the Technische Hochschule of Munich died in January, 1945.

Professor Friedrich Hartogs, formerly of the University of Munich, died August 15, 1943.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE WATSON SCIENTIFIC COMPUTING LABORATORY

In the early part of 1945 the International Business Machines Corporation created a Department of Pure Science under the direction of Dr. W. J. Eckert. One of the first projects proposed by the new department was the establishment of a computing laboratory to carry on research involving complicated computational techniques and to give instruction in modern numerical methods. Columbia University invited the International Business Machines Corporation to locate the proposed laboratory upon the Columbia University campus, and to occupy a building at 612 West 116th Street; this invitation was accepted. The new laboratory was named The Watson Scientific Computing Laboratory. For Dr. Eckert, who became director of the new laboratory, the affiliation with Columbia University represented a continuation of a relation started several years ago. In 1933, he developed the first computing laboratory in which general scientific calculations were automatically performed. This laboratory was made available to astronomers through The Thomas J. Watson Astronomical Computing Bureau, a joint enterprise of the American Astronomical Society, Columbia University and the International Business Machines Corporation. Some of the rapid computing methods developed there were perfected at the United States Naval Observatory during the period of the recent World War when Dr. Eckert was Director of the Nautical Almanac. All members of the staff of the Watson Laboratory are employees of the International Business Machines Corporation, and some have been appointed to the teaching staff of the University.

The purpose of the Watson Laboratory is research and instruction in the various branches of science, especially those involving applied mathematics and numerical calculation. A free computing service is to be offered to a worker in any field of science if he has a problem that is interesting to the members of the Laboratory staff. Members of the staff offer courses of instruction in their fields of interest under the auspices of various departments of the University; this instruction is designed for graduate students. Academic credit for the courses may be obtained by registering with the University in the usual manner and may be counted toward a University degree when approved by the department and the faculty concerned. An important function of the Laboratory is to serve as a center for the exchange of information concerning mathematical and machine methods and mathematical tables.

The Laboratory is equipped with the most modern calculating machines. These machines are of a wide variety to permit not only the most efficient

handling of a given problem but the comparative study of different methods. Included are standard keyboard and electric punched-card machines as well as special calculators which operate by means of relay networks and electronic circuits. The following are the principal units:

Keyboard machines: calculators, printing-adding machines.

Standard electric punched-card machines: 2 alphabetic accounting machines, 6 multiplying punches, 2 collators, 3 reproducing summary punches, sorters, key punches. These units are equipped with many special devices.

Relay calculators: (*a*) Two automatic calculators of large capacity. These machines have automatic plugboard sequence control, 30 counters of 6- to 12-digit capacity and facilities for consulting mathematical tables. They have provision for division and square root and each is capable of performing and recording 16000 6×6 multiplications per hour. (*b*) Experimental multiplying and dividing machine for sequence calculation. This is a flexible unit which can be attached to other machines for special purposes. (*c*) A small digital calculator for special problems. Its operation is controlled by means of a program card. It operates in the decimal, duodecimal, binary, and other systems.

Electronic calculator: an all-electronic multiplying-dividing machine designed for high speed. Results are recorded on punched cards.

Table printing device: this device reads numbers from punched cards and records them for printing by means of lithography or line cuts. It combines the reliability of punched-card operation with the flexibility and elegance of conventional printing.

Card-controlled measuring and recording machine: an automatic machine designed primarily for the measurement of astronomical photographs. It locates the star image by means of coordinates on a punched card, measures its position, and records the measurement on the card. The punched card record is then available for mathematical treatment.

Continuous calculator: an electronic device for the solution of simultaneous linear equations up to and including the twelfth order.

Facilities for the construction of new equipment required by the Laboratory are available in the Department of Engineering of IBM and in the Laboratory itself.

A technical library is maintained in the Laboratory. It will be highly specialized in the field of activity of the Laboratory because of the proximity of the extensive libraries of the University and of the American Mathematical Society.

NEW PRELIMINARY ACTUARIAL EXAMINATIONS

On June 7, 1947, three new preliminary actuarial examinations will be given to undergraduate students of mathematics and others who may be interested in going into the actuarial profession. These new examinations are sponsored jointly by the Actuarial Society of America and the American Institute of Actuaries.

The new series of examinations will replace Parts 1, 2, and 3 of the actuarial examinations which have been given heretofore and will carry the same credit

toward Associateship in the two actuarial organizations. These examinations have been prepared under the direction of a joint committee of actuaries and mathematicians. They will be administered by the College Entrance Examination Board at centers through the United States and Canada.

Descriptions of the three new examinations are as follows:

1. *Language Aptitude Examination.* This is a three-hour aptitude examination testing reading comprehension and precise knowledge of the meaning of words. It is similar to the well-known Scholastic Aptitude Test of the College Entrance Examination Board, except that it is pitched at approximately the college sophomore level. Verbal facility and command of the English language, as well as mathematical ability, are important in the actuarial profession. This is not the type of examination for which specific preparation can be made; it is an aptitude rather than an achievement examination.

2. *General Mathematics Examination.* This is a three-hour achievement examination on material usually covered in the first two years of mathematics in colleges and universities in the United States and Canada. More specifically, it is based on college algebra, trigonometry, coordinate geometry, and differential and integral calculus. It is designed to be taken by the mathematically talented undergraduate at the end of his sophomore year, although it is not restricted to this group.

3. *Special Mathematics Examination.* This is a three-hour achievement examination based on the material usually covered in undergraduate courses in finite differences, probability, and statistics. It is designed to be given at the end of the junior or senior year to college mathematics majors and others who have either taken courses or done concentrated reading in these fields, but it is not restricted to this group.

The two actuarial bodies will jointly award one $200 and eight $100 prizes to the nine undergraduates ranking highest in combined score on the first two of the examinations described above. In determining these awards the General Mathematics Examination will be weighted twice as much as the Language Aptitude Examination.

Information regarding these new examinations, and applications for taking them, may be obtained from either of the following organizations:

The Actuarial Society of America
393 Seventh Avenue
New York 1, New York

American Institute of Actuaries
135 South La Salle Street
Chicago 3, Illinois

REVISED SELECTIVE SERVICE MEMORANDUM NO. 115

The revised Selective Service Local Board Memorandum No. 115, as released to the public on August 23, 1946, contained the following provisions of interest to mathematicians:

Pursuant to the provision of Part II, paragraph I of this memorandum,* local boards will give the most serious consideration to the occupational deferment of registrants engaged in the specialized, technical and scientific fields set forth in Part IV of this memorandum.

PART IV—STANDARDS AND PROCEDURES

1. *Advanced studies in the physical sciences and engineering.* Registrants accepted by an accredited college or university for a Master's or Doctor's degree in the physical sciences or engineering will be certified by the Office of Scientific Research and Development to the Director of Selective Service.

2. *University research in the physical sciences or engineering.* Registrants employed by or attached to the staff of an accredited college or university for research in the physical sciences or engineering will be certified by the Office of Scientific Research and Development to the Director of Selective Service.

3. *Submission of information in certain cases.* Any registrant who wishes to be certified under the provisions of paragraphs 1 and 2 above must present to the Office of Scientific Research and Development, 1530 P Street, N.W., Washington 25, D.C., the following documents in triplicate:

(a) A notarized statement of his intention:
  (1) To undertake graduate studies leading to a Master's or Doctor's degree in the physical sciences or engineering; or
  (2) To undertake advanced research in the physical sciences or engineering.

(b) A statement from an accredited college or university signed by a responsible official of the college or university, indicating that the registrant:
  (1) Has been accepted as a candidate for a Master's or Doctor's degree in the physical sciences or engineering; or
  (2) Is to be employed by or attached to the staff of the college or university for research in the physical sciences or engineering.

(c) A statement showing the registrant's name, address, age, Selective Service local board number and address, classification, and educational qualifications.

4. *Industrial and foundational research in the physical sciences or engineering.* Scientific personnel necessary to and engaged in basic research in the physical sciences or engineering in the employ of laboratories not connected with colleges or universities, and scientific personnel in industry who are not engaged directly in production, will be certified by the Office of Scientific Research and Development to the Director of Selective Service. Employers will submit Form

* A registrant age 19 through 29 may be retained or placed in Class II-A if the local board finds that he is irreplaceable in and indispensable to an activity essential to the national existence. An activity essential to the national existence is one of such importance that any disruption thereof would adversely affect the physical well-being, the public safety or economic life of the community or the Nation.

42A (Special-Revised) in triplicate, in each individual case, to the Office of Scientific Research and Development for certification.

5. *Teachers.* Teachers employed by an accredited college or university will be certified by the United States Office of Education to the Director of Selective Service. Employers will submit Form 42A (Special-Revised) in triplicate, in each individual case, to the United States Office of Education, Temporary Building "M," 26th and Water Streets, N.W., Washington 25, D. C.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE TWENTY-EIGHTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The twenty-eighth summer meeting of the Mathematical Association of America was held at Cornell University, Ithaca, New York, on Monday and Tuesday, August 19–20, 1946, in conjunction with the summer meeting and colloquium of the American Mathematical Society and the meeting of the Institute of Mathematical Statistics. Five hundred and eighty-four persons were in attendance at the meetings, including the following two hundred and seventy-seven members of the Association:

C. R. ADAMS, Brown University
R. P. AGNEW, Cornell University
A. A. ALBERT, University of Chicago
E. B. ALLEN, Rensselaer Polytechnic Institute
C. B. ALLENDOERFER, Haverford College
R. C. ARCHIBALD, Brown University
SILVIO AURORA, Columbia University
W. L. AYRES, Purdue University

D. H. BALLOU, Middlebury College
P. T. BATEMAN, Yale University
GRACE E. BATES, Mount Holyoke College
HELEN P. BEARD, Newcomb College
F. P. BEER, Williams College
R. L. BEINERT, Cornell University
J. A. BENNER, Lafayette College
DOROTHY L. BERNSTEIN, University of Rochester
FELIX BERNSTEIN, New York University
GARRETT BIRKHOFF, Harvard University
C. J. BLACKALL, North Baltimore, Ohio
E. E. BLANCHE, War Department
R. P. BOAS, JR., Brown University
S. G. BOURNE, Johns Hopkins University
JULIA W. BOWER, Connecticut College
C. C. BRAMBLE, U. S. Naval Academy
R. W. BRINK, University of Minnesota
B. K. BROWN, James Millikin University
H. E. BUCHANAN, Tulane University
R. C. BUCK, Harvard University
E. L. BUELL, Northwestern University
R. S. BURINGTON, Bureau of Ordnance, Navy Department
SISTER LEONARDA BURKE, Regis College, Massachusetts
F. J. H. BURKETT, Union College
HERBERT BUSEMANN, Smith College
W. H. BUSSEY, University of Minnesota

S. S. CAIRNS, Queens College
W. D. CAIRNS, Oberlin College
R. H. CAMERON, University of Minnesota
C. C. CAMP, University of Nebraska
H. H. CAMPAIGNE, U. S. Navy
W. B. CAMPBELL, Philadelphia Textile Institute
I. S. CARROLL, Syracuse University
W. B. CARVER, Cornell University
W. F. CHENEY, JR., University of Connecticut
J. A. CLARKSON, University of Pennsylvania
MARY D. CLEMENT, Wells College
G. R. CLEMENTS, U. S. Naval Academy
NANCY COLE, Connecticut College

R. H. COLE, University of Western Ontario
E. P. COLEMAN, U. S. Military Academy
J. B. COLEMAN, Columbia, S. C.
J. A. COOLEY, University of Tennessee
BYRON COSBY, JR., U. S. Naval Academy
N. A. COURT, University of Oklahoma
W. H. H. COWLES, Pratt Institute
H. S. M. COXETER, University of Toronto
H. B. CURRY, Pennsylvania State College
J. H. CURTISS, National Bureau of Standards

D. R. DAVIS, State Teachers College, Montclair, N. J.
F. F. DECKER, Syracuse University
L. S. DEDERICK, Aberdeen Proving Ground
H. L. DORWART, Washington and Jefferson College
ARNOLD DRESDEN, Swarthmore College
W. L. DUREN, JR., Tulane University
P. S. DWYER, University of Michigan
L. A. DYE, The Citadel

E. D. EAVES, University of Tennessee
P. D. EDWARDS, Ball State Teachers College
SAMUEL EILENBERG, Indiana University
BENJAMIN EPSTEIN, Carnegie Institute of Technology
W. S. ERICKSON, Bell Aircraft Corporation
H. S. EVERETT, University of Chicago

WILLIAM FELLER, Cornell University
F. A. FICKEN, University of Tennessee
N. J. FINE, Washington, D. C.
C. D. FIRESTONE, Cornell University
K. W. FOLLEY, Wayne University
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Rooms for members of the organizations and their families were provided in two of the University dormitories, and meals were served in the cafeterias in Willard Straight Hall and Martha Van Rensselaer Hall. Tea was served on Monday and Thursday afternoons in Willard Straight Hall by the ladies of the Cornell Mathematics Department.

On Tuesday afternoon Professor Emeritus W. B. Ford and Mrs. Ford invited the visitors for tea at their beautiful home on the west shore of Cayuga Lake about twenty-five miles north of Ithaca. A very enjoyable concert was given on Tuesday evening in the Memorial Room of Willard Straight Hall by three members of the Walden String Quartette, Messrs. Homer Schmitt, Bernard Goodman, and Eugene Weigel, and a pianist, Miss Margaret Squire, all of the Department of Music of Cornell University. On Wednesday afternoon some of the visitors journeyed to Skaneateles for dinner at The Krebs, while others attended a picnic at the Robert H. Treman State Park (Enfield Glen).

A joint dinner for the three organizations was held on Thursday evening in the Ivy Room of Willard Straight Hall, after which the guests moved to the Memorial Room for a brief program of talks. Professor Allendoerfer acted as toastmaster. An address of welcome to the visitors by Provost Arthur S.

Adams of Cornell University was followed by short and interesting talks by Professors P. A. Smith, L. R. Ford, and Harold Hotelling, representing the Society, the Association, and the Institute respectively. Resolutions were presented by Professor C. C. Camp, and adopted by a rising vote, expressing the thanks and appreciation of the three visiting organizations to President Day and the administration of Cornell University; to the members of the Walden String Quartette and Miss Margaret Squire, pianist; to Professor and Mrs. W. B. Ford; to the ladies of the Mathematics Department of Cornell University; and to Professor Jones and the other members of the Committee on Arrangements.

The American Mathematical Society held its sessions between noon Tuesday and noon Friday. The twenty-eighth Colloquium consisted of four lectures on "Topology of smooth manifolds," by Professor Hassler Whitney of Harvard University. Professor W. H. Stone of Harvard gave his retiring presidential address on "Some general principles of functional representation," and Professor J. L. Doob of the University of Illinois gave an address by invitation on "Probability in function space."

The Institute of Mathematical Statistics held a joint session with the Society on Thursday morning, and a session on Friday morning.

The program committee for the Association consisted of Professors J. J. Gergen, chairman, H. L. Dorwart, and E. H. C. Hildebrandt, and this committee arranged the programs for the two sessions held Monday afternoon and Tuesday morning.

First Session of the Association

1. "The Army Universities in Europe, a preview of the veteran's problem," by Professor H. M. Gehman, University of Buffalo.
2. "Curves," by Lt. Col. R. C. Yates, A.U.S., U. S. Military Academy.
3. "Visual aids to the teaching of calculus," by Professor E. A. Whitman, Carnegie Institute of Technology.

Second Session of the Association

1. "Some identities in the theory of determinants," by Professor G. B. Price, University of Kansas.
2. "An elementary proof of Cauchy's integral theorem," by Professor D. V. Widder, Harvard University.
3. "Analytic extension and summability," by Professor R. P. Agnew, Cornell University.

Meeting of the Board of Governors

The Board met on Monday at 8:00 P.M., in the North Room of Willard Straight Hall, thirteen members of the Board being present. Among the more important items of business transacted were the following:

A list of sixty-two applicants for membership, as given below, was presented, and the applicants were elected by the Board.

J. J. ANDREWS, A.M.(St. Louis Univ.) Lecturer, St. Louis Univ., St. Louis, Mo.

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MAX LONNER, B.S.(C.C.N.Y.) Admin. Asst.; Part-time teacher, Hebrew Parochial High School, Brooklyn, N. Y.

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PHILIP PEAK, M.S.(Iowa) Instr., Indiana Univ., Bloomington, Ind.

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FRANCIS REGAN, Ph.D.(Michigan) Prof., St. Louis Univ., St. Louis, Mo.

N. L. RICHARDSON, A.B.(Westminster Coll.) Lecturer, St. Louis Univ., St. Louis, Mo.

T. E. RINE, M.S.(Iowa) Asst. Prof., Teaching of Math., Illinois State Normal Univ., Normal, Ill.

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W. A. SMITH, B.S.(Vermont) Project Supervisor, Johns Hopkins Univ., Appl. Physics Lab., Silver Spring, Md.

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R. L. VITALE, M.E.(Poly. Inst. of Brooklyn) 1st Lt., U. S. Army Signal Corps. 366 Avenue T, Brooklyn, N. Y.

REV. G. W. WALKER, B.D.(Auburn Theol. Sem.), A.M.(Hamilton Coll.) Pastor, Presbyterian Church, Perry, N. Y.

MARY S. WILSON, B.S.(Georgia) Grad. Student, Univ. of Georgia, Athens, Ga.

R. L. WILSON, A.M.(Wisconsin) Grad. Asst., Univ. of Wisconsin, Madison, Wis.

MRS. REBECCA W. WINCHESTER, A.B.(Greensboro Coll.) Grad. Student, Univ. of Michigan, Ann Arbor, Mich.

E. A. ZUBAY, A.M.(Minnesota) Instr., Drake Univ., Des Moines, Iowa

The Board voted to accept the invitation of Yale University to hold the Summer Meeting of 1947 at New Haven, Connecticut, and the invitation from the University of Georgia to hold the Annual Meeting of 1947 at Athens, Georgia.

Professor Dresden described the work of the Committee on Aid to Devastated Libraries, and the Board voted to donate back numbers of the MONTHLY, of value not to exceed $250, to complete broken sets which might be given to the Committee, and that copies of Carus Monographs might be included in this donation.

A letter was read from Chairman R. G. Harrison of the National Research Council concerning the adherence of the United States to the United Nations Educational, Scientific, and Cultural Organization, UNESCO. The Board authorized our representative on the National Research Council, R. E. Langer, to write a letter expressing the interest of the Association in the organization and our desire to cooperate in establishing good international relations in the field of mathematics.

The Secretary reported that the Editors of the Annals of Mathematics were announcing an increase in their subscription rate, and that they would no longer be able to offer the special rate of $2.50 per year to members of the

Association. The Board voted their thanks to the Annals for their generous treatment of members of the Association in the past. Since the Annals would be offered to members of the American Mathematical Society at a slightly reduced rate, it was felt that no special arrangement with the Association would be necessary.

The Board voted to authorize the Carus Monograph Committee to contract with the George Banta Publishing Company for a thousand copies of the sixth monograph, Fourier Series and Orthogonal Polynomials, by Dunham Jackson, to be made by the photo-offset process.

President MacDuffee and Professor A. A. Bennett as representatives of the Association had attended a recent conference in Washington, sponsored by the American Council on Education, on the relations of higher education to the federal government. They presented to the Board a report of the conference, and expressed the opinion that it had been very effective, and that our membership in the American Council on Education was of definite value to us.

The Board voted an appropriation of an amount not to exceed $50.00 to the Wisconsin Section to cover the expense of clerical help, postage, *etc.*, in the work of a committee and several sub-committees which are undertaking a study of mathematical education in that state.

WALTER B. CARVER, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Thirtieth Annual Meeting, Swarthmore, Pa., December 26–27, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Peoria, May 9–10, 1947
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., December 7, 1946
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Claremont, March 8, 1947
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Madison, May, 1947

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FERMAT'S LAST THEOREM

ITS HISTORY AND THE NATURE OF THE KNOWN RESULTS CONCERNING IT

H. S. VANDIVER, University of Texas

1. Introduction. Fermat's famous theorem [13b], dating from about the year 1637, that

$$x^n + y^n = z^n \tag{1}$$

has no solution in integers x, y, and z, none zero, if n is an integer greater than 2, still awaits a proof or disproof.* Euler, Legendre, Gauss, Abel, Cauchy, Dirichlet, Lamé, Kummer, Frobenius, Furtwängler, and Dickson are among the well-known mathematicians who have worked on the problem, at least for particular values of n. A number of investigators have obtained criteria for the solution of (1) independent of x, y, and z. In other words they have shown that (1) is impossible for a certain class of exponents n. We shall be interested here in examining the nature of these criteria in the cases where it may be shown that none of the classes referred to is the null class. Also, many individuals have given criteria for (1) which cannot, apparently, be applied to any particular exponent in a finite number of steps. No conditions of this latter character will be considered here.†

Fermat [13c] left a proof for $n=4$; and for $n=3$ Euler [13a] gave a proof in the year 1770. The latter proof is incomplete but the missing steps were supplied by later writers.‡

* Concerning the theorem, Kronecker, Vorlesungen über Zahlentheorie, p. 23, expresses himself as follows: "Mit diesem Satze . . . haben sich die Mathematiker . . . vielleicht mehr beschäftigt, als mit irgend einem andern, und wohl keiner hat, abgesehen etwa von der Quadratur des Kreises, zu so vielen falschen und irrtümlichen Deductionen Veranlassung gegeben."

† Many criteria of this latter character have been given in connection with the theorem. In particular, in recent years Morishima, Krasner (particularly interesting is [24]), and the writer have obtained such results. For the history of the problem prior to the year 1927 *cf.* [12] and [64]. Expositions of some of the principal results are given in [2], [11], [21], [32], [39]. The Fermat relation (2) has been generalized in various ways, but none of them will be considered here. Referring to the exposition of some of the theory of algebraic numbers given in this paper, it may be noted that the simplest method for developing this subject in general is based on a theory of semi-groups and groups. However, we shall not use such ideas here, as it would be necessary to develop them to some length before we could begin the present exposition and this would require more space than we have at our disposal. Of course, if we would proceed from the group standpoint, the results in this subject would enable us to derive a great deal more of the theory of algebraic numbers than we consider here. We give few proofs in full. Our idea is to bring out some of the extraordinary features in the proofs, for example, the method of infinite descent. Usually, we give references to the literature which will enable the reader to locate complete proofs of results stated. Not a great deal of the theory of algebraic numbers or of analysis is needed to follow the proof of the statements in this paper (excepting Theorem IV) until we reach Theorem X. But then the situation is different. The argument needed to derive the class number of the cyclotomic field requires considerable analysis and the complete proofs of the Theorems X to XIV inclusive, are all, unfortunately, complicated and require a great deal of the theory of algebraic numbers.

‡ A complete proof from Euler's point of view is not very simple. *cf.* Uspensky [50], pp. 394–5, 408–11.

To prove Fermat's statement in general, it is then sufficient to demonstrate the impossibility of

$$x^l + y^l + z^l = 0 \tag{2}$$

in non-zero integers x, y, and z, for any odd prime $l>3$. This was proved by Legendre [33] in 1823 for $l=5$ and in 1839 by Lamé [30] for $l=7$. Stimulated by these results a number of mathematicians of the time made many efforts to extend the methods of proof to other values of l. For example, Cauchy [7] published a number of results concerning (2), as did Lamé, based on assumptions which were later proved false. The study of these mistakes led Kummer to the notion of *ideal numbers* which we shall discuss later.

The condition that (2) be satisfied in integers may also be expressed in the form that if r is a proper rational fraction then $\sqrt[l]{1+r^l}$ is rational.* The problem seems never to have been successfully attacked, in any special case, by using this form of (2). All known methods begin by taking the equation (2) and assuming that integers x, y, and z satisfy it. The properties of these integers are then investigated. It is important to make use of the fact that x, y, and z are integers since our theorem does not hold if we substitute for integers, quantities in certain more general algebraic systems. For example, if $a+b=c$ then

$$(\sqrt[n]{a})^n + (\sqrt[n]{b})^n = (\sqrt[n]{c})^n.$$

Now in particular, if in (2), x, y, and z are prime to each other and to l, this condition *will be referred to as Case* I *of Fermat's last theorem*; if x, y, and z are prime to each other and one of them is divisible by l, the condition *will be called Case* II *of the theorem.* So far many more classes of criteria have been given for Case I than Case II and for simplicity we shall now consider the former type of criteria. *But first we must signalize a very important theorem of arithmetic stating that a composite integer decomposes uniquely into prime factors,*† as this result and its generalizations have apparently played a fundamental role in all contributions which have been made to our knowledge of (2).

The last result does not carry over to all arithmetical systems. We first note that it is possible to select a subset of the set of all positive integers which has the property that it contains the product of any two of its elements and also their sum. For example, we can take all positive multiples of 3. We may define a prime in this system as one which can not be expressed as the product of two elements in the system, as this is quite analogous to our definition of prime in connection with the entire set of positive integers. We note that our theorem falls down since the number 3×36 can be expressed as $(3\times4)\times(3\times1)^2$ where each factor is a prime by definition. But also

* Geometrically, the curve $u^n+v^n=1$, n odd, in the (u, v) plane has the asymptote $u=-v$ and the only rational points on it are (0, 1) and (1, 0) for the values of n for which Fermat's statement is true.

† Unique decomposition of an integer or an ideal into prime factors means unique aside from the order of the factors.

$$(3 \times 2)^2 \times (3 \times 1) = (3 \times 4) \times (3 \times 1)^2, \tag{3}$$

and each factor on the left is also prime. The two decompositions are obviously different. We can also enlarge our original system of the set of all rational integers by considering all expressions of the form $a+b\sqrt{-5}$, where a and b are rational integers. We may define a prime in this system as an element which contains no factor aside from itself and ± 1. We have

$$2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}), \tag{4}$$

in which each factor may easily be shown to be prime according to the definition of prime, and unique decomposition does not hold in this domain.

In the discussion which follows, the main theorems, from a number of standpoints, are I, III, V, X, XI, XII, XIII, XIV. In the first place, none of the first four criteria are known to be satisfied by any value of l although a considerable number of values of l have been tested in each case. This is also true for theorems XII, XIII, and XIV when we take them together with XI. Although the theorems III and V involve x, y, and z in their statements they lead to a variety of criteria which are independent of x, y, and z for Case I of the theorem, as will be noted on reading further in this paper. By the use of theorem X we are enabled to state theorems XII, XIII, and XIV without any limitation to Case II of the theorem. *In connection with Case* I, *we shall also call particular attention to the relation* (10). *Theorems* III *and* V *are derived directly from it, and it is the principal tool employed in the proof of Theorem* X.

2. Criteria for Case I obtained by elementary methods. Legendre [33] in 1823 published the following result which he attributed to Sophie Germain.

THEOREM I. *If there exists an odd prime p such that*

$$\xi^l + \eta^l + \zeta^l \equiv 0 \pmod{p} \tag{5}$$

has no set of integral solutions ξ, η, ζ, each not divisible by p and such that l is not the residue of the lth power of any integer modulo p, then (2) *has no integral solutions each prime to l.*

We shall indicate a proof of this result. (For full proof *cf.* [12], pp. 734–735.) Set

$$\phi(y, z) = y^{l-1} - y^{l-2}z + y^{l-3}z^2 \cdots + z^{l-1},$$

then $(y+z)\phi(y, z) = -x^l$. It is known that the factors on the left are prime to each other because $x \not\equiv 0 \pmod{l}$, hence

$$y + z = a^l, \qquad z + x = b^l, \qquad x + y = c^l, \tag{6}$$

whence

$$2x = b^l + c^l - a^l, \qquad 2y = a^l + c^l - b^l, \qquad 2z = a^l + b^l - c^l. \tag{7}$$

If x, y, z are integers satisfying (2), they satisfy congruence (5), so that one of them, say x, is divisible by p. Then, by (7), $b^l+c^l+(-a)^l\equiv 0 \pmod{p}$. Consequently a, b, or c is divisible by p, employing the assumption in the theorem. It may be shown that b or $c\equiv 0 \pmod{p}$ leads to a contradiction. Hence $a\equiv 0$, $x\equiv 0$, $z\equiv -y$, $\phi(x, y)\equiv y^{l-1}$, $\phi(y, z)\equiv ly^{l-1} \pmod{p}$. And these give $l\gamma^l\equiv\alpha^l \pmod{p}$, for $\gamma^l=\phi(x, y)$ and $\alpha^l=\phi(y, z)$. From this the result easily follows.

Employing the theorem it is easy to see that (2) is impossible in Case I if $2l+1$ is a prime. By using this theorem Legendre [33] also proved that (2) is not satisfied for any $l<197$ in case I. (Proved for any $l<100$ by Sophie Germain.)

Dickson [8, 9] made an extended examination of the possible primes p such that (5) has no solutions in integers ξ, η, and ζ prime to p for a given l. On dividing the congruence (5) through by ζ^l, prime to p, we may write the result in the form $u+v+1\equiv 0 \pmod{p}$ where u and v are lth powers modulo p. Hence $(v+1)^m\equiv v^m\equiv 1 \pmod{p}$, if $p=1+lm$.

These congruences are also satisfied by

$$v, \quad 1/v, \quad (-1-v), \quad -1/(1+v), \quad -1-1/v, \quad -v/(1+v)$$

in place of v. They are incongruent if $2^m\not\equiv 1 \pmod{p}$. They are the roots of a reciprocal sextic equation. Set $w=v+1/v$. Then the sextic reduces to

$$c(w) = w^3 + 3w^2 + \beta w + 2\beta - 5$$

for β some function of v. This must divide $f(w)=(w+1/w)^m-1$. By proceeding along these lines Dickson found that (5) is impossible if $\xi\eta\zeta\not\equiv 0 \pmod{p}$; $p=lm+1$ for all values of $m<74$ which are not divisible by 3, as well as $m=76$ and $m=128$, except for a finite number of pairs of values l and m which pairs he was able to determine. Using these results, he [8, 9] was able to prove* that (2) is impossible in integers x, y, and z prime to l for $l<7{,}000$. As also shown by Dickson [10], (5) has integral solutions ξ, η, and ζ each prime to p for l fixed; for any prime p exceeding a certain limit. (*cf.* also Pellet, [42, 43] Hurwitz [22]).

Wendt [70] (1894) proved the following, which may also be derived from Theorem I.

THEOREM II. *If* (5) *has no solutions prime to* p, *and* $p=1+ml, m=2^vl^k, v$ *not divisible by* l, *then* (2) *has no solutions prime to* l.

3. Introduction to algebraic numbers and applications to Case I. Let us consider for a moment the problem of finding the integers prime each to each α,β, and γ such that $\alpha^2-\beta^2=\gamma^2$ is satisfied. We note immediately that the left hand member can be factored, and using the unique decomposition theorem, we infer that if integers exist satisfying this equation, then integers δ and ϵ exist such that $\alpha-\beta=\delta^2$, $\alpha+\beta=\epsilon^2$ unless both α and β are odd, in which case 2 must be inserted as a factor in each right-hand member. Now suppose we try to find a method if possible, analogous to this, for attacking the problem of finding integers x, y, and z to satisfy

* In these papers Dickson excepted the value $l=6857$, but he has stated that a proof of this case was carried out by him in manuscript.

$$x^l + y^l = -z^l.$$

As above, the left-hand member can be factored so that we have

$$(x+y)(x+\zeta y)\cdots(x+\zeta^{l-1}y) = -z^l, \tag{8}$$

where ζ is a primitive lth root of unity. But obviously we are leaving the realm of rational integers in such a factorization. However, it is possible to extend the properties of rational integers to expressions involving the lth root of unity; in short we can set up what is called the theory of cyclotomic fields, define integers in such a field and attack (2) with more generality than has so far appeared possible by means of the use of rational integers only. We introduce this topic by defining algebraic numbers in general. For expositions of the general theory *cf.* [17], [20], [21], [69e], [71].

Any subset of the set of complex numbers which contains the sum, the difference, the product and the quotient (division by zero excluded) of any two of its elements is said to form a *field.* If we have a subset of the set of complex numbers which contains the sum, the difference and the product of any two of its elements, but not the quotient in every case, the system is called a *ring.*

Let

$$f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0,$$

where the c's are rational integers, be irreducible in the rational field (that is, $f(x) \neq g(x)h(x)$ where g and h actually contain x and all coefficients are rational) and suppose that the distinct roots of $f(x)=0$ are $\alpha_1, \alpha_2, \cdots, \alpha_n$. Then $\alpha=\alpha_1$ generates an algebraic field which is said to be of *degree n.* The other roots $\alpha_2, \alpha_3, \cdots, \alpha_n$ each generate an algebraic field and these fields are said to be the algebraic fields conjugate to that generated by α. We shall designate the field generated by α by $k(\alpha)$.

Since $f(x)$ is irreducible in the rational field, then any number θ in $k(\alpha)$ may be written uniquely in the form

$$\theta = \theta_1 = a_{n-1}\alpha_1^{n-1} + a_{n-2}\alpha_1^{n-2} + \cdots + a_0,$$

where the a's are rational. If

$$\begin{aligned}
\theta_2 &= a_{n-1}\alpha_2^{n-1} + a_{n-2}\alpha_2^{n-2} + \cdots + a_0,\\
\theta_3 &= a_{n-1}\alpha_3^{n-1} + a_{n-2}\alpha_3^{n-2} + \cdots + a_0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\theta_n &= a_{n-1}\alpha_n^{n-1} + a_{n-2}\alpha_n^{n-2} + \cdots + a_0,
\end{aligned}$$

then $\theta_2, \theta_3, \cdots, \theta_n$ are called the *conjugates* of θ_1 in $k(\alpha)$.

If $c_n=1$ then α is said to be an *algebraic integer.* The numbers in $k(\alpha)$ which are algebraic integers may be shown to form a ring, which we shall call $R(\alpha)$ or R. (In this connection we see that if $n=1$ then our definition of algebraic integer coincides with the notion of rational integer.)

Ideals. A sub-ring S of the ring R of integers in k is said to be an *ideal* in R if it is such that if θ is any integer in S then $\lambda\theta$ is in S where λ is any integer in R. It may be proved that any ideal in R may be reduced to the form

$$\lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_n\omega_n, \tag{8a}$$

where n is the degree of k, and where the ω's are fixed integers and the λ's range independently over all elements of R.

An ideal of the form $\lambda\theta$ where θ is a fixed integer in k and λ ranges over the integers in k is called a *principal ideal.* If the ideal $\mathfrak{m}$ has the form (8a) we write

$$\mathfrak{m} = (\omega_1, \omega_2, \cdots, \omega_n).$$

If $\mathfrak{a} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\mathfrak{b} = (\beta_1, \beta_2, \cdots, \beta_s)$ then

$$\mathfrak{ab} = (\alpha_1\beta_1, \cdots, \alpha_r\beta_1, \cdots, \alpha_1\beta_s, \cdots, \alpha_r\beta_s, \cdots)$$

is called the product of $\mathfrak{a}$ and $\mathfrak{b}$. An ideal $\mathfrak{c}$ in k is said to be *divisible* by an ideal $\mathfrak{a}$ in k when an ideal exists in k such that $\mathfrak{c} = \mathfrak{ab}$. An ideal $\mathfrak{p}$ is said to be *prime* if it has no divisor other than $\mathfrak{p}$ and the ideal consisting of all elements of R or the ideal (1). We may prove the fundamental theorem:—

Any ideal in R aside from (0) *and* (1) *has a unique decomposition into prime ideal factors.*

From this follows the remarkable result:—

Any ideal $\mathfrak{m}$ in R may be reduced to the form $\mathfrak{m} = (\alpha, \beta)$ where α and β are in R.

The last result shows that the principal ideals in R form an important subclass of R. As an example, we shall consider the ring of rational integers and any ideal in that ring. Suppose that d is the smallest integer, >0, in this ideal. Then by definition the ideal necessarily contains all integers in the form md where m ranges over all rational integers. Suppose the ideal contained an integer different from any of these, say q; by the euclidean algorithm we have $q = ad + r$, $0 < r < d$. Now q belongs to our ideal and so do a and d and consequently $q - ad$ belongs to the ideal, hence r belongs to the ideal but $r < d$. This is a contradiction, however; consequently, all the integers in the ideal have the form md and the ideal is principal. Hence all the ideals in the rational ring are principal. *However, not all ideals are principal in an algebraic field.* It may be shown that the numbers $a + b\sqrt{-5}$ with a and b rational integers are integral algebraic numbers which form a ring R. Consider the ideal $(2, 1+\sqrt{-5})$ in R. We have already noted in our discussion of the fundamental theorem of arithmetic that 2 and $1+\sqrt{-5}$ are primes. If we assume that $(2, 1+\sqrt{-5}) = (\alpha)$ it then follows that 2 is divisible by α and so is $1+\sqrt{-5}$. Since they are both primes then $\alpha = \pm 1$. Hence rational integers u, v, u_1, and v_1 exist such that

$$(1 + \sqrt{-5})(u + v\sqrt{-5}) + 2(u_1 + v_1\sqrt{-5}) = 1,$$

and this is easily shown to lead to a contradiction (*cf.* also Dickson [11], for an interesting discussion of the ideals in R). As previously noted

$$6 = 2\cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5}), \tag{4}$$

where $2, 3, 1+\sqrt{-5}, 1-\sqrt{-5}$, are all primes. However, we can now point out that the *ideal* (6) in the ring R of integers of the form $a+b\sqrt{-5}$ *has the unique decomposition into ideals*, each of which is prime,

$$(6) = (2, 1+\sqrt{-5})^2(3, 1+\sqrt{-5})(3, 1-\sqrt{-5}). \tag{9}$$

To reduce the right hand member we note, by definition of multiplication that

$$(3, 1+\sqrt{-5})(3, 1-\sqrt{-5}) = (9, 3-3\sqrt{-5}, 3+3\sqrt{-5}, 6),$$

and since $3=9-6$, the right hand member equals

$$(9, 3-3\sqrt{-5}, 3+3\sqrt{-5}, 6, 3) = (3).$$

Similarly $(2, 1+\sqrt{-5})^2=(2)$.

The relation (4) *gives two different decompositions of the number* 6 *in* R, *but* (9) *shows how we may replace this by the unique decomposition of the ideal* (6) *in* R.

We now consider in some detail the cyclotomic field (For expositions of the theory of cyclotomic fields *cf.* [3] [15], [17] I, [21].) defined by ζ where ζ is a primitive lth root of unity. Following the usual custom we do not always refer to the ring R of integers or ideals in R. We say "ideals in $k(\zeta)$" meaning ideals formed by integers in $k(\zeta)$. Any integer in this field $k(\zeta)$ can be put in the form

$$c_0 + c_1\zeta + \cdots + c_{l-2}\zeta^{l-2},$$

where the c's are rational integers. (A similar result is not true in all algebraic fields.) It would be very convenient in attacking our problem if we took (8) and supposed, in analogy with rational integers we could say that

$$x + \zeta y = (a_0 + a_1\zeta + \cdots + a_{l-2}\zeta^{l-2})^l,$$

but the situation turns out to be not nearly as simple as this. *However, in the case* $l=3$ *it is possible to show that if*

$$\lambda = 1 - \zeta,$$

then

$$x + \zeta y = \lambda\zeta^c(a_0 + a_1\zeta)^3, \qquad x + \zeta^2 y = \lambda\zeta^b(a_0' + a_1'\zeta)^3, \tag{9b}$$

$$x + y = \zeta^d\lambda^{3m-2}v^3(a_0'' + a_1''\zeta)^3,$$

where b, c, d, m, *and the* a's *are rational integers,* $m>0$, *and where* z *is divisible by* 3, (the case $xyz\not\equiv 0 \pmod 3$ can be easily disposed of without the use of relations of this type), *and it is possible to prove* (2) *impossible for* $l=3$ *proceeding from*

these relations. The examination of other values of l shows that in general the problem is more complicated. However, we find that there are products of factors in the left-hand member of (8) which take on comparatively simple forms. *This is because certain products of ideals in the cyclotomic field turn out to be principal ideals.* That is,* if $\mathfrak{p}(\zeta)$ represents an ideal in the field $k(\zeta)$ then $\prod_s \mathfrak{p}(\zeta^{s_1})$ is a principal ideal where $ss_1 \equiv 1 \pmod{l}$ and s ranges over the integers $h<l$ such that $h+|ch|<l$ where c is any integer prime to l and $|ch|$ is the least residue of ch modulo l (*cf.* Weber, [69e], pp. 748–751; Landau, [32], pp. 292–295). As examples, for $l=7$, $c=1$ we have $s=1, 2, 3$ and $s_1=1, 4, 5$ so $\mathfrak{p}(\zeta)\mathfrak{p}(\zeta^4)\mathfrak{p}(\zeta^5)$ is a principal ideal while if $l=7$, $c=2$ we have $s=1, 2, 4$ and $s_1=1, 4, 2$ showing that $\mathfrak{p}(\zeta)\mathfrak{p}(\zeta^2)\mathfrak{p}(\zeta^4)$ is a principal ideal.) Application of this gives (Kummer [29])

$$\prod_s (x + \zeta^{s_1} y) = \zeta^g (\omega(\zeta))^l, \tag{10}$$

where ω is an integer in $k(\zeta)$, and g is a rational integer. Since $(x^l-1)/(x-1)$ is irreducible in the rational field it then follows that

$$\prod_s (x + w^{s_1} y) = w^g (\omega(w))^l + A(w)\frac{(w^l-1)}{(w-1)},$$

where w is an indeterminate and $A(w)$ is a polynomial in w with integral coefficients. Setting $w=e^v$, Kummer obtains from this formula the congruences

$$B_n\left[\frac{d^{l-2n}\log(x+e^v y)}{dv^{l-2n}}\right]_{v=0} \equiv 0 \pmod{l}, \tag{11}$$

for n any integer in the set $1, 2, \cdots, (l-3)/2$ and B_n is the nth Bernoulli number defined as follows:— Consider the recursion formula $(b+1)^k = b_k$ for $k>1$. If we expand the left-hand member using the binomial theorem and substitute b_i for b^i, take $k=2$, $k=3$, *etc.* in turn, we can calculate the values of b_a, $a=1, 2, \cdots$. Thus after defining $b_0=1$, we get $b_1=-\frac{1}{2}$, $b_2=1/6$, $b_3=0$, *etc.* We then set $(-1)^{n-1}b_{2n}=B_n$. The relations (11), or (12) which follow, are known as the *Kummer criteria.* Cauchy [7, p. 362] had previously obtained the relation (10) for $c=1$ but gave no proof. He also gave without proof the result that (2) is impossible in Case I if (*cf.* Kummer [29], p. 74)

$$1 + 2^{l-4} + 3^{l-4} + \cdots + [(l-1)/2]^{l-4} \not\equiv 0 \pmod{l}.$$

The relations (11) received little attention from mathematicians until Mirimanoff [36] in 1905 proved that they are equivalent to the conditions (if we use z and x in place of x and y, *etc.*, in (11)) given by the following theorem.

* If $\omega(\zeta)$ is an integer in $k(\zeta)$ we represent by $\omega(\zeta^a)$ the integer obtained from $\omega(\zeta)$ by substituting ζ^a for ζ. Also if $\mathfrak{a}(\zeta)$ represents an ideal in $k(\zeta)$ then $\mathfrak{a}(\zeta^a)$, $a \not\equiv 0 \pmod{l}$, represents the ideal obtained from $\mathfrak{a}(\zeta)$ by substituting ζ^a for ζ in each number of the ideal $\mathfrak{a}(\zeta)$. From this we see that the result follows that if $\mathfrak{a}(\zeta)=(\alpha_1(\zeta), \alpha_2(\zeta))$ then $\mathfrak{a}(\zeta^a)=(\alpha_1(\zeta^a), \alpha_2(\zeta^a))$.

THEOREM III. *If* (2) *is satisfied in Case* I *then*

$$B_n f_{l-2n}(t) \equiv 0 \pmod{l}; \qquad n = 1, 2, \cdots, (l-3)/2; \tag{12}$$

where if $m>1$,

$$f_m(t) = \sum_{i=1}^{l-1} i^{m-1} t^i; \quad t = -x/y, -y/x, -y/z, -z/y, -x/z, -z/x,$$

and B_n *is the nth Bernoulli number.*

The six values of t in the above are all incongruent, modulo l, except when $t=-1, \frac{1}{2}$ or 2, as the case $t^2-t+1\equiv 0$ is impossible (Pollaczek [44]).
He also showed that

$$f_{l-1}(t) \equiv 0 \pmod{l}, \tag{12a}$$

and, using (12), $B_d \equiv B_{d-1} \equiv B_{d-2} \equiv B_{d-3} \equiv 0 \pmod{l}$; $d=(l-3)/2$. (The first two conditions are due to Kummer [29]). Mirimanoff [*l.c.*], also derived from (12) and (12a) the congruences

$$\begin{gathered} f_n(t) f_{l-n}(t) \equiv 0 \pmod{l}, \\ n = 1, 2, \cdots, l-1; \qquad f_1(t) = 1 + t + \cdots + t^{l-1}, \end{gathered} \tag{13}$$

which are equivalent to (12) and (12a).

In 1909 *Wieferich* [72], *employing* (12) *and* (12a), *proved that if we set*

$$\frac{m^{l-1}-1}{l} = q(m),$$

then

$$q(2) \equiv 0 \pmod{l},$$

if (2) *is satisfied in integers* x, y, *and* z *prime to* l. This remarkable result led to a series of investigations. Wieferich's argument was complicated. It was simplified by Mirimanoff [37, 38] who obtained the criterion

$$q(3) \equiv 0 \pmod{l}.$$

In 1914 Vandiver ([62b], footnote pp. 174–175) derived in a bit more complicated form the identical congruence

$$(w^l - 1)G_m^{(k)}(w) - (w^m - 1)K_m^{(k)}(w) \equiv H_m^{(k)}(w) \pmod{l}, \tag{13a}$$

where, if $b_{2s} = (-1)^{s+1}B_s$; $b_{2r+1}=0$, $r>1$; $b_1 = -\frac{1}{2}$, $b_0 = 1$;

$$K_m^{(k)}(w) = -\sum_{n=0}^{l-2} \left(-\frac{m}{k}\right)^n b_n f_{l-n}(w),$$

$$G_m^{(k)}(w) = \frac{w(w^m-1)}{w-1} \sum_{\rho} \frac{F_l(\rho)(1-\rho^k)}{ml(w-\rho^k)},$$

$$H_m^{(k)}(w) = w(w^m - 1)\sum_{l=0}^{k-1}\sum_{\rho}\frac{\rho^l h_{l-1}^{(k,l)}(w)}{w - \rho^k},$$

$$h_n^{(k,l)}(w) = \sum_{s=0}^{p-1}(ks + l)^{n-1}w^s,$$

and where ρ is any mth root of unity, w is an indeterminate and

$$F_n(v) = \sum_{r=0}^{pm-1} r^{n-1}v^r, \quad (n > 1).$$

This relation was found by Mirimanoff [37, 38] for $k=1$ and used by him to prove $q(2)\equiv q(3)\equiv 0 \pmod{l}$. The extension to any integer k is due to Vandiver. It was this step that made the derivation of the criteria, $q(m)\equiv 0 \pmod{l}$ for $m>3$, possible. Use of the criteria (12) and (12a) with (13a) gives

$$(t^p - 1)G_m^{(k)}(t) \equiv H_m^{(k)}(t) \pmod{l}.$$

Employing this in 1914 Vandiver obtained ([52], and footnote in [62b] pp. 174–175), the criteria $q(5)\equiv q(11)\equiv 0 \pmod{l}$ and if $l\equiv 2 \pmod 3$, $q(7)\equiv q(13)\equiv 0 \pmod{l}$.

By employing a method quite different from that of Kummer, involving the law of reciprocity for lth powers, a powerful tool in the theory of algebraic numbers, we may prove

THEOREM IV. *If* (2) *is satisfied in Case* I, *then*

$$f_n(t)f_{l-n}(1 - t) \equiv 0 \pmod{l}, \tag{14}$$

$n=1, 2, \cdots, l-1$, *with* t *and* $f(t)$ *defined as in Theorem III.* (Vandiver, [59], (1925)).

Frobenius [14] (1914), Pollaczek [44] (1917), Morishima [41] (1931), and Rosser [45, 47] (1939, 1941), employing (12), (12a), (13a) and (14) proved that if (2) is satisfied in Case I then

$$q(m) \equiv 0 \pmod{l}, \tag{14a}$$

for each prime $m<44$. Using $q(2)\equiv 0 \pmod{l}$, Beeger [5] in 1939 found that this criterion does not hold for any $l<16{,}000$ except 1093 and 3511 and using Dickson's [9] results, he concluded that (2) is impossible for all primes less than 16,000. *Interpreting the results in* (14a) *by means of a new and powerful analytic method, Rosser* [46] (1941) *showed that* (2) *does not hold in Case* I *for any* $l<41{,}000{,}000$. *By extending the method of Rosser, D. H. and Emma Lehmer* [35] *in* 1941 *carried this result to* $l<253{,}747{,}889$. This is the best result to date, as far as special exponents are concerned, in Case I. In the year 1912 Furtwängler [16] proved the following.

THEOREM V. *If* (2) *is satisfied, where* x, y, *and* z *are integers prime to each other, then*

$$q(r) \equiv 0 \pmod{l}, \tag{15}$$

where r *is any factor of* x *if* x *is prime to* l *and similarly for* y *and* z. *Also the relation* (15) *holds if* r *is a factor of* $x \pm y$, *provided* $x^2 - y^2$ *is prime to* l.

It is possible to prove ([55], 78–79) this by use of the relation (10). The theorem gives easily a proof of (Vandiver [62a] (1926))

THEOREM VI. *If there exists an odd prime* p *such that* (5) *has no set of integral solutions each not divisible by* p *and such that* $p \not\equiv 1 \pmod{l^2}$ *then* (2) *has no solutions each prime to* l.

Employing the theorem of Sophie Germain together with the Theorems II and VI, Vandiver [62a] was able to prove in 1926

THEOREM VII. *If* (5) *has no set of integral solutions each prime to* p, *where* $p = 1 + ml$ *and* $m < 10\ l$, *then* (2) *has no solutions in integers each prime to* l.

If we denote by $h(l)$ the number of primes p for which (5) has no solutions with $m < 10l$ then $h(5) = 4$, $h(7) = 3$ (Dickson [8]); $h(11) = 3$, $h(13) = 4$, $h(17) = 4$, $h(19) = 6$ (N. G. W. H. Beeger, unpublished).

In 1940 Krasner [24a] proved

THEOREM VIII. *Equation* (2) *is impossible in Case* I *if there exists a prime* $p = 1 + cl$ *with* $p > 3^{c/4}$ *and* $2^c \not\equiv 1 \pmod{p}$, $c \not\equiv 0 \pmod{3}$.

Employing a result due to H. H. Mitchell and theorem VI, Vandiver [69a] proved in 1944 the following.

THEOREM IX. *If* c *is a given even integer not divisible by* 3 *and it is possible to define two primes* l *and* p *such that* $p = 1 + cl$; $l > c$ *and* $p > 3^{\phi(c)}$ *then* (2) *is impossible in integers* x, y, *and* z, *prime to* l. *Here* $\phi(c)$ *is the number of positive integers less than* c *and prime to it.*

Using (10) it may be shown that $B_s \equiv 0 \pmod{l^2}$, $s = (rl+1)/2$, $r = l-4$, $l-6$, $l-8$, $l-10$, (Vandiver [55]); and similarly for $r = l-12$, $l-14$, (Morishima [40], D. H. Lehmer [35a]) if (2) is satisfied in Case I.

If (2) is satisfied in Case I then we also have $E_{(l-3)/2} \equiv 0 \pmod{l}$ where $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, *etc.* are the Euler numbers and the related criterion

$$1 + 1/2^2 + \cdots + 1/[l/6]^2 \equiv 0 \pmod{l},$$

where $[l/6]$ is the greatest integer in $l/6$ (Vandiver [61, 69b] Schwindt [48]).

By the use of (10), and some of the concepts explained in Section 4, which follows, we may prove the

THEOREM X. *If* (2) *is satisfied in integers* x, y, *and* z *prime to* l *then the second factor of the class number of* $k(\zeta)$ (defined in connection with (17) of this paper) *is divisible by* l. (Vandiver [68] (1934)).

4. Criteria governing Cases I and II of the Theorem. We shall now obtain results containing both cases of the theorem and this will enable us to give criteria for the solution of (2) without any restrictions on x, y, and z. Referring again to the relation (8) *we examine the problem: for what values of* l *may we obtain simple relations analogous to* (9b) *which we noted held in the case* $l=3$? So far, this has proved to be very difficult and the solution depends on certain concepts and results in the theory of algebraic numbers which we shall now describe. The first notion we shall consider is that of *congruence* with respect to an ideal modulus. If α and β are integers in an algebraic field k, and $\alpha-\beta$ belongs to an ideal $\mathfrak{a}(\neq(0))$, then we write $\alpha\equiv\beta \pmod{\mathfrak{a}}$. If we are dealing with rational integers $a\equiv b \pmod{m}$ means that $a-b$ is a multiple of m and this is equivalent to saying that $a-b$ belongs to the ideal (m). So this idea of congruence in an algebraic field is an extension of our ordinary idea of congruence.

The *finite number* of integers in an algebraic field k which are incongruent with respect to a given ideal $\mathfrak{m}$, is called the *norm* of $\mathfrak{m}$.

We define a *unit* in the algebraic field as an integer α such that there exists an integer α_1 in the same field with $\alpha\alpha_1=1$. Thus, ± 1 are units in the rational field, and ± 1, ζ, are units in $k(\zeta)$. The units in any algebraic field have the following properties. First, there exists a finite set, say s of them, called a *fundamental system of units*, such that any unit can be expressed as power products of these s units (that is, these units are *generators under multiplication*) multiplied by a root of unity. Also this set of units $\epsilon_1, \epsilon_2, \cdots, \epsilon_s$ is *independent*, that is from $\epsilon_1^{a_1}\epsilon_2^{a_2}\cdots\epsilon_s^{a_s}=1$ we may infer $a_1=a_2=\cdots=a_s=0$. For the cyclotomic field defined by ζ we have $s=(l-3)/2$. It is known ([15], pp. 179–184, 208–209) that

$$(15a)\qquad e_i = e(\zeta^{r^i}) = \left(\frac{(1-\zeta^{r^{i+1}})(1-\zeta^{-r^{i+1}})}{(1-\zeta^{r^i})(1-\zeta^{-r^i})}\right)^{1/2} = \frac{\zeta^{r^{i+1}/2}-\zeta^{-r^{i+1}/2}}{\zeta^{r^i/2}-\zeta^{-r^i/2}}$$

for $i=0, 1, \cdots, (l-5)/2$, and r a primitive root of l form a system of independent but not necessarily fundamental units. It is then easy to show ([17], II, p. 270) that integers none zero, $m_1, m_2, \cdots, m_d$; $d=(l-3)/2$, exist, such that

$$(16)\qquad \epsilon_i^{m_i} = e_1^{c_{1i}} e_2^{c_{2i}} \cdots e_d^{c_{di}};\quad i = 1, 2, \cdots, d.$$

We shall set

$$\begin{vmatrix} c_{11} & c_{21} \cdots c_{d1} \\ c_{12} & c_{22} \cdots c_{d2} \\ \cdots & \cdots \\ c_{1d} & c_{2d} \cdots c_{dd} \end{vmatrix} = D.$$

If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in an algebraic field k, then $\mathfrak{a}$ is said to be *equivalent* to $\mathfrak{b}$ if and only if integers α and β with $\alpha\beta \neq 0$, exist in k such that

$$(\alpha)\mathfrak{b} = (\beta)\mathfrak{a},$$

and $\mathfrak{a}$ and $\mathfrak{b}$ are said to belong to the same *class of ideals.* According to this definition, all principal ideals in k belong to the same class which will be called the *principal class.* It is obvious that all principal ideals are equivalent to the ideal (1). If $\mathfrak{a}$ is equivalent to $\mathfrak{b}$, we write

$$\mathfrak{a} \sim \mathfrak{b}.$$

As an example, we may verify that if $\mathfrak{c} = (3, 1+\sqrt{-5})$, then

$$(1 - \sqrt{-5})\mathfrak{c} = (1 + \sqrt{-5})(3, 1 - \sqrt{-5}),$$

hence

$$\mathfrak{c} \sim (3, 1 - \sqrt{-5}).$$

Now there is an infinity of ideals equivalent to $\mathfrak{c}$ since we may multiply both sides of the last equation by any principal ideal, say (δ), and the definition of equivalent shows that $\mathfrak{c} \sim (3, 1-\sqrt{-5})(\delta)$. All these ideals belong to a particular class. It is possible to show that there are just two ideal classes in the algebraic field defined by $\sqrt{-5}$ and they are the classes* to which the ideal (1) and the ideal $\mathfrak{c}$ belong.

We now state the fundamental result ([20], p. 120; [32], p. 153).

In each ideal class there is an ideal whose norm does not exceed a certain finite limit. The number of ideal classes for a given algebraic field is finite.

Since the number of classes is finite, then if $\mathfrak{a}$ is an ideal, $\mathfrak{a}, \mathfrak{a}^2, \cdots, \mathfrak{a}^k \cdots$ cannot all belong to different classes so $\mathfrak{a}^m \backsim \mathfrak{a}^n$ for some integers m and n, $m > n$.

From this it is easy to prove that

If h is the number of ideal classes in the algebraic field k, then the hth power of any ideal is a principal ideal.

In further developments we shall find that the class number h of a cyclotomic field plays a fundamental role. It is possible to find h in this particular case in finite form, in fact we find (Kummer [26])

For the class number h of the field of $k(\zeta)$, where ζ is a primitive lth root of unity, we have

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* It will be illuminating to the reader for him to examine Dickson ([11], pp. 175–8) where he proves directly, without the use of the general theory, that there are exactly two ideal classes in the field defined by $\sqrt{-5}$.

$$h = \left| \frac{1}{(2l)^d} \prod_k \left[\sum_{n=0}^{l-2} r_n Z^{nk} \right] \right| \left| \frac{m_1 m_2 \cdots}{D} \right|, \tag{17}$$

where

$$d = (l-3)/2;$$

k ranges over $1, 3, 5, \cdots, l-2$; r_n *is defined by*

$$r_n \equiv r^n \pmod{l}; \quad 0 < r_n < l;$$

r is a primitive root of l; Z is a primitive root of

$$x^{l-1} - 1 = 0,$$

and where the m's and D are defined as in (15a) *and* (16) (*cf.* also Dirichlet-Dedekind, Zahlentheorie, 4th ed., pp. 603–634; Fueter [15] pp. 191–226). The *first factor* in h will be called h_1, the *second factor* h_2. Each is a natural number.

As an example let $l=3$; there is no fundamental unit so that we easily verify that $h=1$. For $l=5$, we find that h_1 is, since $Z=i$,

$$\left| \frac{1}{10}(1 + 2i + 4i^2 + 3i^3)(1 + 2i^3 + 4i^2 + 3i) \right| = 1.$$

We find that h_2 is unity for $l=5$, since $e(\zeta)=\zeta^2+\zeta^3$, which is known to be a fundamental unit (Fueter [15], p. 189). The class number has been found to be unity for each $l<23$. For $l=23$ the first factor is 3, and induction would indicate that it increases rapidly from 23 onward.

Using the fact, already discussed, that the e's in (15a) form a set of independent units, we may prove the ([66], p. 749) following lemma

LEMMA 1. *If*

$$E_n = \sum_{i=0}^{(l-3)/2} e(\zeta^{r^i}) r^{-2in}, \tag{17a}$$

then

$$E_1, E_2, \cdots, E_d$$

form a set of independent units in $k(\zeta)$.

If

$$B_1, B_2, \cdots$$

are the Bernoulli numbers as in Section 3 and if

$$B_1, B_2, \cdots, B_d; \quad l>3,$$

expressed in their lowest terms have numerators prime to l, then the prime l is said to be *regular*, otherwise *irregular*. The prime 3 is also said to be regular.

From (17) it is possible to prove, by reducing h_1 and h_2 modulo l, introducing the Bernoulli numbers and using the units in (17a),

LEMMA 2. *In order that h be prime to l it is necessary and sufficient that l be a regular prime.* (Kummer [26]).

We may also prove in a similar way

LEMMA 3. *If we have the congruence in $k(\zeta)$,*

$$\eta \equiv c \pmod{l},$$

with η a unit in $k(\zeta)$ and c a rational integer, then η is the lth power of a unit in $k(\zeta)$ provided l is regular. (Kummer [26], [27]).

An integer α in $k(\zeta)$ is said to be *primary* if an integer β exists in k such that $\alpha \equiv \beta^l \pmod{\lambda^l}$; $\lambda = (1-\zeta)$. The integer α is said to be *semi-primary* if $\alpha \equiv c \pmod{\lambda^2}$, where c is a rational integer. It is easy to show that it is possible to multiply any integer prime to λ by a power of ζ and obtain a semi-primary integer.

We shall now employ the results given above in indicating a proof of the following (Kummer [26], [27])

THEOREM XI. *If l is a given regular prime, then*

$$x^l + y^l + z^l = 0$$

is impossible for rational integers x, y, and z, none zero.

We consider the equation

$$\alpha^l + \beta^l + \gamma^l = 0, \tag{18}$$

where α, β, and γ are integers in $k(\zeta)$ such that the ideals (α), (β), and (γ) are prime each to each. We may choose α, β, and γ to be semi-primary because of the property of semi-primary integers which we mentioned in connection with their definition. If we show that this equation is impossible for l a regular prime and $\alpha\beta\gamma \neq 0$ then it will follow that (2) is also impossible for non-zero integers x, y, and z with the same restriction on l. The relation (18) gives

$$(\alpha + \beta)(\alpha + \zeta\beta) \cdots (\alpha + \zeta^{l-1}\beta) = -\gamma^l. \tag{19}$$

Now if we consider the two ideals $(\alpha+\zeta^a\beta)$ and $(\alpha+\zeta^b\beta)$ with $a \not\equiv b \pmod{l}$ and suppose them each divisible by a prime ideal $\mathfrak{p}$ then these two relations give $(\zeta^a-\zeta^b)\beta \equiv 0 \pmod{\mathfrak{p}}$. Since (β) is prime to (γ) then it follows that $\zeta^a-\zeta^b \equiv 0 \pmod{\mathfrak{p}}$ or $1-\zeta^{b-a} \equiv 0 \pmod{\mathfrak{p}}$. We divide our discussion into two cases as this congruence is not satisfied or is satisfied. In the first case we then find, since the ideals corresponding to each factor in the left hand member of (19) are all prime to each other, noting that $1-\zeta^{b-a} \not\equiv 0 \pmod{\mathfrak{p}}$ makes γ prime to λ,

$$\begin{cases} (\alpha + \beta) & = \mathfrak{a}_0^l, \\ (\alpha + \zeta\beta) & = \mathfrak{a}_1^l, \\ \cdots\cdots\cdots & \\ (\alpha + \zeta^{l-1}\beta) & = \mathfrak{a}_{l-1}^l. \end{cases} \tag{20}$$

If now h is the number of ideal classes in $k(\zeta)$ then

$$(\alpha + \zeta^a\beta)^h = (\mathfrak{a}_a^l)^h = (\mathfrak{a}_a^h)^l = (\rho_a^l)$$

since h by hypothesis is prime to l and the hth power of any ideal of $k(\zeta)$ is a principal ideal as we have already noted. Now $(\omega)=(\delta)$ gives $\omega=\epsilon\delta$ with ϵ a unit, so

$$(\alpha + \zeta^a\beta)^h = \eta_a'\rho_a^l,$$

where η_a' is a unit in $k(\zeta)$. And since h is prime to l there exists an integer h_1 such that $hh_1=1+lt$ and

$$(\alpha + \zeta^a\beta)^{hh_1} = \eta_a''\rho_a^{lh_1},$$

then

$$(\alpha + \zeta^a\beta)^{1+lt} = \eta_a''\rho_a^{lh_1},$$

and

$$\alpha + \zeta^a\beta = \eta_a''\frac{\rho_a^{h_1 l}}{(\alpha + \zeta^a\beta)^{tl}}.$$

It will then follow that*

$$\alpha + \zeta^a\beta = \eta_a\theta_a^l \tag{21}$$

where θ is an integer in $k(\zeta)$, $a=0, 1, \cdots, l-1$. It is known ([32], pp. 225–227) that $\eta_a=\zeta^s\delta$ where δ is a real unit in $k(\zeta)$. Reducing (21) with respect to the ideal $(1-\zeta)^l$ for various values of a this may be shown to be impossible for $l>3$.

For the second case, where γ is divisible by $(1-\zeta)=(\lambda)$, we assume that γ is divisible by $(\lambda)^m$ and hence we may write, if $m>1$,

$$\alpha^l + \beta^l = \mu\lambda^{lm}\omega^l \tag{22}$$

in general, where μ is a unit in $k(\zeta)$.

Proceeding by a method somewhat similar to that already used in obtaining (21) we obtain from (22)

$$\alpha + \beta = \lambda^{l(m-1)+1}\gamma_1^l,$$
$$\alpha + \zeta\beta = \nu_1\lambda\alpha_1^l,$$
$$\alpha + \zeta^2\beta = \nu_2\lambda\phi^l,$$

where ν_1 and ν_2 are units in $k(\zeta)$. Elimination of α and β from these relations gives

$$\alpha_1^l + \tau\phi^l = \tau_1\lambda^{l(m-1)}\gamma_1^l, \tag{23}$$

where τ and τ_1 are units in $k(\zeta)$. Using lemma 2 we find that $\tau=\xi^l$ where ξ is a unit in $k(\zeta)$. Set $\xi\phi=\beta_1$, in (23) we obtain

* The derivation of (21) for regular primes l is a triumph for the theory of ideals. If we had taken (19) and considered only the decomposition of γ^l into prime integers in the field $k(\zeta)$ we would at most have been able to obtain (21) for $l<23$.

$$\alpha_1^l + \beta_1^l = \tau_1 \lambda^{l(m-1)} \gamma_1^l. \tag{24}$$

This equation has the same form as (22), except the exponent of λ has been decreased by l. Proceeding with this equation as we did with (22) we obtain

$$\alpha_2^l + \beta_2^l = \tau_2 \lambda^{l(m-2)} \gamma_2^l,$$

and finally

$$\alpha_{m-1}^l + \beta_{m-1}^l = \tau_{m-1} \lambda^l \gamma_{m-1}^l,$$

with α_{m-1} and β_{m-1} semi-primary, but this is easily shown to be impossible.

The celebrated Theorem XI stands as one of the deepest results in number theory. Landau devotes the third volume of his *Vorlesungen über Zahlentheorie* to developing, in the main, the part of algebraic number theory which is needed in the proof of the theorem that (2) is impossible if l is prime to the class number of $k(\zeta)$. It would take, in the writer's opinion, at least fifty more pages, written in the style Landau employs, to prove Kummer's Theorem XI. This is mainly because of the length of the argument required to derive the closed form of the class number of $k(\zeta)$ as we gave it in (17). The argument depends on finding the class number of any algebraic field as the limit of an infinite series and showing that this limit may be given in finite form when applied to our cyclotomic field. Such able mathematicians as Dedekind, Weber, Hilbert and Fueter have given expositions of this without any of them materially simplifying the original work of Kummer.

All primes less than 619 are regular except 37, 59, 67, 101, 103, 131, 149, 157, Kummer [26, 27, 28] 233, 257, 263, 271, 283, 293, 307, 311, 347, 353, 379, 401, 409, 421, 433, 461, 463, 467, 491, 523, 541, 547, 557, 577, 587, 593, 607, 617 ([69], pp. 573–576). Jensen [23] proved that there is an infinity of irregular primes of the form $4n+3$.

We now examine equation (2) when l is an irregular prime. In all the results treated further in this paper the second factor of the class number h_2 as defined in connection with (17) occupies a central place. This number has the property (Kummer),

LEMMA 4. *The second factor h_2 of the class number of $k(\zeta)$ equals the class number of the sub-field defined by $(\zeta+\zeta^{-1})$.*

Also all the following results stem from one due to Kummer [29] (1857), which is as follows:—

The relation $x^l+y^l+z^l=0$ is impossible for rational integers x, y, and z none zero, l a given odd prime, under the following three assumptions.

1. The first factor of the class number of $k(\zeta)$ is divisible by l but not by l^2.

2. If $B_n \equiv 0 \pmod{l}$, $n<(l-1)/2$, there exists an ideal in $k(\zeta)$ with respect to which as a modulus the unit

$$E_n = \sum_{i=1}^{(l-3)/2} e(\zeta^{r^i}) r^{-2in}$$

is not congruent to the lth power of an integer in $k(\zeta)$. Here $e(\zeta^{r^i})$ is defined as in (15a).

3. The Bernoulli number B_{nl} is not divisible by l^3.

Based on the above, Kummer proved that Fermat's Last Theorem is true for $l = 37$, 59, and 67, which are irregular primes. Since the other primes less than 100 are regular, he concluded that the theorem was true for all prime exponents less than 100. We do not give Kummer's result, stated above, as a separate theorem, or discuss his argument, as it is entirely included in the following

THEOREM XII. (Vandiver [54] (1929)). *Under the following assumptions:*

(1) *the second factor of the class number of the field $k(\zeta)$ is prime to l;*

(2) *none of the Bernoulli numbers B_{nl}, $n = 1, 2, \cdots, (l-3)/2$, is divisible by l^3;*

then the equation (2) *is impossible in non-zero integers x, y, and z.*

In Case I the result follows immediately from theorem X.

In Case II the argument is a bit complicated and depends on the lemmas which follow.

LEMMA 5. *If the ideal (ω) in $k(\zeta)$ is the lth power of an ideal in that field and ω is a primary number, then (ω) is the lth power of a principal ideal in the field provided that the second factor of the class number of $k(\zeta)$ is prime to l. If the second factor of the class number is divisible by l, then at least one of the units*

$$E_n: \quad n = 1, 2, \cdots, (l-3)/2,$$

is the lth power of a unit in $k(\zeta)$. (Takagi [49a] p. 236; Vandiver [60]; [64]; [67], p. 673).

LEMMA 6. *If a unit η exists in $k(\zeta)$ which is of the form*

$$\eta \equiv c^l \pmod{\lambda^{2l}},$$

where c is a rational integer, then η is the lth power of a unit in $k(\zeta)$ except possibly when

$$B_{nl} \equiv 0 \pmod{l^3},$$

for n some integer in the set $1, 2, \cdots, (l-3)/2$. (Vandiver [54], pp. 616–621)

These lemmas were obtained by Kummer for the case where the first factor of the class number of $k(\zeta)$ is divisible by l but not by l^2. The published proofs of lemma 5 involve the use of what is called the class field theory and the law of reciprocity in the theory of algebraic numbers which concepts we cannot discuss here. The proof of Lemma 6 is direct and depends on the properties of Bernoulli numbers and the fact the E's in (17a) are independent. To handle Case II in connection with Theorem XII we write the equation in the generalized form

$$\omega^l + \theta^l = \eta\kappa^{ml}\xi^l, \tag{25}$$

where η is a unit and where (ω), (θ) and (ξ) are ideals in the field defined by $(\zeta+\zeta^{-1})$ prime each to each, $(\lambda, \xi)=1$, $m>0$. If this is proved impossible then it follows that (2) is impossible in Case II. We find, as in the proof of Theorem XI,

$$\left(\frac{\omega + \zeta^a\theta}{1-\zeta^a}\right) = \mathfrak{j}_a^l, \qquad (a = 1, 2, \cdots, l-1)$$

$$(\omega + \theta) = (\kappa)^{ml-(l-1)/2}\mathfrak{j}_0^l\,; \kappa = (1-\zeta)(1-\zeta^{-1}),$$

where $\mathfrak{j}_a$ and $\mathfrak{j}_0$ are ideals in $k(\zeta)$. Lemma 5 then gives

$$\left(\frac{\omega + \zeta^a\theta}{1-\zeta^a}\right)\left(\frac{\omega + \zeta^{-a}\theta}{1-\zeta^{-a}}\right)^{l-1} = (\rho_a')^l, \tag{26}$$

since the left-hand member is primary. But we also have since h_2 is prime to l and because of Lemma 4, if β is a real unit,

$$\left(\frac{\omega + \zeta^a\theta}{1-\zeta^a}\right)\left(\frac{\omega + \zeta^{-a}\theta}{1-\zeta^{-a}}\right) = (\rho_a'')^l\beta, \tag{27}$$

proceeding as we did in obtaining (21) from (20). Comparison of (26) and (27) gives, if η_a' is a real unit,

$$\frac{\omega + \zeta^a\theta}{1-\zeta^a} = \eta_a'\rho_a^l, \qquad (a = 1, 2, \cdots, l-1), \tag{28}$$

and since h_2 is prime to l, we may again use Lemma 4 and obtain

$$\omega + \theta = \eta_0'\kappa^{ml-(l-1)/2}\rho_0^l. \tag{28a}$$

Our proof consists of two main parts and we have finished one part with the derivation of the last relation. The next main step is to show that we can multiply ρ_a and ρ_b, the latter number obtained by substituting b for a in (28), by certain units, so that we can write

$$\frac{\omega + \zeta^a\theta}{1-\zeta^a} = \eta_a\rho_a^{*l}, \tag{28b}$$

and

$$\frac{\omega + \zeta^b\theta}{1-\zeta^b} = \eta_b\rho_b^{*l}, \tag{28c}$$

with

$$\eta_a/\eta_b = c^l \pmod{\lambda^{2l}}. \tag{28d}$$

We shall not give the steps necessary to prove the last statements. From (28d), using Lemma 6, η_a/η_b is then the lth power of a unit, using the second as-

sumption in our Theorem. In (28b) we now substitute ζ^{-1} for ζ, and multiply the two resulting relations together. This gives

$$\omega^2 + (\zeta^a + \zeta^{-a})\omega\theta + \theta^2 = \eta_a(2 - \zeta^a - \zeta^{-a})\omega^{*l}. \tag{29}$$

Also carrying out the same operation for b in place of a, where $a \not\equiv \pm b \pmod{l}$,

$$\omega^2 + (\zeta^b + \zeta^{-b})\omega\theta + \theta^2 = \eta_b(2 - \zeta^b - \zeta^{-b})\theta^{*l}. \tag{29a}$$

Then from (28a) we also have

$$\omega^2 + 2\omega\theta + \theta^2 = \eta_0 \kappa^{2\,2ml-l+1}\,\rho_0^{2l}. \tag{29b}$$

The elimination of $\omega^2+\theta^2$ and $\omega\theta$ from the last three equations yields a relation which reduces, after we employ Lemma 6 with (28d), to the form

$$\omega_1^l + \theta_1^l = \delta\kappa^{(2m-1)l}\xi_1^l, \tag{30}$$

with δ a unit in $k(\zeta)$ and ω_1, θ_1, ξ_1, are integers in $k(\zeta)$. This equation has the same form as (25) with ω_1, θ_1, ξ_1, integers in $k(\zeta)$ prime to each other but $2m-1$, >0, replaces m. Comparing the value of ξ_1 with the value of ξ we see, however, that since $\xi_1=\rho_0^2$, then (ξ_1) necessarily contains a lesser number of distinct prime ideal factors than does (ξ), (aside from the exceptional case when ξ/ρ_0 is a unit in $k(\zeta)$, which is easily shown to lead to a contradiction).

Hence we obtain by repetition of the process used in connection with (25) an unlimited series of ideals in the field $k(\zeta)$, (ξ_1), (ξ_2), (ξ_3), $\cdots$, in each of which the number of distinct ideal factors is less than in the preceding, which is impossible.

The above argument was a direct extension of the argument employed by Kummer in obtaining his result given at the beginning of Section 4. Vandiver [54] in (1929) proved the

THEOREM XIII. *Under the following assumption*:

None of the units E_a, $a=a_1, a_2, \cdots, a_s$, is congruent to the lth power of an integer in the field $k(\zeta)$, modulo $\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal divisor of p, p is a prime $<(l^2-l)$ of the form 1 (mod l), *and $a_1, a_2, \cdots, a_s$ are the subscripts in the Bernoulli numbers in the set*

$$B_1, B_2, \cdots, B_{(l-3)/2} \tag{31}$$

which are divisible by l;

then the relation (2) *is impossible in non zero rational integers.*

As in the proof of Case II of Theorem XII the argument to prove the present theorem consists of two parts. The first part is the same as in the proof of the first part of the previous theorem (derivation of 28a) since if the first assumption in Theorem XIII holds then none of E_a's can be lth powers so h_2 is prime to l by Lemma 5. This also yields a proof for Case I by Theorem X. Proof of the second part for Case II is quite different and the method is due to Vandiver. It

consists in taking (25) and proving that ξ is divisible by a certain rational prime $p=1+cl$, and then showing that $\omega+\theta$ is divisible by p. In view of this it is possible to show that η_a/η_b in (28d) is the lth power of a unit without employing assumption 2 of Theorem XII. We can also show in (30) that $\omega_1+\theta_1$ is divisible by p and this property holds throughout our process of infinite descent which will prove Theorem XIII. In a somewhat similar way we may prove the

THEOREM XIV. (Vandiver, [54], (1929)). *Under the following assumptions:* (1) *there exists a rational prime integer p such that the congruence*

$$u^l + v^l + w^l \equiv 0 \pmod{p}$$

has no solution u, v and w all rational integers prime to p, and $p \not\equiv 1 \pmod{l^2}$;

(2) *None of the units E_a, $a=a_1, a_2, \cdots, a_s$, is congruent to the lth power of an integer in the field $k(\zeta)$ modulo $\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal divisor of p, with p defined as in the first assumption and $a_1, a_2, \cdots, a_s$ are the subscripts in the Bernoulli numbers in the set* (31) *which are divisible by l;*

then the equation (2) *is impossible in rational integers none zero.*

A number of collaborators of the writer have tested the Theorems XII, XIII, and XIV for various primes l, particularly Mr. M. M. Abernathy, Dr. D. H. Lehmer, and Mr. M. E. Tittle. In the first place the assumption in Theorem XIII was tested ([67], pp. 667–670) for all irregular primes $l<619$. In each case the least prime p of the form $1+ml$ was used, and in each case it turned out that the assumption in Theorem XIII held. That is, E_a was not congruent to the lth power of an integer in $k(\zeta)$. *Hence, it was concluded* [69] [69c] *that* (2) *is impossible with x, y, and z rational integers none zero for all primes $l<619$,* since if l is regular Theorem XI provides a proof.

The fact that Theorem XIII did not break down within the limits of our computations indicates the difference in the present status of Fermat's Last Theorem from what it was as Kummer left it in his 1857 paper. One of his assumptions failed for $l=157$. If our present criteria do fail for some value of l it is unfortunate, of course, that this value did not come within the range of our calculations.

5. Discussion. I have often been asked my opinion as to the truth of the Theorem, and I should like to give here my ideas as they stand at the present time. First let us discuss Case I. I am definitely of the opinion that the theorem is true in Case I, but not because it has been verified for a great number of special cases. It is mainly because of Theorems VI and VII. Based on a great deal of experience with trinomial congruences, I think it is very likely that the number of congruences of type (5), with $p=1+ml$, $m<10l$, and those in which $p \not\equiv 1 \pmod{l^2}$ with no limitation on m, which have no solutions increase with l. Hence, in general, we could prove the first case of the theorem by the use of any one of a number of primes p. Also in the formula (13a) for $k=1$ and using (12) and (12a) we obtain

$$\frac{t(t^m-1)}{t-1} \sum_{\rho} \frac{F_i(\rho)(1-\rho)}{ml(t-\rho)} \equiv 0 \pmod{l}$$

as criteria for Case I of the theorem. The values $m=2$ and $m=3$ yield $q(2) \equiv q(3) \equiv 0 \pmod{l}$. We have left $l-3$ congruences which must also be satisfied. There is no value less than 16,000 which satisfies the first two mentioned.

In Case II I think the theorem is also true, but the indications of this are not so overwhelming. At least one of the assumptions in each of the Theorems XII to XIV fail if the second factor of the class number of the field defined by ζ, as given in (17), is divisible by l. However, about twenty-five years ago I conjectured that this number was never divisible by l. Later on, when I discovered how closely the question was related to Fermat's Last Theorem, I began to have my doubts, recalling how often conjectures concerning the theorem turned out to be incorrect. When I visited Furtwängler in Vienna in 1928, he mentioned that he had conjectured the same thing before I had brought up any such topic with him. As he had probably more experience with algebraic numbers than any mathematician of his generation, I felt a little more confident. With the second factor prime to l, I think that at least one of the Theorems XIII or XIV, if applied to the theorem for a particular l, gives a proof for l irregular with Theorem XI supplying a proof when l is regular. This is for the same reason that I mentioned in connection with certain criteria for Case I, that is, considerable experience with trinomial congruences. *However it would probably be best if I were wrong about this. I can think of nothing more interesting from the standpoint of the development of number theory, than to have it turn out that the Fermat relation has solutions, for a finite number* >0, *of primes l.*

Many mathematicians are often interested in ascertaining how a particular topic connects up with other parts of mathematics. In the case of Fermat's Last Theorem it is well known that Kummer's attempts to prove it gave rise to the theory of ideals which is now of fundamental importance in many parts of mathematics. The remarkable character of Kummer's achievement has tended, however, to minimize the great number of connections which the theorem has with other subjects. Efforts on my part to clear up the question have led me into the following topics: Bernoulli numbers and polynomials and generalizations; Euler and Genocchi numbers; Euler and Mirimanoff polynomials; partitions modulo m; finite fields and rings, including a great many types of congruences; the Dirichlet Zeta Function and the related Dedekind Function; the Lagrange resolvent and Jacobi ϕ number and various generalizations including generalized Gauss sums; theory of Kummer fields, class number, class fields, power characters and laws of reciprocity in the theory of algebraic fields; transformations of Elliptic functions and complex multiplication; Fermat's quotient and other arithmetic quotient forms; congruence theories as applied to power series; abstract algebra including, particularly, group theory and semigroups; and many types of Diophantine equations aside from the Fermat relation itself.

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THE FIRST PROFESSORSHIP OF MATHEMATICS IN THE COLONIES

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It appears that the identity of the holder of the first professorship of mathematics in what is now the United States is a moot question long since regarded as hopeless of solution by the more eminent historians in the subject. In a brochure entitled, "The Teaching and History of Mathematics in the United States," published in 1890, Florian Cajori properly gives credit to the College of William and Mary but states (page 33) that: "Owing to the repeated destruction of the college buildings and records by fire, not even the succession of the professors has been preserved . . . . The earliest professor at William and Mary whose name has come down to us was the Rev. Hugh Jones. The college had a professorship of mathematics from its very beginning, and at a date when mathematical teaching at Harvard was still in the hands of tutors only. The names of the predecessor or predecessors of Hugh Jones are not known."

The above is not a statement of fact at present although it may well have been so in 1890. President Tyler reopened the college in 1888 after a recess of seven years and at that time it is very probable that the archives, such as had survived, were in sad disorder. Now, however, due to the ceaseless researches of the librarian emeritus, Dr. Earl G. Swem, the facts are available and they portray an interesting episode in American academic history.

In casting about for a professor of mathematics the authorities very properly corresponded with the Lord Bishop of London who as chancellor of the college acted in an intimate capacity in matters of import. Upon his recommendation, according to the diary of William Byrd, the first professor of mathematics and philosophy was the Rev. Tanaquil Lefevre, who was chosen on April 25, 1711 and voted a salary of eighty pounds a year. He was a man of excellent education and background. His father, who spelled his name Tanaguy Lefebre, was for thirty years a French minister to Switzerland and England. Furthermore the records seem to indicate that he had a competent knowledge of mathematics and in 1714 published a work on algebra jointly with his son. Under the circumstances it was to be expected that the new addition to the college faculty would have brought scholarship and dignity as the handmaidens of success. Apparently, however, his interests were not wholly mathematical in view of the fact that on May 8, 1712 we find the authorities addressing the following information to the honorable Lord Bishop:

"I gave your lordship an account of Mr. Lefevre's admission into the college upon your lordship's recommendation and aim to acquaint you now that after a tryal of three quarters of a year he appeared so negligent in all of the posts of duty and guilty of some other very great irregularities, that the governors of the college could no longer bear with him, and were obliged to remove him from office, though at the same time out of regard to the honorable recom-

mendations he brought with him they continued his salary for four months longer than he officiated. I am apt to believe that most of his irregularities were owing to an idle hussy he brought over with him, because since she left him (I got her a passage back to England last February) he has left off that scandalous custom of drinking and appears quite another man, being now settled at a gentleman's house for teaching his son, and has a competent salary enough to keep him from being any more burdensome to your lordship or his other friends . . . ."

Just how long Lefevre remained in the colonies is not known but that his dismissal caused him both pecuniary and mental grief is evidenced by a petition he submits on August 17, 1735 in which through the medium of rather ponderous and obscure rhetoric his embarrassment is obvious:

"I humbly recommend myself, since I have got a sham employment who hath ruined me, let by the laws of the Universal God a distressed clergyman be relieved by a more successful warrant from the Society Justice and Charity are Petitioners that I might be indemnified so that I be able to have bread till I be sent by the Society to a real business about Divinity, Philosophy, Mathematics or Grammar, either in Greek or Latin,

"I will be sufferer if I do not find again ye included-papers when I pleased a Society for ye propagation of the faith is altogether for the Propagation of Charity. This is the only foundation of the humble petition presented to you.

"By your most humble Svt. Tanaguy Le Fevre
Tanaquillus Febre

"The employment of head master of Wm and Mary College hath been given several years ago to one Blackmore by ye bare hog drivers Authority & Power."

Successive perusals of the above cause increasing query as to whether the writer had fully relinquished some of his weaknesses of former years. In any event the document stands for whatever it may be worth.

The foregoing details are particularly eloquent when taken together with the fact that the first Hollis professor of mathematics at Harvard, Isaac Greenwood, who took office in 1727, was finally haled before the exasperated officials, to whom by 1738 forbearance had ceased to be a virtue, and likewise expelled for causes including intemperance. It took Greenwood longer in the more austere New England climate to liquidate himself but in any event the evidence is conclusive that mathematics got off to a rollicking, if not wholly dignified, start in the colonies. Those who today assert that of yore the teachers of the queen of the sciences lacked the human touch are grossly unaware of what might be termed the initial conditions.

After the Lefevre debacle the authorities moved with great deliberation, so much so in fact that it was not until 1717 that the Rev. Hugh Jones mentioned by Cajori was appointed. He was a Cambridge graduate of outstanding culture and scholarship. His monograph on the "Present State of Virginia" is acknowledged to be one of the best sources of information respecting the colony in the early part of the eighteenth century.

In 1729 there were six professors, all graduates of Edinburgh, Oxford or Cambridge.

By its charter the college was empowered to grant licenses to the county land surveyors in return for which the recipient was to pay one sixth of his fees received. It is a point of interest that in 1749 such a license was granted to George Washington. Such an appointment was at that time the equivalent of a degree in civil engineering. He was doubtless examined in part by John Graeme then professor of mathematics.

In 1770 the professor of natural philosophy also taught "physicks, metaphysics and mathematicks" at a salary of eighty pounds a year plus twenty shillings for each scholar. In addition each professor was the incumbent of a neighboring church and received therefrom an additional consideration of sixteen thousand pounds of tobacco.

It would appear that as time went on there arose an alarming demand for permission to take only courses in mathematics. In consequence we find the professors framing the following opinion as of May 2, 1770:

"With respect to the proposal of the visitors and governors for setting all such youths, whether resident in or out of the college, who have acquired a competent knowledge of common or vulgar arithmetic, and whose parents or guardians may desire it, to be received into the mathematical school. The president and masters or professors beg leave to represent that the college is not designed to be the sole place of resort for education in the colony, but the best place for training up youth who are intended to be qualified for any of the three learned professions, and accomplished citizens, in a regular course of study.

That the plan or method for this regular progress in study hitherto approved in the most famous universities as well as in the Statute of William and Mary College consists in the pursuit first of classical knowledge 2ndly of philosophy natural and moral and lastly of such sciences as are to become the business of the students during the remainder of their lives.

That the above method cannot be departed from or occasionally altered even for the sake of extraordinary geniuses who are able to shine in a learned profession without the foundation of classical learning, much less can it be accomodated to the views of those who aim at no more than a skill in Vulgare Arithmetic and some practical branch of mathematics to qualify them for an inferior office in life, without doing abundantly more injury than benefit to the public.

. . . . . .

That the number of those whom we call irregulars from our unwillingness to use a more adequate term, may soon grow great enough to employ the whole time of the professor of N.P. and Math. in teaching a practical branch or two of the mathematics, and all such irregular and desultary students after quitting the college, will be very apt to seize every opportunity to boast of their University education, under which disadvantage alone we think it almost impossible that the college should ever rise to any very high place of renoun."

However narrow the foregoing may appear as viewed from the present it nevertheless possesses a pith and lucidity that commands admiration. One can imagine that the document stirred up somewhat of an academic tempest concern ing matters curricular in view of the fact that in the local paper, the Virginia Gazette, under date of June 9, 1774 one signing himself "A.B." comments as follows: "I shall not ask why algebra is to precede the first elements of geometry, why fluxions must precede spherical trigonometry, or conic sections: it would be as useless as absurd. The method which has been for three or four years adopted as such has received the sanction of a Barrow, a Newton, a McLauren, a Whiston, and a Sanderson."

A complete list of the professors of mathematics up to the time of the closing of the college by the Civil War is as follows:

Tanaquil Lefevre	1711	Thomas Gwatkin	1769
Hugh Jones	1717	James Madison	1773
Alexander Irvine	1728	Robert Andrews	1784
Joshua Fry	1732	George Blackburn	1805
John Graeme	1737	Ferdinand Campbell	1811
Richard Graham	1749	Robert Saunders	1833
William Small	1758	Benjamin S. Ewell	1849
Richard Graham	1763	Robert Gatewood	1854
John Camm	1766	James M. Wise	1855
		Thomas T. L. Snead	1856

Robert Andrews was probably the first to teach mathematics only.

RETROSPECT

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1. These five years. This is the last issue of the MONTHLY to appear under my editorship. As I pause and look back over the past five years and while the details are fresh in my mind, it seems appropriate to relate for the readers something of the story of this eventful period.

My editorship has spanned the war years. The Japanese attack on Pearl Harbor occurred while the first issue was being set into type. We are reading proof on the last number while treaties of peace are being debated in a broken world. The four dozen issues between them were produced amidst all sorts of difficulties engendered by the war. If the editor who took up his duties so lightheartedly five years ago could have foreseen the future, would he have recoiled from the task? I am not sure.

2. Secretarial troubles. The most serious single difficulty arose from the scarcity of secretarial help. Chicago developed rapidly as a center of war industry and hundreds of office staffs were hastily built up. Any young woman with a

slight acquaintance with a typewriter could get a position at an inflated salary. One after another the secretaries left us after a stay of a few weeks each. I have no idea how many there were and their very names are forgotten.

It takes a secretary several months to learn the details connected with the editorship of the MONTHLY. For nearly three years the work was done by girls who never got beyond the first few lessons in the duties of the position. There came finally one dark time when, for weeks, it seemed that another secretary was nowhere to be found. The editor seriously considered employing a mathematician to teach his classes—mathematicians being somewhat plentiful at the moment—and serving as his own typist. It never quite came to that desperate improvisation. The best of all secretaries appeared magically upon the scene. She remains to this day.

3. Printing difficulties. I want to explain to our readers that for years it was never possible to get the MONTHLY out on time. This was never anybody's fault: it was inherent in the war effort. We had to stand aside, of course, while work for the armed services was rushed through. From first to last the George Banta Publishing Company printed ten million books for the Army and Navy.

The following are comments of a member of the Banta staff.* "Before the echoes of the Pearl Harbor catastrophe had died down, we were beset with a series of countless headaches which are concomitant with any such abrupt transition. But these headaches had to be endured with patience, and the new work proceed as best and with what speed it might, for the word had gone out from Washington and from Annapolis that this or that book, in such and such quantities, had to be put on our presses—and *now*!"

"Production plans and schedules were impossible of maintenance; a working schedule planned in the morning would be disrupted and changed through a directive or a priority plea by noon. And so it went; customers given a promise made in good faith were at times sadly disappointed—their work had to be delayed when war requirements demanded."

During the latter part of the war the shortage of skilled workmen made itself felt. The young printers, like the young of all callings, went into the service. The older men carried on. Some of our best proof came from the hands of these experienced men, but they had too much to do. There were delays and more delays.

The amount of paper available to us was reduced by governmental regulation. Consequently, it was necessary to reduce the number of pages printed during the year. Also, it was required that we adopt a cheaper grade of paper than we had used formerly. The result was a somewhat less attractive product. Photographic reproduction, for example, became impossible. These were small sacrifices.

The censorship was never a burden. Mathematical articles are seldom of a sort to give either aid or comfort to the enemy. Our readers had to forego a couple of papers on cryptography and one or two other small things.

* A. J. Hyson in the Banta shop newspaper, *Nightmare*, September-October, 1945.

4. Staples and the war. The most amusing of all the war regulations was never put into effect. The pages from which you are reading are held together by two wire staples. It occurred to some planner that half of this metal might be saved for the making of munitions and a regulation was promulgated that only one staple should be used. Many of us felt that the havoc done to our magazines could not possibly be justified by the gain in war materials, which must indeed have been infinitesimal. The order was withdrawn.

5. The stream of papers. The mathematicians, like others, put their shoulders to the wheel. The younger men went into the service. Many of all ages entered technical fields in connection with the armed forces or in war industries. Oldsters continued to teach, working with V-12 or ASTP, devoting long hours to the hurried instruction of heterogeneous groups, foregoing the usual summer vacations. Some followed all this with a period of teaching at Biarritz, Shrivenham, or Florence.

It is difficult to see who had time to write mathematical papers. Yet the stream of articles continued to flow. In all, some 450 longer papers passed through the hands of the editor-in-chief. There was never a time when there was a lack of material for the printer and there was never a conscious sacrifice of standards.

Some papers came, strangely enough, from abroad. There were some from India and several from the interior of China. France and Spain were represented by the indefatigable M. Thébault. We printed articles from Great Britain, Belgium, Australia, Brazil, and Puerto Rico. But most of the papers came from the United States and Canada.

6. Rejected papers. After careful study by competent referees about forty per cent of the papers presented were adjudged unsuitable for publication in the MONTHLY. Why? Perhaps an attempt to analyze the reasons for rejection will be of some help to future authors. Certainly a higher rate of acceptability will lighten the tasks of my successor.

There were a few papers which had little to do with mathematics and there were some which were simply wrong. But the main causes of rejection were three in number.

(1) Some papers are too technical for our readers. They are sound mathematically but they lie in fields of advanced research and narrow specialization. They would be read by few in the collegiate field. A paper which lacks some universality of appeal probably should be sent to a research journal.

(2) Some papers are what might, in not too derogatory a sense, be called trivial. They develop at length results that might be considered as mathematical exercises. They might better form contributions to the problem departments or to Discussions and Notes.

(3) Very often papers are poorly written. Fifty per cent of the papers which are ultimately accepted have to be revised. Nobody who has not been an editor can realize how much bad writing college professors can do. Many a paper composed in a lengthy, prolix, and awkward prose might have been saved by a

sprightly and fluent style. Those who write on the teaching of mathematics are still the chief sinners in this group. My advice to you would be to swallow your pride and show your paper to a friend in the English department.

7. Angle trisections and Fermat's theorem. I have not included in the previous count the long list of angle trisections and proofs of Fermat's last theorem. The diverting correspondence connected with these has done much to lighten the editorial load. The authors are blest with a boundless confidence and enthusiasm. "The new discovery is so important," says one trisector, "that I am in hopes of being invited to attend the Annual Meeting of the Association." Much of their work is copyrighted.

Sometimes the writer seeks profit, but this is exceptional. "For the disclosure of my discovery I should like to have some financial benefit" runs a letter from the West Indies. Another hopes that our publication of his results will help him "to find some one to finance the construction of my machine for interplanetary navigation."

My letters telling the authors firmly but not too bluntly that their work is erroneous have elicited varied responses. Some are angry. "I have become the dupe of bad incompetency." "I am very suspicious of your qualifications as to judging fairly the merits and demerits of my paper." Usually, however, the news is received in a spirit of resignation, born, no doubt, of oft-repeated experience. "I cannot make this any plainer and shall leave the matter in the hands of the university experts." "You know what happened to the men who first proclaimed that the earth is round." Usually we have parted friends, but I am sure that I have never convinced anybody of his errors.

This issue of the MONTHLY contains a scholarly article on Fermat's last theorem. I have no doubt that its author will received many "proofs" from these old friends of mine.

8. The associate editors. The publication of the MONTHLY has been a cooperative undertaking. Foremost among the collaborators have been the dozen and a half associate editors. Those who have been in charge of departments have met and solved their own considerable editorial problems. All have given unstintingly of their time year after year to the end that the magazine might eventually reach your desk. My most pleasant recollections during these five years are of their unselfish devotion.

One of these will now become editor-in-chief for the period 1947–1951. Professor C. V. Newsom, editor during the war of the widely read department of War Information, brings to his task both editorial experience and familiarity with the affairs of the Association. He will be spared many of the difficulties of the war years and he can devote his full thought to the building of a greater MONTHLY. May he achieve a matchless success!

DISCUSSIONS AND NOTES

EDITED BY E. F. BECKENBACH, University of California, Los Angeles 24, Calif.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

AN INTEGRAL

C. I. LUBIN, University of Cincinnati

It is pointed out early in the study of integrals in a course in advanced calculus that an integral of an integrable function is a continuous function of the upper limit.

Consider the integral,

$$F(x) = \int_{-1}^{x} \frac{e^{1/y}}{y^2(e^{1/y}+1)^2}\,dy. \tag{1}$$

The integrand,

$$f(y) = \frac{e^{1/y}}{y^2(e^{1/y}+1)^2}, \qquad (y \neq 0),$$

$$f(0) = 0,$$

is continuous in the interval $-1 \leqq y \leqq k$, k some positive number, and so the function $F(x)$ must be continuous in the same interval.

Now

$$\frac{d}{dx}\left(\frac{1}{e^{1/x}+1}\right) = f(x), \qquad (x \neq 0),$$

so that at first glance one might be tempted to write the integral (1) as

$$\frac{1}{e^{1/x}+1} - \frac{e}{e+1}. \tag{2}$$

The function (2), however, is discontinuous at $x=0$. This means that the continuous function $F(x)$, for the interval $-1 \leqq x \leqq k$, cannot be represented by (2).

The appropriate representation is as follows:

$$F(x) = \frac{1}{e^{1/x}+1} - \frac{e}{e+1}, \qquad (-1 \leqq x < 0),$$

$$F(0) = \frac{1}{e+1},$$

$$F(x) = \frac{1}{e^{1/x}+1} + \frac{1}{e+1}, \qquad (0 < x \leqq k).$$

A WELL-KNOWN INTEGRAL

H. F. SANDHAM, Trinity College, Ireland

If we let

$$\theta = \int_0^\infty e^{-x^2} dx,$$

then we have

$$\theta = \int_0^\infty e^{-x^2y^2} y dx,$$

or

$$2e^{-y^2}\theta = \int_0^\infty e^{-y^2(1+x^2)} 2y dx.$$

Integrating from 0 to ∞ with respect to y, and interchanging the order of integration, we get

$$2\theta^2 = \int_0^\infty \frac{dx}{1+x^2} = \tan^{-1} x \Big]_0^\infty = \frac{\pi}{2},$$

whence

$$\int_0^\infty e^{-x^2} dx = \tfrac{1}{2}\sqrt{\pi}.$$

A REMARK ON THE THEORY OF PARTITIONS

RAYMOND REDHEFFER, Massachusetts Institute of Technology

1. The partition function $p(m, r, n)$ represents the number of ways of distributing n indistinguishable objects in r indistinguishable boxes without putting more than m objects in any box. In this note we use very elementary methods to establish that

$$\sum_{n=0}^{mr} p(m, r, n) = \frac{(m+r)!}{m!r!} \tag{1}$$

for all positive integers m, r, both sides of the equation being defined to be unity when $mr = 0$.

2. *Proof.* Denoting the sum in (1) by $A(m, r)$, we make all possible distributions of 0, 1, 2, $\cdots$ objects in the r boxes, without putting more than $m-1$ objects in any box. The number of distributions of this type is $A(m-1, r)$, by definition. Next, let us put m objects in some one box and make all possible distributions of 0, 1, 2, $\cdots$ objects in the remaining $r-1$ boxes, this time allowing as many as m objects in any box. The number of different arrangements is now $A(m, r-1)$. We shall show that all arrangements of the first type, plus all of the second type, will give the total number of arrangements $A(m, r)$, so that

$$A(m, r) = A(m - 1, r) + A(m, r - 1). \tag{2}$$

a) All arrangements are included. For the first type includes all in which no box has more than $m-1$ objects, while the second includes all in which at least one box has m objects.

b) The same arrangement is not counted twice. For no arrangement of the first type is equal to an arrangement of the second type, since in the first case no box has more than $m-1$ objects, while in the second case one box always has m objects. Moreover, the arrangements of the first type are all different from each other, by the definition of $A(m, r)$, and the same is true of the arrangements of the second type.

Having established (2), let us suppose that (1) holds for $r+m=k$. Then it must hold for $r+m=k+1$; for (2) is simply the well-known law of formation for the binomial coefficients, as used in constructing Pascal's triangle. Equation (1) obviously holds for $r+m=0$ or 1, and thus the induction is complete.

3. The result may also be proved from the well-known fact that successive summation, starting with 1, will lead to the binomial coefficient; or, alternately, this latter result can be established by means of (1).

ON A CONJECTURE OF ERDÖS

F. BAGEMIHL, University of Rochester

In a paper appearing in this MONTHLY, vol. 53, 1946, p. 248, P. Erdös made the conjecture that on every convex curve there exists a point P such that every circle with center P intersects the curve in at most two points. We shall show that this conjecture is false by taking as our convex curve an equilateral triangle. If we describe about a vertex of the triangle a circle of radius greater than an altitude but less than a side, about the midpoint of a side of the triangle a circle of radius equal to one half a side, about any other point of the triangle a circle of radius equal to the maximum of the distances of the point from the two adjacent vertices, then in each case the circle in question will intersect the triangle in more than two points.

It is obvious, by continuity, that we can obtain from our triangle a convex curve containing no rectilinear arc and not having the conjectured property.

QUATERNIONS AND REFLECTIONS (POSTSCRIPT)

H. S. M. COXETER, University of Toronto

Theorem 5.1* was given five years earlier (with an even simpler proof) by Ernst Witt, Spiegelungsgruppen und Aufzählung halbeinfacher Liescher Ringe, Abhandlungen aus dem Mathematischen Seminar der Hansischen Universität, vol. 14, 1941, p. 308.

* This MONTHLY, vol. 53, 1946, p. 141.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to Loyal F. Ollmann, Hofstra College, Hempstead, N. Y.

CLUB REPORTS 1945-46

Mathematics Club, University of Dayton

Biweekly meetings were held during the year at which members of the club presented the following papers:

Heuristic reasoning, by Catherine Ens

Women in mathematics, by Virginia Johnston

Points of intersection of concentric circles, by Alice Blaeser

Unsolved riddles in mathematics, by George Falkenbach

Mathematics and navigation, by Robert Huels

The Bernoulli family, by Margaret Magin

Diophantine analysis, by Alice Blaeser

Mathematics and genetics, by Thomas McCarthy.

Faculty talks presented during the year were:

Are irrational numbers rational? by Dr. K. C. Schraut

A consideration of the wine glass problem of calculus, by Dr. Schraut

The statistical analysis of tests, by Professor C. G. Peckham

What is an infinite series?, by Dr. Otto Szasz of the University of Cincinnati.

A picnic and wiener roast was held in a Dayton City Park during the first semester, and an annual banquet at the Wishing Well Inn during the second semester.

At the formal initiation ceremony, King Bradow delivered the Vice President's charge to the five new members on the theme

The Mathematics Club member, his privileges and responsibilities.

The *Dean of Science Awards* were conferred each semester for the most interesting student paper. The recipients, George Falkenbach and Thomas McCarthy, received copies of Men of Mathematics by E. T. Bell. The *Mathematics Club Alumni Awards of Excellence in Advanced Mathematics* were conferred at Commencement upon a senior, Catherine Ens, and a junior, George Falkenbach.

The sum of $100.00 was contributed by the Mathematics Club Alumni Fund to the Albert Emmanuel Library for the purchase of books, and a suitable book plate was designed for these books.

The year was marked by progress in the National Mathematics Honor Society of Secondary Schools which was founded under the auspices of the Mathematics Club. Club officers participated in Chapter installations in Our Lady of Angels High School in St. Bernard and Our Mother of Mercy High School in Cincinnati.

The officers of the Club during the year 1945–46 were: President, Catherine Ens; Vice-President and Secretary, King Bradow; Treasurer, James Noll. Dr. K. C. Schraut was Faculty Adviser.

Mathematical Society, University of Adelaide

Apart from a preliminary meeting, on March 22, thirteen regular meetings were held during the academic year 1945 when the following papers were presented:

Fields of complex numbers with real coordinates, by M. C. Gray.

Rings with divisors of zero, by G. E. Wall.

Groups and ornaments, by Mrs. M. Sved.

An application of the method of least squares, by E. A. Cornish.

The linear oscillator, a boundary and eigen-value problem, by W. O. Gibberd.

An eigen-value problem of quantum mechanics, by H. Schwerdtfeger.

In how many ways can a 2-shilling piece be changed into smaller coins?, by Mrs. H. Schwerdtfeger.

Constructions with ruler and compass, by G. E. Wall.

The irrationality of pi, by M. C. Gray.

Some problems of particle motion in viscous fluids, by W. H. Schneider.

On transcendental numbers, by Mrs. M. Sved.

A distribution problem in mathematical statistics, by A. T. James.

In order to meet the interests of students of Mathematical Statistics and to relieve some third- and fourth-year courses, the *Mathematical Society*, during the year 1946, will be conducted as a *Colloquium on the theory of the gamma-function and its applications.*

Kappa Mu Epsilon, Mount St. Scholastica College

The Kansas Gamma Chapter numbered 19 active members and 13 active pledges. Meetings were held semi-monthly. The theme for the year's activities was

Various types of mathematical presentation: to the general public, to the practical mathematician, to the scientist, and to the theoretical mathematician.

Mathematics and the college student, by the Reverend Pius Pretz, O.S.B., Professor of mathematics at St. Benedict's College, Atchison, Kansas, was the lecture given at an open meeting.

The chapter continued the quarterly publication of the *Exponent*, the chapter paper.

Following the initiation in May of nine new members, a buffet supper was served. It was announced that Mary Lou Maloney received the annual award for having contributed the most to the fraternity during the year.

Officers for the year 1946–1947 are: President, Mary Jane Fox; Vice-President, Joanne Shuey; Secretary, Victoria Fritton; Treasurer, Jean Moran; Chapter Publicity, Frances Knightley; Chapter Musician, Elizabeth Gulde; Corresponding Secretary and Faculty Sponsor, Sister Helen Sullivan, O.S.B.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 746. *Proposed by H. F. Sandham, Trinity College, Ireland*

Show that for all positive integers r

$$r\{1 - (n+1)^{-1/r}\} < 1/1 + 1/2 + \cdots + 1/n \leqq r(n^{1/r} - 1) + 1.$$

E 747. *Proposed by H. W. Becker, Omaha, Nebraska*

Can we decompose an integer force F into n integer forces $F_1, \cdots, F_n$, such that the sum of any number of the components is also an integer force, where $n>2$?

E 748. *Proposed by George Pólya, Stanford University*

Let α be the angle between the axis and any element of a right circular cone. If the cone is cut through by a plane (not necessarily perpendicular to the axis) show that the lateral area of the part remaining between the plane and the vertex is given by $\pi AG \sin \alpha$, where A and G are respectively the arithmetic and the geometric mean of the longest and the shortest remaining elements.

E 749. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a given sphere inscribe a right circular cone whose lateral area is equal to the area of the zone beneath its base. Show that the total area of the cone is equal to the area of the zone in which it is inscribed.

E 750. *Proposed by Paul Erdös, Stanford University*

Find the number of intersections of the diagonals of a convex polygon of n sides.

SOLUTIONS

Iterated Factorials

E 684 [1945, 395 and 1946, 271]. *Proposed by Paul Erdös, Stanford University*

Prove that $n!^{(n-1)!}$ divides $(n!)!$; and that if n is not a power of a prime, $n!^{n(n-2)!}$ divides $(n!)!$.

Note by Fritz Herzog, Michigan State College. I believe that the solution as given (*cf.* [1946, 271]) is in error, as far as the proof of the second part of the problem is concerned. The statement that "since n is not a power of a prime, the number of times that p divides $n!$ is the same as the number of times that p divides $(n-1)!$" holds true if and only if p does not divide n. Thus, in order to complete the proof, the case in which p is a prime divisor of n ought to be considered.

Let p^a be the highest power of p that divides n and let the integer b be defined by $p^b<n<p^{b+1}$. (Here the fact is used that n is not a power of a prime.) Obviously, $1 \leqq a \leqq b$. We have to show that

$$\sum_{i=1}^{\infty} \{[n!/p^i] - n(n-2)![n/p^i]\} \geqq 0. \tag{1}$$

The terms of the series in (1), corresponding to $i=1, 2, \cdots, a$, are actually negative, namely, since n/p^i and $n!/p^i$ are integers, equal to $-n(n-2)!/p^i$. Thus their sum equals

$$-n(n-2)!\sum_{i=1}^{a} 1/p^i.$$

To show that this negative number is offset by a positive number of equal or greater absolute value, I remark first that for $i>a$, p^i does not divide n, hence $[n/p^i]=[(n-1)/p^i]$ and the method employed in the published solution shows that no terms with $i>a$ of the series in (1) can be negative. Moreover, consider the terms of that series, corresponding to $i=b+1, b+2, \cdots, b+a$, We have for $j=1, 2, \cdots, a$, by the use of $p^b \leqq n-1$,

$$n!/p^{b+i} = \{n(n-2)!/p^i\}\{(n-1)/p^b\} \geqq n(n-2)!/p^i.$$

The last number being an integer, we obtain $[n!/p^{b+i}] \geqq n(n-2)!/p^i$ for $j=1, 2, \cdots, a$. On the other hand, $p^{b+i}>n$ and hence $[n/p^{b+i}]=0$. Thus the sum of the terms in (1), corresponding to $i=b+1, b+2, \cdots, b+a$, is greater than or equal to

$$+n(n-2)!\sum_{j=1}^{a} i/p^i.$$

I also remark that, *when n is the power of a prime, then* $n!^{n(n-2)!}$ *does not divide* $(n!)!$; in other words, the condition of the second past of the problem is necessary as well as sufficient,

Let $n=p^a$. It suffices to show (see (1)) that

$$\sum_{i=1}^{\infty} \{[p^a!/p^i] - p^a(p^a-2)![p^a/p^i]\} < 0. \tag{2}$$

The sum of the first a terms in (2) is again, as above, equal to

$$-p^a(p^a-2)!\sum_{i=1}^{a} 1/p^i = -(p^a-1)!/(p-1).$$

For $i>a$, we have $[p^a/p^i]=0$ and $[p^a!/p^i]\leqq p^a!/p^i$, but actually $<$ for sufficiently large i, Hence the sum of the remaining terms of the series in (2), beginning with $i=a+1$, is less than

$$\sum_{i=a+1}^{\infty} p^a!/p^i = (p^a - 1)!/(p - 1).$$

This completes the proof of (2).

Sir William Hamilton's Icosian Game

E 711 [1946, 156]. *Proposed by H. S. M. Coxeter, University of Toronto*

Suppose that the vertices of a polyhedron represent places that we wish to visit, while the edges represent the only possible routes. Hamilton considered the problem of visiting all the places, without repetition, on a single journey. (See, *e.g.*, W. W. R. Ball, *Mathematical Recreations and Essays*, London, 1939, p. 262.) This is easily solved for the pentagonal dodecahedron. Prove that it cannot be done for the rhombic dodecahedron.

Solution by Arthur Rosenthal, University of New Mexico. The rhombic dodecahedron has eight three-edged vertices T and six four-edged vertices F. The neighbors of each T are F's and neighbors of each F are T's. Hence for each journey the T's and F's have to alternate. But eight T's and six F's cannot be written as one alternating sequence, whether we wish to return to the starting point or not.

Also solved by Norman Anning, J. B. Kelly, N. D. Lane, Leo Moser, and E. D. Schell.

A Set of Three Numbers

E 718 [1946, 219]. *Proposed by Roy Dubisch, Syracuse University*

Let A be a positive integer of r digits given by $A=\sum_{i=1}^{r} x_i 10^{i-1}$ and define $D(A)=(\sum_{i=1}^{r} x_i)^2$. By $D^n(A)$, where n is a positive integer, we mean the result of applying the operator, D, n successive times to A. Prove that for every A there exists an n such that $D^n(A)=1$, 81, or 169. (*Cf.* Porges, *A set of eight numbers*, this MONTHLY [1945, 379–384]).

Solution by Paul Bateman, Philadelphia, Pa. It is easily seen that for $A\geqq 400$, $D(A)<A$. Hence, if $A\geqq 400$, there exists an n such that $D^n(A)<400$. Hence it suffices to prove the assertion for $A<400$. If $A<400$, then clearly $D(A)\leqq(3+9+9)^2=21^2$ and $D(A)$ is one of the squares $1^2, 2^2, \cdots, 21^2$. Verification of the theorem for these twenty-one numbers completes the proof.

Also solved by Paul Brock, R. C. Buck, Daniel Finkel, Free Jamison, W. J. LeVeque, E. P. Starke, and the proposer

Starke also showed that if A has four or less digits, then $n\leqq 4$. He also showed that the end result is 1 whenever $A\pm 1$ is divisible by 9, 81 when A is divisible by 3, and 169 otherwise.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, N. J. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extension of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS AND SOLUTION

4225. *Proposed by H. F. Sandham, Trinity College, Ireland*

There are given N points in a plane, the roots of $f(z)=0$. A conic is constructed with center at the mean center, and is such that the sum of the squares of the perpendiculars from the points on to a line through the center is $N(N-1)$ times the square of the perpendicular from the center on to a parallel tangent. Prove that the foci are given by $f^{(N-2)}(z)=0$.

4226. *Proposed by Paul Erdös, University of Michigan*

Let $n!=\Pi_{p\leqq n}p^{k_p}$ (p, primes). Assume that $p_1>p_2$ and $k_{p_1}<k_{p_2}$. Prove that $p_1^{k_{p_1}}<p_2^{k_{p_2}}$.

4227. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In what number systems is the square of a number of three digits of the form bcb of the form $aabbcc$?

4228. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $ABCD$ the straight lines joining each vertex to the points of intersection V and W of the six spheres of similitude of four given spheres with centers A, B, C, D, meet the circumsphere in the vertices of two equal tetrahedrons $A'B'C'D'$ and $A''B''C''D''$.

SOLUTIONS

Probability of an Uncrowded Road

4161 [1945, 345]. *Proposed by R. E. Gaines, University of Richmond*

Along a straight road a miles long are n persons. What is the probability that no two persons are less than b miles apart?

Solution by A. G. Clark, Colorado A & M College.

With one end of the road as an origin, let the position of any one person be represented by x, $0\leqq x\leqq a$. Then x is a random variable with a distribution law:

$$p(x)=\begin{cases}1/a; & 0\leqq x\leqq a\\ 0; & \text{elsewhere.}\end{cases}$$

If x_i; $i=1, 2, \cdots, n$; give the positions along the road of some ordered arrangement of the n persons, then the probability that $x_{i+1} \geqq x_i+b$; $i=1, 2, \cdots, n-1$; for this one arrangement of the n persons is given by the iterated integral,

$$I = \int_0^{a-(n-1)b} p(x_1)dx_1 \int_{x_1+b}^{a-(n-2)b} p(x_2)dx_2 \cdots \int_{x_{n-2}+b}^{a-b} p(x_{n-1})dx_{n-1} \int_{x_{n-1}+b}^{a} p(x_n)dx_n.$$

The evaluation of I is easy, giving

$$I = \left[\frac{a-(n-1)b}{a}\right]^n \Big/ n!.$$

Since there are $n!$ possible ordered arrangements of the n persons along the road, the desired probability is

$$P = n!I = [1-(n-1)b/a]^n.$$

Of course it is clear that the condition $(n-1)b \leqq a$ must be imposed.

Solved also by D. W. Alling, N. J. Fine, J. B. Kelly, G. W. Petrie, and the proposer.

Editorial Note. A slight variation of the problem is as follows:

There are n equal cylinders each of length c strung on a horizontal rod with stops at its ends which give a length of a on which the n cylinders can take various positions. For random positions of the cylinders what is the probability P_n that no two cylinders are less than b apart?

Denote by x_i the abscissa of the left end of the ith cylinder counting from the left stop, then we have the following inequalities

$$(i-1)(c+b) \leqq x_i \leqq x_{i+1}-(c+b), \qquad 1 \leqq i \leqq n-1;$$
$$(n-1)(c+b) \leqq x_n \leqq a-c.$$

If x_1 alone varies, then the measure of the favorable cases is $x_2-(c+b)$; if x_1, x_2 alone vary, the measure is $[x_3-2(c+b]^2/2$; if x_1, x_2, x_3 alone vary, the measure is $[x_4-3(c+b)]^3/3!$; and so on up to and including the case where $x_1, x_2, \cdots, x_{n-1}$ alone vary and the measure is $[x_n-(n-1)(c+b)]^{n-1}/(n-1)!$. Finally if all vary, the measure is $[a-nc-(n-1)b]^n/n!$.

If no condition is imposed, then $b=0$, and the measure is now $[a-nc]^n/n!$; and we have

$$P_n = \left[1-\frac{(n-1)b}{a-nc}\right]^n, \qquad a \geqq nc+(n-1)b.$$

Refinement of the Triangle Inequality

4171 [1945, 462]. *Proposed by Tibor Radó, Ohio State University*

Let $\mathbf{x}_n$, $n=0, 1, 2, \cdots$ be a sequence of vectors in euclidean three-space such that $|\mathbf{x}_n| > \delta$ for all n, where δ is a fixed positive constant, and absolute

value signs designate the length of the vector involved. Prove that the relation

$$|\mathbf{x}_n| + |\mathbf{x}_0| - |\mathbf{x}_n + \mathbf{x}_0| \to 0 \quad \text{as} \quad n \to \infty$$

holds if and only if there exists a sequence of positive scalars a_n such that $|a_n\mathbf{x}_n - \mathbf{x}_0| \to 0$ as $n \to \infty$.

Solution by R. C. Buck, Harvard University. Let $\mathbf{x}_n$ be a sequence of points of a normed linear space, for example, 3-space. We suppose that $\|\mathbf{x}_n\| \geqq \delta$ for all n.

First, if there is a sequence of positive scalars c_n such that $\lim \|c_n\mathbf{x}_n - \mathbf{x}_0\| = 0$, then $\lim \|\mathbf{x}_0\| + \|\mathbf{x}_n\| - \|\mathbf{x}_0 + \mathbf{x}_n\| = 0$. For, $\|\mathbf{x}_0\| = \|\mathbf{x}_0 - c_n\mathbf{x}_n + c_n\mathbf{x}_n\| \leqq c_n\|\mathbf{x}_n\| + \|\mathbf{x}_0 - c_n\mathbf{x}_n\|$ and $\|\mathbf{x}_0 + \mathbf{x}_n\| = \|(c_n+1)\mathbf{x}_n + \mathbf{x}_0 - c_n\mathbf{x}_n\| \geqq (c_n+1)\|\mathbf{x}_n\| - \|\mathbf{x}_n - c_n\mathbf{x}_n\|$ so that

$$0 \leqq \|\mathbf{x}_0\| + \|\mathbf{x}_n\| - \|\mathbf{x}_0 + \mathbf{x}_n\| \leqq 2\|\mathbf{x}_0 - c_n\mathbf{x}_n\| \to 0.$$

Now if, in addition, the space has an inner product, true of Euclidean n-space for example, then

$$\left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} - \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\|^2 = \frac{\|\mathbf{x}\| + \|\mathbf{y}\| + \|\mathbf{x} + \mathbf{y}\|}{\|\mathbf{x}\|\,\|\mathbf{y}\|} (\|\mathbf{x}\| + \|\mathbf{y}\| - \|\mathbf{x} + \mathbf{y}\|)$$

so that, with $\|\mathbf{x}_n\| \geqq \delta > 0$, and $\lim \|\mathbf{x}_0\| + \|\mathbf{x}_n\| - \|\mathbf{x}_0 + \mathbf{x}_n\| = 0$, we have

$$\lim \left\| \frac{\mathbf{x}_0}{\|\mathbf{x}_0\|} - \frac{\mathbf{x}_n}{\|\mathbf{x}_n\|} \right\| = 0$$

or, putting $\|\mathbf{x}_0\| / \|\mathbf{x}_n\| = c_n > 0$, $\lim \|\mathbf{x}_0 - c_n\mathbf{x}_n\| = 0$.

This remains true if the space lacks an inner product, but does satisfy a certain convexity relation, sufficient to infer the convergence of $\|\mathbf{x}_0/\|\mathbf{x}_0\| - \mathbf{x}_n/\|\mathbf{x}_n\|\|$ from that of $\|\mathbf{x}_0\| + \|\mathbf{x}_n\| - \|\mathbf{x}_0 + \mathbf{x}_n\|$. For example, we might require that for any positive δ there is an M such that for all $\mathbf{x}$, $\mathbf{y}$, $\|\mathbf{x}\| > \delta$, $\|\mathbf{y}\| > \delta$, we have

$$\left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} - \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\| \leqq M\{\|\mathbf{x}\| + \|\mathbf{y}\| - \|\mathbf{x} + \mathbf{y}\|\}.$$

Solved also by Truman Botts, Peter Frank, J. W. Green, P. C. Hammer, and Hans Samelson.

Finite Cosine Series

4172 [1945, 462]. *Proposed by R. P. Agnew, Cornell University*

Prove or disprove the following statement involving cosine series. If $a_1, a_2, a_3, \cdots$ is a sequence of real constants, and if L_n is the length of the part of the graph of the function

$$f_n(x) = a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx$$

lying in the interval $-\pi \leqq x \leqq \pi$ then $L_1 \leqq L_2 \leqq L_3 \leqq \cdots$.

I. *Solution by Fritz Herzog, Michigan State College.* The statement is not true. More specifically, for each $n > 2$ there exist values of $a_1, a_2, \cdots, a_n$ such

that $L_{n-1} > L_n$; however, $L_1 \leqq L_2$ for all values of a_1 and a_2.

Let n be a fixed integer >2. We consider L_n as a function of the n real variables $a_1, a_2, \cdots, a_n$. The statement that $L_{n-1} \leqq L_n$ would imply that if $a_1, a_2, \cdots, a_{n-1}$ are arbitrarily fixed L_n would have a minimum at $a_n = 0$. In order to prove that this cannot be true, it suffices to show that the expansion of L_n in a Taylor series about the origin ($a_1 = a_2 = \cdots = a_n = 0$) contains non-zero terms which are linear in a_n. We assume throughout the following that $a_1, a_2, \cdots, a_n$ are restricted to a sufficiently small neighborhood of the origin, for instance, the region defined by $|a_1| + 2|a_2| + \cdots + n|a_n| \leqq 1/2$.

We have

$$L_n = \int_{-\pi}^{\pi} [1 + (a_1 \sin x + 2a_2 \sin 2x + \cdots + na_n \sin nx)^2]^{1/2} dx,$$

$$L_n = \int_{-\pi}^{\pi} \sum_{r=0}^{\infty} \binom{1/2}{r} (a_1 \sin x + 2a_2 \sin 2x + \cdots + na_n \sin nx)^{2r} dx. \tag{1}$$

Expanding each term in the infinite series in (1) by the polynomial theorem, we obtain the expansion of the integrand in a Taylor series about the origin. It is easily seen that this Taylor series converges absolutely and uniformly for $-\pi \leqq x \leqq \pi$. Integrating term by term, we thus obtain from (1) the expansion of L_n itself in a Taylor series about the origin.

In order to show the presence in this Taylor series of non-zero terms, linear in a_n, let i, j, k be any integers with

$$0 < i \leqq j \leqq k < n. \tag{2}$$

The coefficient of $a_i a_j a_k a_n$ in the Taylor series for L_n is easily obtained from (1) as being equal to

$$\int_{-\pi}^{\pi} (-1/8) W (i \sin ix)(j \sin jx)(k \sin kx)(n \sin nx) dx$$

$$= -(Wijkn/32) \int_{-\pi}^{\pi} [\cos (j-i)x - \cos (j+i)x][\cos (n-k)x - \cos (n+k)x] dx,$$

where W equals 24 or 12 or 4, respectively, according to whether no two or exactly two or all three of the numbers i, j, k are equal to each other. By the use of the well-known orthogonality relations between the functions $\cos px$ ($p = 0, 1, 2, \cdots$) and since by (2) $n+k > j+i > j-i \geqq 0$, we conclude that the coefficient of the term $a_i a_j a_k a_n$, which by (2) is linear in a_n, will be different from zero if and only if

$$n - k = j - i \quad \text{or} \quad n - k = j + i. \tag{3}$$

Clearly, values of i, j, k satisfying (2) and (3) exist for each $n > 2$ (but not for $n = 2$). This concludes the proof of the first part of our statement.

To show that $L_1 \leqq L_2$ for all real values of a_1 and a_2, we make use of the fact that $(1+t^2)^{1/2}$ is a convex function of t and hence for real u and v

$$(4) \qquad [(1+u^2)^{1/2}+(1+v^2)^{1/2}]/2 \geqq \{1+[(u+v)/2]^2\}^{1/2}.$$

Put $u=a_1 \sin x+2a_2 \sin 2x$ and $v=a_1 \sin x-2a_2 \sin 2x$, so that $(u+v)/2=a_1 \sin x$. Then integrating from $x=0$ to $x=\pi/2$, the left side of (4) yields $L_2/4$, while the right side gives $L_1/4$. Thus the inequality (4) leads to $L_2 \geqq L_1$.

Remark. An investigation of the signs of the coefficients of the terms $a_i a_j a_k a_n$ (see above) reveals the fact that for $n>2$ the statement $L_{n-1} \leqq L_n$ can be disproved even if the a_m are restricted to *non-negative* numbers.

II. *Solution by H. E. Bray, Rice Institute.* It will be proved that $L_1 \leqq L_2 \leqq L_3 \cdots$ is false. Let us define $f_1(x), f_2(x), f_3(x)$ as follows,

$$f_1(x) \equiv f_2(x) \equiv \cos x, \qquad f_3(x) \equiv \cos x + a \cos 3x,$$

where a is a parameter. Then L_3 is a function of a

$$L_3(a) = 2\int_0^{\pi} [1+(\sin x+3a \sin 3x)^2]^{1/2} dx.$$

Note that $L_3(0)=L_2=L_1$. We shall prove that dL_3/da is positive when $a=0$. It will then follow that $L_3(a)$ is less than $L_3(0)=L_2=L_1$ if a is negative and numerically small. We have, after differentiating under the integral sign and setting $a=0$,

$$\begin{aligned} dL_3/da\,|_{a=0} &= 6\int_0^{\pi} \sin x \sin 3x(1+\sin^2 x)^{-1/2} dx \\ &= 6\int_0^{\pi/2} (\cos 2x - \cos 4x)(1+\sin^2 x)^{-1/2} dx \\ &= 6\int_0^{\pi/2} g(x)(1+\sin^2 x)^{-1/2} dx. \end{aligned}$$

In the last integral $g(x) \equiv \cos 2x - \cos 4x$. The function $g(x)$ is positive, $0<x<\pi/3$, and is negative, $\pi/3<x<\pi/2$. Moreover $\int_0^{\pi/2} g(x)dx=0$. But $(1+\sin^2 x)^{-1/2}$ is a positive decreasing function, $0 \leqq x \leqq \pi/2$, which takes on the value $2/\sqrt{7}$ when $x=\pi/3$. Therefore

$$\int_0^{\pi/3} g(x)(1+\sin^2 x)^{-1/2} dx > \frac{2}{\sqrt{7}} \int_0^{\pi/3} g(x)dx$$

$$\int_{\pi/3}^{\pi/2} g(x)(1+\sin^2 x)^{-1/2} dx > \frac{2}{\sqrt{7}} \int_{\pi/3}^{\pi/2} g(x)dx.$$

Hence, on adding, we obtain the inequality

$$(1/6)dL_3/da]_{a=0} = \int_0^{\pi/2} g(x)(1+\sin^2 x)^{-1/2} dx > \frac{2}{\sqrt{7}} \int_0^{\pi/2} g(x)dx = 0$$

from which it follows that dL_3/da is positive when $a=0$. Thus $L_3(a)<L_2=L_1$ when a is negative and numerically small.

Solved also by the proposer.

The proposer stated that it can be shown that $L_1 \leqq L_2$, with equality only when $a_2=0$; and to show that the statement is not true he considered $a_1=1$, $a_2=0$, $a_3=c/3$, and proceeded as in Bray's elegant solution to show that $[dL_3/dc]_{c=0}>0$.

Twelve Point Sphere

4175 [1945, 463]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The twelve point sphere of any tetrahedron is the locus of the points such that the sum of the squares of their distances to the vertices diminished by their powers with respect to the circumsphere, is equal to a third of the sum of the squares of the edges.

Solution by the Proposer. The following generalization will be proved.

Theorem. The locus of the points such that the sum of the squares of their distances to the vertices of a tetrahedron diminished by their powers with respect to its cicumsphere is constant, is a sphere concentric to the twelve point sphere of the tetrahedron.

Proof. Given the tetrahedron $ABCD$, set $BC=a$, $DA=a'$, $CA=b$, $DB=b'$, $AB=c$, $DC=c'$; denote by O and R the circumcenter and circumradius, and by G the centroid. Then the variable point M is such that

$$(1)\qquad (MA)^2+(MB)^2+(MC)^2+(MD)^2-[(MO)^2-R^2]=k\Sigma[a^2+(a')^2],$$

where the right member is the given constant. Since

$$\begin{aligned}(MA)^2+(MB)^2+(MC)^2+(MD)^2&=(GA)^2+(GB)^2+(GC)^2+(GD)^2+4(GM)^2\\&=\tfrac{1}{4}\Sigma[a^2+(a')^2]+4(GM)^2,\end{aligned}$$

the relation (1) can be written

$$4(GM)^2-(OM)^2=-R^2+(k-\tfrac{1}{4})\Sigma[a^2+(a')^2],$$

and this represents a sphere (S) with the center ω such that $\omega G/\omega O=1/4$, and ω coincides with the center of the twelve point sphere of $ABCD$. The square of the radius is equal to

$$(\omega M)^2=\frac{R^2}{9}+\tfrac{1}{8}(k-\tfrac{1}{4})\Sigma[a^2+(a')^2],$$

and conversely. The Stewart theorem applied to triangle $MO\omega$ gives in fact

$$4(GM)^2-(OM)^2=3(\omega M)^2-\tfrac{4}{3}(OG)^2,$$

$$(OG)^2=R^2-\tfrac{1}{16}\Sigma[a^2+(a')^2].$$

For the particular case of the problem where $k=1/3$, the sphere (S) is the twelve point sphere with the radius $R/3$.

Solved also by W. T. Short.

Editorial Note. The solution by Short referred to Court's *Modern Pure Geometry*, p. 251, for theorems as to the twelve point sphere, and used a system of coordinates in the proof.

The generalized theorem can be extended to a simplex in euclidean space of n dimensions as follows. The $3(n+1)$ point sphere of a simplex S in euclidean space of n dimensions is the sphere (L) through the $n+1$ centroids of its faces; if M is the Monge point of S it passes through the $n+1$ points A_i' where $MA_i' = MA_i/n$ where A_i is a vertex of S; and it also passes through the points A_i'' the projections of A_i' on the corresponding faces opposite to A_i. The center of the $3(n+1)$ point sphere is at the point L where $ML = MC/n$, C being the center of the circumsphere (C) of S; and its radius is R/n, R being the radius of (C). The sphere (L) is the inverse of (C) with respect to the polar sphere of S.

Thus if M is taken as the origin of vectors, $\mathbf{a}_i$, $\mathbf{c}$, $\mathbf{g}$ being the vectors of A_i, C, G, the last being the centroid of S, the vector equation of (L) is

$$(L); \qquad \left(\mathbf{x} - \frac{\mathbf{c}}{n}\right)^2 = \frac{R^2}{n^2}, \qquad \mathbf{x}^2 - \frac{2\mathbf{c}\cdot\mathbf{x}}{n} + \frac{\mathbf{c}^2 - R^2}{n^2} = 0.$$

The polar sphere of S has the equation $\mathbf{x}^2 - \rho = 0$, and the equation of (L) may also be written

$$(L) \equiv (C'); \qquad n\mathbf{x}^2 - 2\mathbf{c}\cdot\mathbf{x} + \rho = 0, \quad n\rho = \mathbf{c}^2 - R^2, \quad 2\mathbf{c} = (n+1)\mathbf{g}.$$

Let P be a point such that

$$\Sigma\,(PA_i)^2 - [(CP)^2 - R^2] = k \text{ (constant)}.$$

Then the equation of the locus of P is

$$(n+1)\mathbf{x}^2 - 4\mathbf{c}\cdot\mathbf{x} + \Sigma\,\mathbf{a}_i^2 - [\mathbf{x}^2 - 2\mathbf{c}\cdot\mathbf{x} + \mathbf{c}^2 - R^2] = k,$$
$$n\mathbf{x}^2 - 2\mathbf{c}\cdot\mathbf{x} + [\Sigma\,\mathbf{a}_i^2 - \mathbf{c}^2 + R^2 - k] = 0.$$

Hence the locus of P is a sphere with center at L given by the vector $\mathbf{c}/n$. To find its radius we write the locus equation

$$\left(\mathbf{x} - \frac{\mathbf{c}}{n}\right)^2 = \{(n+1)\mathbf{c}^2 + n[k - R^2 - \Sigma\,\mathbf{a}_i]\}/n^2$$
$$= \frac{1}{n^2}[(n-1)^2\mathbf{c}^2 - n(n+2)R^2 + nk],$$

and it follows that the square of its radius is the right member. For a suitable value of k the locus reduces to (L), and to determine its value we solve

$$(n-1)^2\mathbf{c}^2 - n(n+2)R^2 + nk = R^2$$
$$nk = (n+1)^2R^2 - (n-1)^2\mathbf{c}^2 = \Sigma\,e_{ij}^2 = \Sigma\,(A_iA_j)^2.$$

In the Editorial Note for 4049 [1943, 577] there are given a number of vector equations related to the simplex S where m is used in place of the above ρ.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

The first three graduate scholarships granted by the Sperry Gyroscope Company have been awarded to E. J. Barlow, O. T. Schultz, and W. E. Tinsley.

Michigan State College announces the following appointments: Dr. J. H. Bell of the University of Manitoba, Professor H. L. Black of Westminster College, Dr. G. M. Brown of the Harrow School, England, and Dr. Leo Katz of Wayne University have been appointed to assistant professorships; Assistant Professor C. P. Wells has been promoted to an associate professorship.

Oklahoma Agricultural and Mechanical College makes the following announcements: Dr. J. C. C. McKinsey has been appointed to an associate professorship; Assistant Professors P. E. Lewis and C. E. Marshall have been promoted to associate professorships; R. L. Caskey, N. D. Griffin, and W. L. Shepherd have been promoted to assistant professorships.

Stanford University announces: Associate Professors George Polya, A. C. Schaeffer and D. C. Spencer have been promoted to professorships.

The United States Naval Academy makes the following announcements: Dr. J. C. Abbott and Dr. C. W. Seekins have been appointed to assistant professorships, Dr. E. E. Betz, Dr. T. J. Benac and R. C. Morrow have been promoted to assistant professorships, Dr. C. P. Brady and Dr. T. L. Downs have been appointed to associate professorships. Assistant Professor T. W. Moore has been promoted to an associate professorship. At the Postgraduate School, Assistant Professor Randolph Church has been promoted to an associate professorship and Associate Professor C. C. Torrance of the Case School of Applied Science has been appointed to an associate professorship.

The University of Michigan makes the following announcements: Associate Professor P. S. Dwyer has been promoted to a professorship, Assistant Professors C. J. Nesbitt and E. D. Rainville have been promoted to associate professorships, Assistant Professor Nathaniel Coburn of the University of Texas, Dr. Isaac Opatowski of the University of Chicago, Assistant Professor M. O. Reade of Purdue University and Assistant Professor Hans Samelson of Syracuse University have been appointed to assistant professorships, and Dr. C. L. Dolph has been appointed to a lectureship. Assistant Professor W. A. Ambrose has been granted a leave of absence to serve as Visiting Assistant Professor at Yale University. Professor Peter Field has retired.

The University of Wyoming makes the following announcements: Professor O. H. Rechard has resigned as head of the department to become dean of the

liberal arts college, Professor C. F. Barr has been appointed acting head of the department, Dr. Leonard Bristow, Dr. S. R. Smith and Palmer Steen have been appointed to assistant professorships, Assistant Professor Verne Varineau has returned to the university and Dr. Nathan Schwid has been appointed to an associate professorship.

Associate Professor H. E. Arnold of Wesleyan University has been promoted to a professorship.

Dr. Winifred A. Asprey of Vassar College has been promoted to an assistant professorship.

Dr. S. P. Avann of Oregon State College has been appointed to an assistant professorship at the University of Washington.

Associated Professor H. W. Bailey of the University of Illinois has been appointed Associate Dean for Liberal Arts and Sciences in the Division of Undergraduate Studies of the University of Illinois, Chicago, Illinois.

Assistant Professor J. J. Barron of Marquette University has been promoted to an associate professorship.

Assistant Professor A. E. Basch of Amherst College has been appointed to an associate professorship at the Massachusetts State College at Fort Devens.

J. K. Baumgart has been appointed to an assistant professorship at Elmhurst College, Elmhurst, Illinois.

Dr. Dorothy L. Bernstein of the University of Rochester has been promoted to an assistant professorship.

Dr. E. E. Blanche has accepted the position of principal administrative analyst, Management Branch, Plans and Policy Office, Service, Supply and Procurement Division of the War Department.

Associate Professor M. G. Boyce at Western Reserve University has been appointed to an associate professorship at Vanderbilt University.

Dr. G. W. Brown has been appointed to an associate professorship at the Iowa State College of Agriculture and Mechanical Arts.

Professor C. T. Bumer of Kenyon College has been appointed Head Mathematician, Assistant to the Director of the Evaluation and Analysis Group, Bureau of Ordnance.

Assistant Professor H. E. Burns of Calumet Extension Center, Indiana University, has been appointed director.

K. A. Bush of the United States Naval Academy has been appointed to an assistant professorship at Mohawk College, Utica, New York.

Assistant Professor F. A. Butter, Jr., of Stanford University has been appointed mathematical consultant at the Hughes Aircraft Company, Culver City, California.

Assistant Professor H. H. Campaigne of the University of Minnesota has accepted a position as mathematician with the Navy Department.

Dr. G. F. Carrier has been appointed to an assistant professorship at Brown University.

Dr. A. B. Carson has been appointed to an associate professorship at the Army Air Force Institute of Technology, Wright Field, Dayton, Ohio.

Associate Professor J. W. Cell of the State College of Agriculture and Engineering of the University of North Carolina has been promoted to a professorship.

Assistant Professor C. L. Clark of Oregon State College has been promoted to an associate professorship.

Dr. W. W. S. Claytor has been appointed to a professorship at Hampton Institute.

Assistant Professor A. H. Clifford of the Massachusetts Institute of Technology has been appointed to an associate professorship at Johns Hopkins University.

Professor Lennie P. Copeland of Wellesley College has retired with the title professor emeritus.

Dr. Byron Cosby, Jr., of the United States Naval Academy has been appointed to an assistant professorship at the University of Iowa.

Professor C. C. Craig of the University of Michigan has been appointed director of the Statistical Research Laboratory.

Dr. W. J. R. Crosby of the University of Saskatchewan has been promoted to an assistant professorship.

Dr. E. L. Crow has accepted a position as mathematician at the United States Naval Ordnance Test Station, Inyokern, California.

Dr. J. F. Daly has accepted a position as statistician with the Bureau of the Census.

Dr. Margaret R. Davis has been appointed to an associate professorship at Southeastern Louisiana College, Hammond, Louisiana.

Professor Charlotte I. Davison of Wilson College, Chambersburg, Pennsylvania, has retired with the title professor emeritus. She received the honorary degree of Doctor of Humane Letters from Wilson College in June, 1946.

Dr. W. W. Denton has been appointed principal of the High School, Minden City, Michigan.

Associate Professor Douglas Derry of the University of Saskatchewan has been appointed to an associate professorship at the University of British Columbia.

Dr. J. B. Diaz of Brown University has been appointed to an assistant professorship at Carnegie Institute of Technology.

Sister Agnes T. Dimond of the College of Mount Saint Joseph, Mount Saint Joseph, Ohio, has been promoted to a professorship.

Professor Emeritus L. L. Dines of Carnegie Institute of Technology has been appointed to a visiting professorship at Smith College.

Professor J. E. Dotterer of Manchester College, North Manchester, Indiana, has been appointed professor of mathematics and head of the department.

Dr. D. M. Dribin of the Army Security Agency has been awarded the Legion of Merit.

J. A. H. Duffie has been appointed to a professorship in physics at Assumption College, University of Western Ontario.

Dr. J. N. Eastham of Nazareth College has been appointed to an assistant professorship at the Cooper Union School of Engineering.

Assistant Professor F. G. Fender of Rutgers University has been promoted to a professorship.

Associate Professor E. J. Finan of Catholic University has been promoted to a professorship.

Dr. N. J. Fine has accepted a position with the Operations Evaluation Group, Navy Department.

Assistant Professor F. G. Fisher of the University of Santa Clara has been promoted to an associate professorship.

J. D. Fitzpatrick has been appointed to an assistant professorship at Marquette University.

N. S. Free of Mount Royal College, Calgary, Alberta, Canada, has been appointed to a lectureship at the University of British Columbia.

Dr. Bernard Friedman has been appointed to an assistant professorship at New York University.

Dr. G. N. Garrison of the College of the City of New York has been appointed to an assistant professorship at Lehigh University.

Dr. R. E. Gaskell of Brown University has been appointed to an assistant professorship at the Iowa State College of Agricultural and Mechanical Arts.

Associate Professor F. C. Gentry of the Louisiana Polytechnic Institute has been promoted to a professorship.

M. E. Gillis of the University of Tennessee has been appointed to an acting associate professorship at the University of Florida.

W. H. Glenn of Pasadena Junior College has been appointed head of the division of natural science of Muir Junior College, Pasadena, California.

Professor V. D. Gokhale of the University of the Philippines has been appointed to a visiting lectureship at Rutgers University.

Dr. H. H. Goldstine of the University of Michigan has been promoted to an assistant professorship.

Assistant Professor Michael Golomb of Purdue University has been promoted to an associate professorship.

Assistant Professor S. H. Gould of Victoria College, University of Toronto, has been promoted to an associate professorship.

Dr. A. A. Grau has been appointed to an assistant professorship at the University of Kentucky.

Dr. R. E. Greenwood of the University of Texas has been promoted to an assistant professorship.

Professor W. C. Griffith of Marion Institute, Marion, Alabama, has been appointed to an associate professorship at Centenary College, Shreveport, Louisiana.

Dr. N. A. Hall has been appointed head of the Analysis Section, Research Department, United Aircraft Corporation.

Assistant Professor P. C. Hammer of Oregon State College has been promoted to an associate professorship.

Dr. D. C. Harkin of Brooklyn College has been appointed mathematical consultant at the Naval Research Laboratory.

Assistant Professor F. S. Harper of the University of Nebraska has been appointed to a professorship at Drake University, Des Moines, Iowa.

Assistant Professor O. G. Harrold of Pomona College has been appointed lecturer at Princeton University.

Dr. Philip Hartman has been appointed to an assistant professorship at Johns Hopkins University.

Cecil Hastings has been appointed research analysist with the Douglas Aircraft Company, Santa Monica, California.

Dr. Charles Hatfield, Jr., has been appointed to an assistant professorship at the University of Minnesota.

Professor E. K. Haviland of Lincoln University, Lincoln University, Pennsylvania, has been appointed to an assistant professorship at Johns Hopkins University.

Assistant Professor Euphemia L. Haynes of Miner Teachers College, Washington, D. C., has been promoted to an associate professorship.

Manuel Herschdorfer of Amherst College has been appointed to an assistant professorship at Seton Hall College, South Orange, New Jersey.

J. S. Hickman of Rochester Junior College has been appointed physicist at the United States Naval Electronics Laboratory, San Diego, California.

Assistant Professor E. H. C. Hildebrandt of Northwestern University has been promoted to an associate professorship.

Dr. V. A. Hoersch of the University of Illinois has been promoted to an assistant professorship.

Associate Professor Aughtum S. Howard of Kentucky Wesleyan College, Winchester, Kentucky, has been promoted to a professorship.

Dr. L. C. Hutchinson of the Norden Laboratories has been appointed to an assistant professorship at the Polytechnic Institute of Brooklyn.

Professor J. B. Jackson of the University of South Carolina has been appointed Dean of Men.

A. W. Jacobson has been promoted to an assistant professorship at Wayne University.

Assistant Professor W. C. Janes of the Kansas State College of Agriculture and Applied Science has been promoted to an associate professorship.

J. B. Jeffries has been appointed to a professorship at the Agricultural and Technical College of North Carolina.

Walter Jennings has been appointed to an assistant professorship at Virginia Polytechnic Institute.

Assistant Professor Dora E. Kearney of Iowa State Teachers College has resigned.

J. R. F. Kent has been appointed to an assistant professorship at the University of British Columbia.

Dr. F. L. Kiokemeister of Purdue University has been appointed to an assistant professorship at Mount Holyoke College.

Dr. E. A. Knobelauch of the University of Pennsylvania has been appointed lecturer at Rutgers University.

Dr. G. B. Lang of the University of Florida has been promoted to an assistant professorship.

Associate Professor H. D. Larsen of the University of New Mexico has been promoted to a professorship.

H. R. Leifer has accepted a position as assistant chief, Registration and Research Section, Veterans Administration, Pittsburgh, Pennsylvania.

Associate Professor Hans Lewy of the University of California has been promoted to a professorship.

Dr. F. W. Light of the Johns Hopkins University has been promoted to an assistant professorship.

Assistant Professor H. W. Linscheid of Eastern New Mexico College has been appointed to an associate professorship at the College of Emporia, Emporia, Kansas.

Associate Professor Mayme I. Logsdon of the University of Chicago has retired. During the academic year 1946–47 she will be visiting professor at the University of Miami.

Dr. L. H. Loomis of Harvard University has been promoted to an associate professorship.

R. A. Lytle of the University of Virginia has been appointed to an adjunct professorship at the University of South Carolina.

Associate Professor J. D. Mancill of the University of Alabama has been promoted to a professorship.

Ralph Mansfield of Weidenhoff, Inc., Chicago, Illinois, has been promoted to the rank of chief engineer.

Associate Professor Morris Marden of the University of Wisconsin at Milwaukee has been promoted to a professorship.

Dr. A. J. Maria of Brooklyn College has been promoted to an assistant professorship.

Dr. A. E. May of the University of Wisconsin at Kenosha has been promoted to an assistant professorship.

W. S. McCulley of the Agricultural and Mechanical College of Texas has been promoted to an assistant professorship.

Dr. Janet McDonald of Vassar College has been promoted to an assistant professorship.

Professor W. H. McEwen of Mt. Allison University has been appointed to a professorship at the University of Manitoba.

Dr. A. W. McGaughey of the United States Naval Academy has been appointed to a professorship at Westminster College, New Wilmington, Pennsylvania.

P. E. Meadows has been appointed to an assistant professorship at Washington and Lee University.

Professor H. A. Meyer of Indiana University has been appointed to an associate professorship at the University of Florida.

Associate Professor Herman Meyer of the University of Miami has been promoted to a professorship.

Assistant Professor W. I. Miller of Bucknell University has been promoted to an associate professorship.

Dr. H. J. Miser has been appointed to an assistant professorship at Williams College.

Associate Professor Josephine M. Mitchell of Winthrop College has been appointed to an associate professorship at the Texas State College for Women.

E. C. Molina, formerly with Bell Telephone Laboratories, has been appointed special lecturer at Newark College of Engineering.

Dr. L. J. Monville of John Carroll University has been promoted to an assistant professorship.

Dr. C. W. Moran of Illinois Institute of Technology has accepted a position at Wright Junior College.

Professor Eugenie M. Morenus of Sweet Briar College has retired.

Sister M. Joanne Muggli of Saint Benedict's College, Saint Joseph, Minnesota, has been promoted to an assistant professorship.

Assistant Professor David Nelson of Amherst College has been appointed to an assistant professorship at George Washington University.

Dr. Albert Newhouse of the University of Houston has been promoted to an assistant professorship.

Professor R. E. Norris of State Teachers College, Milwaukee, Wisconsin, has been appointed dean of instruction.

Associate Professor E. P. Northrop of the University of Chicago has been appointed associate dean of the College.

Assistant Professor Emma J. Olson of Kent State University, Kent, Ohio, has been promoted to an associate professorship.

Dr. H. L. Olson of Western Union College, Le Mars, Iowa, has been appointed to a professorship at Indiana Technical College, Fort Wayne, Indiana.

R. I. Pepper has been appointed to an assistant professorship at Winthrop College, Rock Hill, South Carolina.

Dr. A. S. Peters of New York University has been promoted to an assistant professorship.

Associate Professor H. R. Phalen of the College of William and Mary has been promoted to a professorship.

Associate Professor R. J. Pitts of Fort Valley State College, Fort Valley, Georgia, has been promoted to a professorship.

Associate Professor H. S. Pollard of Miami University has been promoted to a professorship.

Assistant Professor A. R. Poole of Montana State College has been appointed to an assistant professorship at Oregon State College.

Assistant Professor E. S. Quade of the University of Florida has been promoted to an associate professorship.

L. J. Quaid of the University of Minnesota has been promoted to an assistant professorship.

Ellen F. Rasor has been appointed professor of mathematics and head of the department at Flora Macdonald College, Red Springs, North Carolina.

Assistant Professor H. W. Raudenbush of Queens College, Flushing, New York, has been promoted to an associate professorship.

Reverend John Raymond of Saint Martin's College, Lacey, Washington, has been promoted to a professorship.

Assistant Professor L. M. Reagan of the Polytechnic Institute of Brooklyn has been appointed to an assistant professorship at the University of Wichita.

Professor L. J. Reed of Johns Hopkins University has been appointed vice president.

Professor P. K. Rees of Southwestern Louisiana Institute, Lafayette, Louisiana, has been appointed to an associate professorship at Louisiana State University.

Assistant Professor Eric Reissner of the Massachusetts Institute of Technology has been promoted to an associate professorship.

Dr. C. E. Rickart of Yale University has been promoted to an assistant professorship.

Assistant Professor R. M. Robinson of the University of California has been promoted to an associate professorship. He will be a visiting lecturer at Princeton University this academic year.

Associate Professor A. E. Ross of Saint Louis University has been appointed to a professorship at the University of Notre Dame.

S. G. Roth of Cooper Union has been appointed to an assistant professorship at New York University.

Assistant Professor Raphael Salem of the Massachusetts Institute of Technology has been promoted to an associate professorship.

Associate Professor George Sauté of Rollins College has been promoted to a professorship.

Dr. Alice T. Schafer has been appointed lecturer at New Jersey College for Women, Rutgers University.

Dr. W. T. Scott of Northwestern University has been promoted to an assistant professorship.

Assistant Professor C. E. Sealander of the University of South Dakota has been appointed to an assistant professorship at the Iowa State College of Agriculture and Mechanical Arts.

M. Anice Seybold of the University of Illinois has been appointed to an assistant professorship at North Central College, Naperville, Illinois.

Dr. M. E. Shanks of the University of Missouri has been appointed to an assistant professorship at Purdue University.

Associate Professor D. T. Sigley of Kansas State College has been appointed research engineer, Institute of Cooperative Research, Johns Hopkins University.

Professor Heinz Simon of Southern Union College has been appointed to a professorship at William Penn College, Oskaloosa, Iowa.

Harold E. Smith of the Junior College of Connecticut has been promoted to an assistant professorship.

Dr. Abraham Spitzbart of the University of Wisconsin has been appointed to an assistant professorship at the University of Wisconsin in Milwaukee.

Assistant Professor D. W. Starr of Southern Methodist University has been promoted to an associate professorship.

Dr. J. C. Stewart of Louisiana State University has been appointed to an assistant professorship at Lawrence College, Appleton, Wisconsin.

Assistant Professor R. R. Stoll of Williams College has been appointed to an assistant professorship at Lehigh University.

Irving Sussman of Johns Hopkins University has been appointed to an assistant professorship at the University of Dayton.

Dr. Ethel Sutherland of East Carolina Teachers College has been appointed to an associate professorship at the State Teachers College, Farmville, Virginia.

Associate Professor H. S. Thurston of the University of Alabama has been promoted to a professorship.

Dr. C. B. Tompkins has been appointed director of research, Engineering Research Associates, Incorporated, Saint Paul, Minnesota.

Dr. G. B. Van Schaack has been appointed to an assistant professorship at Union College, Schenectady, New York.

Dr. T. C. G. Wagner has been appointed to an associate professorship at the University of Maryland.

Associate Professor O. E. Walder of South Dakota State College has been promoted to a professorship.

Dr. P. R. Wallace has been appointed to an associate professorship at McGill University.

Assistant Professor R. M. Walter of New Jersey College for Women, Rutgers University, has been promoted to an associate professorship.

Dr. L. F. Walton of the University of California at Santa Barbara has been promoted to an assistant professorship.

Associate Professor L. E. Ward of the State University of Iowa has been appointed mathematician with the Aircraft Fire Control Section, Research, Development, and Test Organization, Naval Ordnance Test Station, Inyokern, California.

Associate Professor W. G. Warnock of the University of Alabama has been appointed to an associate professorship at Rensselaer Polytechnic Institute.

Dr. J. V. Wehausen has been appointed mathematician at the David Taylor Model Basin.

Professor A. H. Wheeler of Clark University has retired.

Dr. Wilfred Wilson of the University of Illinois has been promoted to an assistant professorship.

Dr. Y. K. Wong of the University of North Carolina has been promoted to an associate professorship.

Associate Professor C. B. Wright of East Texas State Teachers College, Commerce, Texas, has been promoted to a professorship.

Assistant Professor C. R. Wylie of Ohio State University has been appointed to a professorship at the Army Air Force Institute of Technology, Wright Field, Dayton, Ohio.

Assistant Professor S. D. Zeldin of the Massachusetts Institute of Technology has been promoted to an associate professorship.

Sister M. Claudia Zeller of the College of Saint Francis has been promoted to an assistant professorship.

The following appointments to instructorships are announced:

Agricultural and Mechanical College of Texas: W. E. Beeman, J. A. Daum, N. W. Wells

Brooklyn College: Dr. H. S. Kieval

College of the City of New York: J. H. Blau, Dr. Solomon Hurwitz, Herbert Solomon

Cooper Union: M. E. Levenson

Cornell University: Dr. Y. H. Kuo

Creighton University: Helen E. Clarkson

De Paul University: Rev. F. J. Fischer

Michigan State College: Evelyn O. Bychinsky, Laura J. Bullard, Frances E. Davis, H. A. Hanson, Emerson Grindall, H. H. Holloway, C. A. Jacokes, M. S. Miller; in physical science: J. A. Straw

Montana State College: Dr. H. M. Schaerf

Newark College of Arts and Sciences, Rutgers University: J. J. McCarthy

New York University: S. D. Bernardi

Oklahoma Agricultural and Mechanical College: Milton Berg, J. L. Cawood, Helen Dayton, R. D. Morrison, P. C. Stanger

Pennsylvania State College: Helen F. Story

Purdue University: Dr. Lee Byrne

Rutgers University: Dr. L. M. Court, Luis Nanni, Dr. M. A. Hyman, Aaron Galuten

Rice Institute: Dr. H. D. Brunk

Stanford University: A. V. Baez

United States Naval Academy: A. R. Craw, G. J. Mann, Dr. R. C. Rand, Dr. H. K. Sohl, W. J. Strange, E. G. Swafford

The University of California: Dr. C. A. Hayes, Dr. Alfred Horn, Dr. E. L. Lehmann
The University of Chicago: R. S. Fouch, Dr. Daniel Zelinsky
University of Cincinnati: H. D. Lipsich
The University of Michigan: Dr. W. M. Kincaid
University of Pennsylvania: Dr. I. S. Cohen
University of Rochester: Dr. Horace Komm
The University of Wyoming: Rosamond Jones
Washington University: Marlow Sholander, Dr. W. J. Thron
Wellesley College: Kathleen E. Butcher

A. M. Freeman, director of the mathematical laboratory, Boston Fiduciary and Research Association, died May 20, 1946.

Professor Emeritus Dean Townley of Stanford University died March 16, 1946.

The Editor-in-Chief wishes to express his indebtedness to the following persons who have served as referees of papers during 1945 and 1946.

C. B. Allendoerfer, I. A. Barnett, E. F. Beckenbach, Garrett Birkhoff, Henry Blumberg, L. M. Blumenthal, W. C. Brenke, R. H. Cameron, W. B. Caton, F. W. Cheney, R. V. Churchill, Nathaniel Coburn, N. B. Conkwright, J. L. Coolidge, A. H. Copeland, H. S. M. Coxeter, H. V. Craig, John DeCicco, Otto Dunkel, Paul Erdös, G. C. Evans, C. J. Everett, Howard Eves, Tomlinson Fort, J. S. Frame, Philip Franklin, Orrin Frink, Jr., Wallace Givens, J. W. Green, Lois W. Griffiths, P. R. Halmos, M. L. Hartung, G. E. Hay, C. G. Hempel, T. R. Hollcroft, Dunham Jackson, R. A. Johnson, Irving Kaplansky, L. M. Kells, R. E. Langer, D. H. Lehmer.

C. C. MacDuffee, W. T. Martin, C. W. Mendel, F. D. Murnaghan, J. R. Musselman, C. V. Newsom, Ivan Niven, E. P. Northrop, Rufus Oldenburger, Gordon Pall, B. C. Patterson, G. B. Price, Hans Rademacher, Tibor Radó, E. D. Rainville, Haim Reingold, Eric Reissner, M. M. Resnikoff, P. R. Rider, E. D. Schell, O. F. G. Schilling, I. M. Sheffer, T. L. Wade, R. J. Walker, H. S. Wall, Alexander Weinstein, Marie J. Weiss, A. L. Whiteman, D. V. Widder, L. R. Wilcox, John Williamson, and Roscoe Woods.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE ARMY UNIVERSITIES IN EUROPE: A PREVIEW OF THE VETERANS PROBLEM.*

H. M. GEHMAN, University of Buffalo

1. *The Army Education Program in Europe.* After the end of the fighting in Europe in the spring of 1945, many of the soldiers who had fought there were eligible for discharge under the Army's point system and were to be returned to this country as soon as shipping was available. It was evident, however, that many men would have to wait for months in Europe before facilities were available for bringing them home. The Army Education Program was instituted to keep these men occupied, to help them make the transition from military life to civilian life, and in particular to give those who intended to enter educational institutions after discharge a chance to begin their studies while still in the Army.

The program was a comprehensive one. At the apex stood the University Centers, with which this paper is concerned. There were three of these, located at Florence, Italy; at Biarritz, France; and at Shrivenham, England.

The Universities at Shrivenham and Biarritz were planned as comprehensive educational institutions. Each contained eight sections (corresponding to schools in the ordinary civilian university): Agriculture, Commerce, Education, Engineering, Fine Arts (Music and Art), Journalism, Liberal Arts, and Science. Each section in turn was divided into branches (corresponding to departments). The Science section consisted of the five branches: Biology, Chemistry, Geology, Mathematics, and Physics.

2. *Recruiting the faculties.* In the original planning of the Army University Centers, it was contemplated that the faculty would consist largely of civilians brought to Europe just for this purpose. Although many qualified instructors were already in the armed forces, it was difficult to obtain the transfer of enough to staff the universities.

The recruiting of the civilian faculty was done from the Pentagon Building in Washington during June, 1945. I spent most of that month there endeavoring to secure instructors in mathematics for Shrivenham and Biarritz. The qualifications sought by the Army and the restrictions imposed were: Only men with experience in college teaching were desired, and it was hoped to secure men with reputations as outstandingly good teachers. Instructors were to be men, aged

* A portion of an address delivered to the Mathematical Association of America at Ithaca, New York, on August 19, 1946.

42 or over. A physical examination was required, but it was not a very rigorous one. Of course, good character was required. Finally it was necessary that each man be able to obtain a release from his institution and from his family without imposing too many hardships upon either. Since it was contemplated that the universities at Shrivenham and Biarritz would continue for a year, each instructor was asked to agree to stay with the Army for at least seven months, and those who so desired were given contracts for a year. The salary arrangements were liberal, so that no one would suffer financially by accepting a position overseas.

Of those invited to join the mathematics departments of the universities at Shrivenham and Biarritz, about one-third accepted appointments and went overseas during July. Of those who refused, a few could not accept for personal reasons. In most cases, the individual could not be spared by his institution because its mathematics department had already been depleted by leaves of absence for work in connection with the war effort.

The mathematics departments at Shrivenham and Biarritz were among the leading departments in their respective institutions with respect to teaching ability and scientific standing. With one or two exceptions, the civilian instructors in mathematics were full professors, holders of the Ph.D. degree, and members of the Society and the Association. Many were department heads; many were authors of textbooks and contributors to mathematical periodicals; two were members of the editorial staff of this MONTHLY. The military members of the mathematics departments were for the most part younger men and consequently had had less in the way of teaching experience and scientific training, but all those with whom I was acquainted were adequately prepared for the teaching which they were doing.

3. *Shrivenham American University.* The university center in England was located at Shrivenham Barracks, in the village of Shrivenham, Berkshire, about 75 miles due west of London. It was officially known in the beginning as Army University Center Number 1 (AUC #1). Later its name was changed to Shrivenham American University (SAU).

Housing facilities were adequate at Shrivenham, which was one reason for its selection as the site of the university center. The buildings on the post had been erected during the period from 1937 to 1939, with the exception of a 200-year old manor house which was used as the officers' club and mess. The faculty were housed in small brick residences.

There was a shortage of classrooms, and most of the classrooms were small. As for equipment, blackboards were literally boards painted black. Erasers were made by nailing strips of felt to blocks of wood. In the beginning there was a shortage of chalk.

Fortunately there was a sufficient supply of textbooks in mathematics. There were available the Education Manuals,* which are standard textbooks bound in paper covers, and also many textbooks that had been used in the ASTP

* These books are described in this MONTHLY, vol. 52, 1945, pp. 288–289.

courses. After the close of the ASTP, these books had been shipped to England in anticipation of their use in the Army Education Program.

Courses in Mathematics offered at Shrivenham included the usual undergraduate courses: Intermediate Algebra, College Algebra, Plane Trigonometry, Spherical Trigonometry, Plane Analytic Geometry, Differential Calculus, Integral Calculus, Statistics, Advanced Statistics, Astronomy, Review of Engineering Mathematics, Higher Mathematics for Engineers and Physicists, and Theory of Equations.

The course entitled "Review of Engineering Mathematics," taught by C. G. Stipe and D. R. Davis, was open only to students who had completed Calculus. In forty lessons the subjects of Algebra, Trigonometry, Analytic Geometry, and Calculus were reviewed.

Mathematics was a popular subject at Shrivenham. The course registrations were approximately 700 during each of the two terms. In number of registrations, Mathematics was second to Business Administration, and was ahead of English and Foreign Languages, which followed in that order. At Shrivenham there was a free elective system, and mathematics prospered rather than suffered under such a system.

4. *The Shrivenham Mathematics Colloquium.* The Army university centers at Biarritz and Florence have already been described in detail,* and hence this paper is restricted essentially to activities that took place at Shrivenham.

Soon after our arrival at Shrivenham, it was suggested that the mathematics group hold meetings for the presentation of scientific papers. Accordingly on Tuesday, September 4, there was held the first meeting of the Shrivenham Mathematics Colloquium. The speaker at that meeting was C. G. Latimer, his topic being: "The Representation of an Integer as the Sum of Four Squares." Meetings of the Colloquium were held approximately every two weeks. The succeeding speakers were P. R. Rider: "The Analysis of Variance"; J. P. Ballantine: "Triads of Arccotangents"; E. T. Browne: "Concerning a Certain Ring of Homographies." At the next meeting, P. D. Edwards was in charge of a showing of British moving picture films of a mathematical nature. The final meeting on November 27, was addressed by T. C. Benton on the topic: "The Flow of a Perfect Fluid over a Right-Angled Step."

There was a good attendance at all meetings of the Colloquium. Besides the members of the department, there were a few advanced students of mathematics and members of other departments who were regular attendants at the meetings.

5. *Relations of Shrivenham with Oxford University.* In connection with the course in Astronomy at Shrivenham, an attempt was made to arrange a visit of students and faculty to the Greenwich Observatory, but permission for the visit could not be obtained. This was exceptional, because in general the faculty and students of Shrivenham were welcome at all British scientific and educational institutions.

* T. A. Bancroft and R. M. Winger: Mathematics at the American Universities in Europe, this MONTHLY, vol. 53, 1946, pp. 131–135.

Oxford was only twenty miles away and there were frequent contacts between the mathematicians of the two institutions. This was due chiefly to the efforts of T. W. Chaundy of Oxford. Through him the facilities of Oxford University were made available to those who wished to use them. I made frequent visits to the Oxford libraries while preparing the lectures that I delivered at Amsterdam University in December.

On October 26, the members of the Shrivenham Mathematics Department attended the 279th meeting of the Oxford University Mathematical and Physical Society. Professor Whitehead presided and Professor Milne spoke on "The Conservation of Momentum." At a later meeting of the Oxford Society, D. R. Davis and E. T. Browne were the speakers.

6. *Closing of the Army Universities.* The unexpected collapse of Japanese military resistance in August, 1945, changed the plans of the Army for the continued operation of the Army universities. Additional ships were now available for sending men home from Europe. It was decided to concentrate all American troops in Germany and to discontinue the schools in England and in France. Hence, instead of the five two-month terms that were originally planned, Shrivenham was closed at the end of the second term in early December. The university at Biarritz was continued for a third term with half the normal enrollment, and was finally closed on March 15, 1946.

The end of the Japanese war and the resulting influx of students into American colleges in September, 1945, made many faculty members of the Army universities anxious to return to their colleges at the earliest possible date. A number returned to this country in December and January in time to take up their usual teaching duties with the opening of the second semester of 1945–46. Those who wished to carry out the original plans and spend the entire year with the Army were sent to various points in Germany after the universities at Shrivenham and Biarritz had closed. There they were assigned to various educational projects among the troops in the army of occupation.

7. *The Army Education Program in Germany.** During the winter and spring of 1946, army schools giving courses in mathematics of college grade were in operation in Vienna and in the Bremen area. Of the schools conducted at this time by the occupation forces, the most ambitious was the Weihenstephen Agricultural and Technical College conducted by the Third Army at Freising, near Munich. The college was located in buildings erected for a branch of the University of Munich. Courses were given in College Algebra, Plane and Spherical Trigonometry, Analytic Geometry, and Calculus. In addition, Professor Lowenstein conducted an advanced class in the Theory of Numbers.

One characteristic of Weihenstephen was that students studied only one subject. Classes met four hours per day for five days per week, the twenty hours weekly being divided into periods of regular instruction and supervised study. Each course continued for a term of eight weeks.

After April 1, 1946, all courses upon the college level were discontinued. The

* I am indebted to L. L. Lowenstein and P. D. Edwards for information on this program.

schools which continued in operation after that date offered only secondary school subjects.

8. *The Veteran and the Veterans Problem.* Veterans are now presenting many problems to the colleges. While I cannot claim to be an expert on all aspects of the subject of veterans, I and my colleagues in Europe saw these men when they were making their first adjustments from a life of combat to a life of study.

During the first few weeks of the first term at Shrivenham, the Mathematics Office was crowded at all hours of the day (including Saturday morning and afternoon) with soldier students who wished assistance in the preparation of their lessons. During their years of combat duty, many had forgotten how to study, to say nothing of having forgotten the most elementary aspects of mathematics. It was noticeable that at the beginning of the second term there was much less demand for assistance of this sort. During the intervening two months, the soldiers who came to us in the second term had been busy with occupation duty rather than combat duty and consequently had had more leisure time to devote to reading and study. These men had become accustomed to a non-combat type of army life, which is one step toward making an adjustment to civilian life.

In the same way, the present veteran student comes to college either after an interval of civilian life or, if he comes directly from one of the armed services, after a period of non-combat service. In either case he has been able to look forward to college life, has been able to plan for it over a period of time, and if he is wise, has been trying to accustom himself to habits of study. For this reason I believe that the necessity for giving refresher courses has largely passed, although local conditions may make them necessary in some colleges.

In general, the veteran is a normal, well-adjusted person. Travel and contact with persons of various nationalities have given him the ability to adjust easily to new situations. He is serious, earnest, and well-motivated. Many colleges report that the veterans are superior in grades, in reliability, and in studiousness to the general run of students. Often the veteran shows a lack of interest in college life, in social activities and particularly in inter-collegiate athletics. He is interested in finishing college as soon as possible and hence has little time and energy for extra-curricular activities. This is especially true in the case of the veteran with dependents. Because the typical veteran feels the need for making up some of the time he has lost, the increased summer school registration of the past year will probably continue for several years. The veteran feels the necessity for accelerated courses, whether or not the colleges feel that they are educationally desirable.

The great majority of veterans are interested only in professional courses: engineering, science, business, law, medicine, *etc.* In the pursuit of such courses, veterans will take many courses in mathematics. Hence it is safe to predict increased registration in mathematics for the next few years.

I hope that departments of mathematics will not be too rigid in enforcing prerequisites for courses. For example, a veteran may have learned sufficient

trigonometry in his training in navigation or gunnery so that he is prepared for the study of analytic geometry, although no formal credit for a course in trigonometry may appear on his academic record. Compelling him to repeat trigonometry may dampen his ardor for the study of mathematics.

I believe that in spite of the great number of students now in college, faculty members will find that the veterans are anxious to learn, and that many of them are eager to continue their study of mathematics. If the latter point can be kept in mind, we may be able to forget some of our troubles.

THE PREDOCTORAL FELLOWSHIPS

The Rockefeller Foundation made a grant of $335,000 to the National Research Council in the autumn of 1944 to provide predoctoral fellowships in the natural sciences for the benefit of students whose studies were interrupted by war activities. A committee consisting of the following persons was appointed to administer the fellowships: John T. Tate, research professor of physics, University of Minnesota (chairman); Henry A. Barton, director, American Institute of Physics; Charles W. Bray, professor of psychology, Princeton University; Detlev W. Bronk, professor of biophysics, University of Pennsylvania; Luther P. Eisenhart, dean and professor emeritus of mathematics, Princeton University; and W. Albert Noyes, Jr., professor of chemistry, University of Rochester. In January, 1946, Dr. Noyes succeeded Dr. Tate as chairman.

The Committee began its deliberations in June 1945, and applications were first received during the fall of 1945. In a very short time it became apparent that the number of applications would far exceed predictions, and the Rockefeller Foundation subsequently made a grant of $250,000 in addition to the amount originally given.

At each meeting of the committee, awards were apportioned generally according to the number of applications reviewed in each science, but at the final meeting the number awarded was such as to make the distribution among the various sciences approximately proportional to the final percentage of applications in each science, so as to assure the development of men in all fields.

The following table gives the number of applications and the number of awards in each science:

Science	*Applications*	*Awards*
Biology	266	20
Chemistry	733	61
Geology	127	9
Mathematics	207	18
Metallurgy	14	3
Physics	841	75
Psychology	114	10
Total	2,302	196

The stipends are $1,200 per year for single men and $1,800 per year for married men. In addition, an amount up to $500 per year is allowed for tuition and

fees. For a man who receives benefits under the G. I. Bill of Rights the stipend is such as to bring his total income to the appropriate one of the above figures; that is, $420 additional is awarded a single man and $720 additional is given a married man.

Since all awards of fellowships are made for one year and renewal is contingent upon evidence of satisfactory work, the Committee will maintain records of the recipients and receive periodic official reports from the institutions where the men are enrolled.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE ANNUAL MEETING OF THE KENTUCKY SECTION

The twenty-eighth annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Louisville, Louisville, Kentucky, on Saturday, April 27, 1946, in conjunction with the annual meeting of the Kentucky Academy of Science. Dr. Aughtum S. Howard, Chairman of the Section, presided.

There were fifty-nine in attendance, including the following sixteen members of the Association: M. C. Brown, J. C. Eaves, H. P. Fawcett, W. L. Fields, Clarence Ford, Charles Hatfield, Sr., Aughtum S. Howard, W. R. Hutcherson, W. L. Moore, Sister Charles Mary Morrison, S. T. Parker, D. W. Pugsley, W. J. Robinson, J. H. Simester, D. E. South, Guy Stevenson.

At the business meeting the following officers were elected for the coming year: Chairman, Guy Stevenson, University of Louisville; Secretary, M. C. Brown, University of Kentucky.

The following program was presented:

1. *An interesting approach to analytic geometry*, by Dr. W. L. Moore, University of Louisville.

It was pointed out that since analytic geometry is a correspondence between algebra and geometry, and since similar correspondences exist between concepts and science and algebra, then more experience with coördinate systems should be offered to the student. In addition to the rectangular, oblique, and polar coördinate systems, some attention should be given to the theory of algebra of points as outlined in Veblen and Young, and in the Carus Monograph on projective geometry by Young. A brief outline of the basic concepts was then given. Dr. Moore felt that these concepts could be understood by advanced undergraduates.

2. *Color reproduction matrices*, by Sister Mary Charlotte Fowler, Nazareth College, introduced by Sister Charles Mary Morrison.

Matrix algebra was applied to the gelatin matrices of a color print. From a set of equations representing the red, green and blue densities of the reproduction, a color matrix was obtained in which for a set of ideal primaries the elements on the main diagonal were unity, all others zero. In order to reproduce grays correctly, it is often necessary to change these dye concentrations, and this is done by means of the inverse color matrix operating on the vector densities of the reproduction. As a result of this operation, another set of nine coefficients was obtained, and these form the neutralized color matrix. In this matrix the elements on the main diagonal represent the gammas to which the three color elements must be printed to render gray correctly.

3. *Perturbation theory in wave mechanics*, by Professor W. L. Fields, Louisville Municipal College.

The subject of wave mechanics concerns all questions relative to electrons, atoms, systems of electrons and atomic nuclei which lead to the study of Schroedinger's equation, which is

$$\sum \frac{1}{m_k}\left(\frac{\partial^2\psi}{\partial x_k^2}+\frac{\partial^2\psi}{\partial y_k^2}+\frac{\partial^2\psi}{\partial z_k^2}\right)+\frac{8\pi^2}{h^2}(E-V)\psi=0.$$

The solution of this equation cannot be performed rigorously in every case. It is often necessary to resort to perturbations, or other approximate methods for solutions. A small perturbation such as an electrical or magnetic field will change the variables in the equation by amounts which can readily be computed.

4. *Post war mathematics on a college campus*, by Dr. W. R. Hutcherson, Berea College.

On the Berea College campus much recent attention has been devoted to remedial mathematics for the weaker students in the regular mathematics courses. A public mathematical exhibit staged by mathematics majors and the quickened interest in mathematics on the part of veterans were discussed.

5. *The relationship of mathematics to Christianity*, by Professor Beulah Graham, Campbellsville College, introduced by M. C. Brown.

The speaker discussed the relation between mathematical and religious thought.

6. *Some recursion formulas*, by Professor S. T. Parker, University of Louisville.

Certain recursion relations, developed with the aid of Dean W. H. Durfee of Hobart College, were presented. Applications to the solution of Diophantine equations were explained.

7. *The education of mathematics teachers*, by Dr. H. P. Fawcett, Ohio State University.

The speaker cited various qualities which the teacher of mathematics should possess. It was remarked that, in addition to competence in the field of mathe-

matics, the teacher should have a broad cultural background. Courses in the history of mathematics, in fundamental concepts, and in applications, were recommended for teachers, as were other details of teacher training.

8. *Stirling's formula for factorial N*, by Dr. D. E. South, University of Kentucky.

A development of Stirling's formula based upon the well known methods and notations of the calculus of finite differences was discussed. This development had been written to be used in a physics course in which the students had studied no mathematics beyond the integral calculus.

M. C. BROWN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirtieth Annual Meeting, Swarthmore, Pa., December 26–27, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Peoria, May 9–10, 1947
INDIANA
IOWA, Cedar Falls, April 18–19, 1947
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., December 7, 1946
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA
ROCKY MOUNTAIN
SOUTHEASTERN, Columbia, S. C., April, 1947
SOUTHERN CALIFORNIA, Claremont, March 8, 1947
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Madison, May, 1947

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